Geometric Group Theory

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Graphs of groups

Definition

Let Y be an oriented graph such that the corresponding unoriented graph is connected and each of its edges appears with both orientations in Y.

A graph of groups is a pair (G, Y), where G is a map that assigns a group G_v to each vertex $v \in V(Y)$ and a group G_e to each edge $e \in E(Y)$ such that

1. $G_e = G_{\bar{e}}$
2. for all edges e, there exists an injective homomorphism $\alpha_e : G_e \to G_{t(e)}$

where $t(e)$ is the terminus of the edge $e = [o(e), t(e)]$.

Graphs of groups

Definition

The path group of the graph of groups (G, Y) is

$$F(G,Y) =$$
$$\langle \bigcup_{v\in V} G_v \cup E(Y) | \bar{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g), \forall e \in E(Y), g \in G_e\rangle.$$

If $G_v = \langle S_v | R_v\rangle$ then

$$F(G,Y) = \langle \bigcup_{v\in V} S_v \cup E(Y) | \bigcup_{v\in V(Y)} R_v, \bar{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g)\rangle.$$

NB The group we are interested in will be defined first as a subgroup, then as a quotient of $F(G, Y)$.

Graphs of groups

Remarks

1. *If all* $G_v = \{1\}$ *then* $F(G, Y) = F(E^+(Y))$.
2. *There exists an epimorphism* $F(G, Y) \to F(E^+(Y))$ *defined by sending each* G_v *to* $\{1\}$.
3. *If all* $G_e = 1$ *then*

$$F(G, Y) = *_{v \in V(Y)} G_v * F(E^+(Y)).$$

Graphs of groups

Definition

A path in (G, Y) is a sequence

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

such that $t(e_i) = o(e_{i+1})$ and $g_i \in G_{t(e_i)} = G_{o(e_{i+1})}$. If $v_0 = o(e_1)$, $v_n = t(e_n)$ then we call this a path from v_0 to v_n. We call

$$v_0, v_1 = t(e_1) = o(e_2), ..., v_i = t(e_i) = o(e_{i+1}), ..., v_n$$

its sequence of vertices. We define $|c|$ to be the element of the path group $g_0 e_1 g_1 ... e_n g_n$. If $a_0, a_1 \in V(Y)$ then we define

$$\pi[a_0, a_1] = \{|c| : c \text{ a path from } a_0 \text{ to } a_1\}$$

Graphs of groups

Remark

If $a_0, a_1, a_2 \in V(Y)$ and $\gamma \in \pi[a_0, a_1]$, $\delta \in \pi[a_1, a_2]$ then $\gamma\delta \in \pi[a_0, a_2]$.

Proposition

Let (G, Y) be a graph of groups and suppose $a_0 \in V(Y)$. The set $\pi[a_0, a_0]$ is a subgroup of $F(G, Y)$.

Proof.

If

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

is a path from a_0 to a_0 then

$$|c|^{-1} = g_n^{-1}\bar{e}_n g_{n-1}^{-1}...\bar{e}_1 g_0^{-1} \in \pi[a_0, a_0]$$

□

Graphs of groups with basepoint

Proposition

Let (G, Y) be a graph of groups and suppose $a_0 \in V(Y)$. The set $\pi[a_0, a_0]$ is a subgroup of $F(G, Y)$.

We call this subgroup the fundamental group of the graph of groups (G, Y) with basepoint a_0 and denote it $\pi_1(G, Y, a_0)$.

This is the definition as a subgroup. Problem: it seems to depend on a_0.

Fundamental group of (G, Y) wrto a maximal subtree

Definition

Let (G, Y) be a graph of groups, and let T be a maximal subtree of Y. The fundamental group of (G, Y) with respect to T, denoted $\pi_1(G, Y, T)$, is

$$F(G, Y)/\langle\langle\{e : e \in T\}\rangle\rangle$$

This is the definition as a quotient of $F(G, Y)$. Problems: it seems to depend on T. Why isometric to the first?

