Further Partial Differential Equations (2024)
Problem Sheet 4

1. **Linear stability of a two-dimensional Stefan problem**
   Consider the linear stability of the free boundary problem depicted in Figure 2.2 in the limit $\mathrm{St} \to 0$. Assume that the free boundary is moving at constant speed V under a constant temperature gradient $-\lambda_{1,2}$ in each phase before being perturbed, so the solutions take the form

$$u_1(x,y,t) = -\lambda_1(x - Vt) + \tilde{u}_1(x,y,t), \qquad u_2(x,y,t) = -\lambda_2(x - Vt) + \tilde{u}_2(x,y,t)$$

   and the position of the free boundary is given by

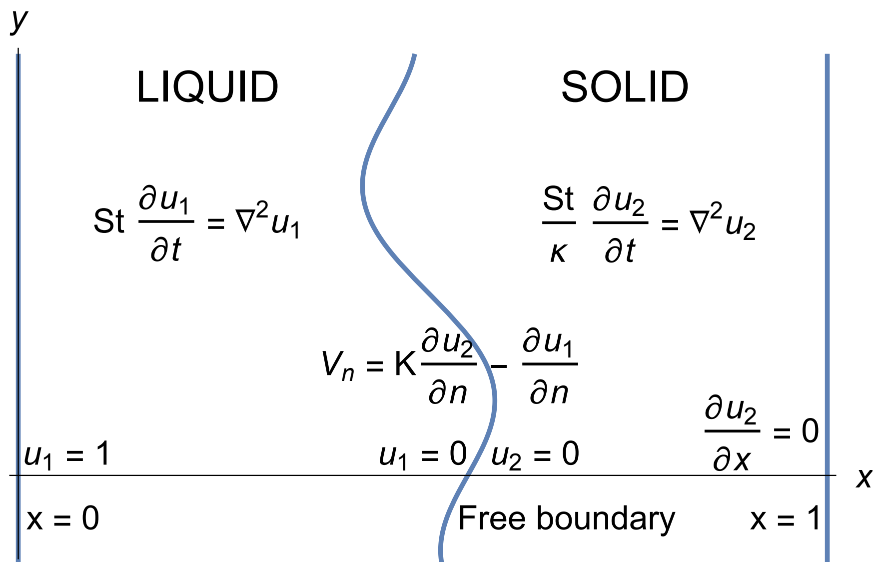
$$x = Vt + \xi(y,t).$$

   By linearising the problem with respect to $\tilde{u}_1$, $\tilde{u}_2$ and ξ, show that perturbations with wavenumber $k > 0$ and growth rate σ are possible provided

$$\frac{\sigma}{Vk} = -\frac{\lambda_1 + K\lambda_2}{\lambda_1 - K\lambda_2}.$$

Solution

We consider the following problem with $\mathrm{St} \to 0$:



We set

$$
\begin{aligned}
u_1 &= -\lambda_1(x - Vt) + \tilde{u}_1, \\
u_2 &= -\lambda_2(x - Vt) + \tilde{u}_2, \\
x &= Vt + \xi(y, t).
\end{aligned}
$$

If the free boundary is $x = f(y, t)$ then the unit normal is

$$
\boldsymbol{n} = \frac{\left(1, -\dfrac{\partial f}{\partial y}\right)}{\sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2}},
$$

the normal derivative is

$$
\frac{\partial u}{\partial n} = \frac{1}{\sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2}} \left(\frac{\partial u}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial u}{\partial y}\right),
$$

and the normal velocity is

$$
V_n = \frac{\dfrac{\partial f}{\partial t}}{\sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2}}.
$$

Now in our case, $f = Vt + \xi(y,t)$, so the free boundary conditions are

$$K\left(-\lambda_2 + \frac{\partial \tilde{u}_2}{\partial x} - \frac{\partial \xi}{\partial y}\frac{\partial \tilde{u}_2}{\partial y}\right) - \left(-\lambda_1 + \frac{\partial \tilde{u}_1}{\partial x} - \frac{\partial \xi}{\partial y}\frac{\partial u_1}{\partial y}\right) = V + \frac{\partial \xi}{\partial t}$$

on $x = Vt + \xi(y,t)$. Considering this at O(1) gives

$$-K\lambda_2 + \lambda_1 = V \qquad \text{on} \quad x = Vt$$

and at next order,

$$K\frac{\partial \tilde{u}_2}{\partial x} - \frac{\partial \tilde{u}_1}{\partial x} = \frac{\partial \xi}{\partial t} \qquad \text{on} \quad x = Vt.$$