Let $q : F(G, Y) \to \pi_1(G, Y, T)$ be the quotient map.

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ is an isomorphism to $\pi_1(G, Y, T)$.

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Proof: We define a homomorphism $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$ as follows.

$\forall a \in V(Y)$, there exists a unique geodesic path $e_1, e_2, ..., e_n$ in T from a_0 to a. Set

$$g_a := e_1 ... e_n \in F(G, Y)$$
$$g_{a_0} := 1$$

We first define $\hat{f} : F(G, Y) \to \pi_1(G, Y, a_0)$:

- $\forall g \in G_a$, set $\hat{f}(g) = g_a g g_a^{-1} \in \pi_1(G, Y, a_0)$
- $\forall e \in E^+(Y)$ with $o(e) = a, t(e) = b$, set $\hat{f}(e) = g_a e g_b^{-1} \in \pi_1(G, Y, a_0)$.

Graphs of groups

The definition of $\hat{f}$ is consistent with the relations:

1. $\bar{e} = e^{-1}$ since for $o(e) = a, t(e) = b$,

$$\hat{f}(\bar{e}) = g_b \bar{e} g_a^{-1} = \left(g_a e g_b^{-1}\right)^{-1} = \hat{f}(e)^{-1}.$$

2. $e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g)$ since if $e = [P, Q]$:

$$\begin{aligned}\hat{f}(e\alpha_e(g)\bar{e}) &= (g_P e g_Q^{-1})(g_Q \alpha_e(g) g_Q^{-1})(g_Q \bar{e} g_P^{-1}) \\ &= g_P(e\alpha_e(g)e^{-1})g_P^{-1} \\ &= g_P \alpha_{\bar{e}}(g) g_P^{-1}\end{aligned}$$

So $\hat{f}$ is defined on $F(G, Y)$.

For every $e = [P, Q] \in T$, $\hat{f}(e) = g_P e g_Q^{-1} = 1$.

Hence $\hat{f}$ defines $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$.

Recall that q is the restriction to $\pi_1(G, Y, a_0)$ of the quotient map $F(G, Y) \to \pi_1(G, Y, T)$.

Consider $q \circ f : \pi_1(G, Y, T) \to \pi_1(G, Y, T)$. For all $g \in G_a$,

$$q \circ f(g) = q(g_a g g_a^{-1}) = g.$$

For all $e \notin T$,

$$q \circ f(e) = q(g_P e g_Q^{-1}) = e.$$

Hence $q \circ f = \mathrm{id}$.

Consider now $f \circ q : \pi_1(G, Y, a_0) \to \pi_1(G, Y, a_0)$.

If $g_0 e_1 \dots e_n g_n$ arbitrary in $\pi_1(G, Y, a_0)$ and $e_i = [P_{i-1}, P_i]$,

$$f \circ q(g_0 e_1 \dots e_n g_n) = g_0 (e_1 g_{P_1}^{-1})(g_{P_1} g_1 g_{P_1}^{-1})(g_{P_1} e_2 g_{P_2}^{-1}) \dots g_{P_{n-1}}^{-1}(g_{P_{n-1}} e_n) g_n$$
$$= g_0 e_1 \dots e_n g_n$$

NB If $e_i \in T$ then $f \circ q(g_{i-1} e_i g_i) = f(g_{i-1} g_i) = g_{P_{i-1}} g_{i-1} g_{P_{i-1}}^{-1} g_{P_i} g_i g_{P_i}^{-1} = g_{P_{i-1}} g_{i-1} e_i g_i g_{P_i}^{-1}$. $\square$

Graphs of groups

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ *is an isomorphism to* $\pi_1(G, Y, T)$.

Corollary

The fundamental group $\pi_1(G, Y, a_0)$ *of the graph of groups* (G, Y) *does not depend on the choice of basepoint* a_0.

Corollary

The quotient $\pi_1(G, Y, T)$ *of the path group does not depend on the choice of the tree* T.