Since $u_1 = u_2 - 0$ on the interface, this gives

$$-\lambda_1\xi + \tilde{u}_1 = -\lambda_2\xi + \tilde{u}_2 = 0 \qquad \text{on} \quad x = Vt.$$

The leading-order equations for St $\to 0$ are

$$\begin{aligned} \nabla^2\tilde{u}_1 &= 0, && x < Vt, \\ \nabla^2\tilde{u}_2 &= 0, && x > Vt, \end{aligned}$$

We no longer need to consider the conditions on $x = 0$ and $x = 1$ since we are now just performing a local analysis. Our only requirement is that the perturbations decay away so we seek solutions of the form

$$\begin{aligned} \tilde{u}_1 &= A\exp(\sigma t + \mathrm{i}ky + k(x - Vt)), \\ \tilde{u}_2 &= B\exp(\sigma t + \mathrm{i}ky - k(x - Vt)), \\ \xi &= C\exp(\sigma t + \mathrm{i}ky). \end{aligned}$$

These satisfy Laplace's equation and decay away from the interface. The interface conditions give

$$-kA - KkB = \sigma C$$

and

$$\begin{pmatrix} k & Kk & \sigma \\ 1 & 0 & -\lambda_1 \\ 0 & 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Non-trivial solutions require the determinant of this matrix to be zero, which gives

$$\frac{\sigma}{kV} = -\frac{1}{V}\left(K\lambda_1 + \lambda_2\right) = -\frac{\lambda_1 + K\lambda_2}{\lambda_1 - K\lambda_2}$$

as required.

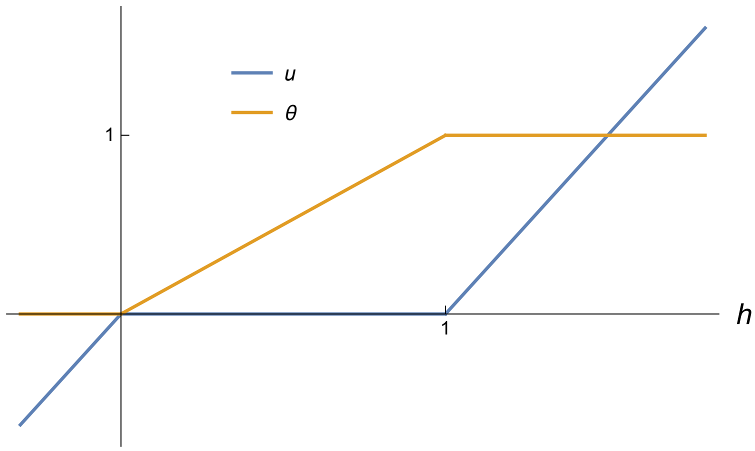


Figure 1: Normalised temperature u and liquid fraction θ versus enthalpy h.

2. **Enthalpy for mushy layers**

Show that the free boundary problem (2.31) may be posed as

$$\frac{\partial h}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q,$$

where $h = \mathrm{St}u + \theta$ is the (dimensionless) *enthalpy*. Deduce that u is a piecewise linear function of h, as indicated in Figure 1.

Solution

This is obtained straightforwardly by substituting in.

3. **Unsteady electropainting**

Consider the unsteady version of the model problem depicted in Figure 2.9, i.e., with the conditions on $y = 0$ replaced by

$$\frac{\partial \phi}{\partial y} = \frac{\phi}{h}, \quad \frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta \qquad y = 0, \ |x| < c, \tag{1}$$

$$\phi = 0 \qquad y = 0, \ |x| > c, \tag{2}$$

where now $c = c(t)$.

(a) By considering the set-up at $t = 0$, show how the boundary conditions (1) simplify and hence find the solution for ϕ at $t = 0$ using the method of images or otherwise.

(b) By substituting this solution into (1) find the early time behaviour for h and thus show that painting commences provided $\delta < 1/\pi$, in which case the layer initially grows over a half-width $c_0 = \sqrt{1/(\delta\pi) - 1}$.

Solution

The unsteady problem is described by

$$\nabla^2 \phi = 0 \tag{3}$$

with

$$\frac{\partial \phi}{\partial y} = \frac{\phi}{h}, \qquad \frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta, \qquad y = 0, \quad |x| < c, \tag{4}$$

$$\phi = 0 \qquad y = 0 \quad |x| > c, \tag{5}$$

$$\phi \sim -\frac{1}{4\pi} \log\left(x^2 + (y-1)^2\right) \qquad \text{as} \quad (x, y) \to (0, 1). \tag{6}$$

(a) At $t = 0$, $h = 0$ so (4) gives $\phi = 0$ and so we have

$$\nabla^2 \phi = 0 \tag{7}$$

with

$$\phi = 0 \qquad y = 0, \tag{8}$$

$$\phi \sim -\frac{1}{4\pi} \log\left(x^2 + (y-1)^2\right) \qquad \text{as} \quad (x, y) \to (0, 1). \tag{9}$$

The solution to this problem is

$$\phi = \frac{1}{4\pi} \log\left(\frac{x^2 + (y+1)^2}{x^2 + (y-1)^2}\right), \tag{10}$$

using the method of images.

(b) So the growth is initially given by

$$\frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta \tag{11}$$

$$= \frac{1}{\pi(1 + x^2)} - \delta, \tag{12}$$

and so

$$h(x, t) \sim \left(\frac{1}{\pi(1 + x^2)} - \delta\right) t. \tag{13}$$

This is valid provided $h \geq 0$ so

$$\frac{1}{\pi(1 + x^2)} \geq \delta \qquad \Rightarrow \qquad |x| \leq \sqrt{\frac{1}{\delta\pi} - 1} \tag{14}$$

as required.

4. **One-dimensional welding**

(a) Derive the dimensionless one-dimensional welding problem (2.31).

(b) Show that the normalised heating coefficient is given by

$$q = \frac{a^2 J^2}{\sigma k(T_{\mathrm{m}} - T_0)} = \frac{\sigma V^2}{k(T_{\mathrm{m}} - T_0)},$$

where V is the applied voltage. Assuming that we require $q = O(1)$ to melt the plate, roughly how high must the voltage be to achieve melting?

Solution

(a) The dimensional problem is

$$\begin{aligned} \rho c \frac{\partial T}{\partial t} &= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{J^2}{\sigma} && 0 \leq x \leq a, \\ \frac{\partial T}{\partial x} &= 0 && \text{on} \quad x = 0, t > 0, \\ T &= T_0 (< T_{\mathrm{m}}) && \text{on} \quad x = a, t > 0, \\ T &= T_0 (< T_{\mathrm{m}}) && 0 < x < a, t = 0. \end{aligned}$$

Non-dimensionalize via

$$\begin{aligned} T &= T_{\mathrm{m}} + (T_m - T_0)\, u, \\ x &= a x', \\ t &= \left(\frac{\rho L a^2}{k (T_{\mathrm{m}} - T0)} \right) t'. \end{aligned}$$

This gives the dimensionless problem (2.31) with

$$q = \frac{J^2 a^2}{k \sigma (T_{\mathrm{m}} - T_0)}.$$

J = current per unit area = I/A.
$V = IR$.
$R = a/\sigma A$ where a is the length of the material.
So $J = V\sigma/a$. So

$$q = \frac{V^2 \sigma}{k (T_{\mathrm{m}} - T_0)}.$$

We need $q = O(1)$ for a chance to melt the plate, so we need

$$V \gtrsim \sqrt{\frac{k (T_{\mathrm{m}} - T_0)}{\sigma}}.$$

Additional questions for practice (will not be marked)

5. **One-dimensional welding**

(a) Derive the dimensionless one-dimensional welding problem (2.31).

(b) Show that the normalised heating coefficient is given by

$$q = \frac{a^2 J^2}{\sigma k (T_\mathrm{m} - T_0)} = \frac{\sigma V^2}{k(T_\mathrm{m} - T_0)},$$

where V is the applied voltage. Assuming that we require $q = O(1)$ to melt the plate, roughly how high must the voltage be to achieve melting?

(c) Consider the dimensionless one-dimensional welding problem (2.31). Show that, before melting occurs, the solution is given by

$$u(x,t) = -1 + \frac{q}{2}\left(1 - x^2\right) + \sum_{n=0}^{\infty} c_n \cos\left[\left(n + \tfrac{1}{2}\right)\pi x\right] \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 \pi^2 t/\mathrm{St}} \tag{15}$$

and use Fourier series to evaluate the constants c_n.

(d) Deduce that the sample will eventually melt provided $q > 2$, at a time t_m that satisfies

$$q = \left(\frac{1}{2} - 2\sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 \pi^2 t_\mathrm{m}/\mathrm{St}}}{\left(n+\frac{1}{2}\right)^3 \pi^3}\right)^{-1}. \tag{16}$$

(e) Show that the leading-order asymptotic dependence of equation (16) between t_m/St and q is

$$\frac{t_\mathrm{m}}{\mathrm{St}} \sim \frac{1}{q} \qquad \text{as} \quad t_\mathrm{m}/\mathrm{St} \to 0,$$

$$\frac{t_\mathrm{m}}{\mathrm{St}} \sim \frac{4}{\pi^2} \log\left(\frac{64}{\pi^3(q-2)}\right) \qquad \text{as} \quad t_\mathrm{m}/\mathrm{St} \to \infty.$$

(Hint: for the second limit, split up the summation (16) *into* $0 \le n \le m$ *and* $m \le n < \infty$ *where* $m^2 t_\mathrm{m}/St \ll 1$ *and* $m \gg 1$*.)*

(f) For $t > t_\mathrm{m}$, consider the free boundary problem (2.31). Explain why $s_2(t) = 0$ until $t = t_\mathrm{m} + 1/q$.

(g) Now consider the limit $\mathrm{St} \to 0$. Show that the plate will have melted entirely to a depth $x = 1 - \sqrt{2/q}$ (so the mush has disappeared) after a time $t_\mathrm{c} \sim t_\mathrm{m} + 1/q + O(\mathrm{St})$.

(h) Show that the subsequent leading-order behaviour of the solid–liquid free boundary $x = s(t)$ is governed by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{q}{2}(1+s) - \frac{1}{1-s}, \qquad\qquad s(t_{\mathrm{c}}) = 1 - \sqrt{\frac{2}{q}}.$$

(i) Deduce that the solid ahead of the free boundary is not superheated, and that the system approaches a steady state with the plate melted to a depth $x = \sqrt{1 - 2/q}$.

Solution

(a) The dimensional problem is

$$\begin{aligned}
\rho c \frac{\partial T}{\partial t} &= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{J^2}{\sigma} && 0 \le x \le a, \\
\frac{\partial T}{\partial x} &= 0 && \text{on} \quad x = 0, t > 0, \\
T &= T_0 (< T_{\mathrm{m}}) && \text{on} \quad x = a, t > 0, \\
T &= T_0 (< T_{\mathrm{m}}) && 0 < x < a, t = 0.
\end{aligned}$$

Non-dimensionalize via

$$\begin{aligned}
T &= T_{\mathrm{m}} + (T_m - T_0)\, u, \\
x &= a x', \\
t &= \left(\frac{\rho L a^2}{k(T_{\mathrm{m}} - T_0)}\right) t'.
\end{aligned}$$

This gives the dimensionless problem (2.31) with

$$q = \frac{J^2 a^2}{k\sigma(T_{\mathrm{m}} - T_0)}.$$

$J =$ current per unit area $= I/A$.
$V = IR$.
$R = a/\sigma A$ where a is the length of the material.
So $J = V\sigma/a$.

(b) From (a) we have

$$q = \frac{V^2 \sigma}{k(T_{\mathrm{m}} - T_0)}.$$

We need $q = O(1)$ for a chance to melt the plate, so we need

$$V \gtrsim \sqrt{\frac{k(T_{\mathrm{m}} - T_0)}{\sigma}}.$$

(c) A particular solution to (2.31) is $u_p = -1 + q/2(1 - x^2)$. We then seek a solution $u = u_p + v$ where v satisfies

$$\begin{aligned}
\mathrm{St} \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2}, && 0 \le x \le 1, && (17) \\
\frac{\partial v}{\partial x} &= 0 && \text{on} \quad x = 0, && (18) \\
v &= 0 && \text{on} \quad x = 1, && (19) \\
v &= -\frac{q}{2}\left(1 - x^2\right) && \text{at} \quad t = 0. && (20)
\end{aligned}$$

Separation of variables gives the general homogeneous solution to this problem as

$$v(x,t) = \sum_{n=0}^{\infty} c_n \cos((n + 1/2)\pi x) \exp\left(-(n + 1/2)^2 \pi^2 t/\mathrm{St}\right)$$

where

$$c_n = -q\int_0^1 (1-x^2)\cos((n+1/2)\pi x) = -\frac{2q(-1)^n}{(n+1/2)^3\pi^3}.$$

is obtained by multiplying $v(x,t)$ by $\cos((m+1/2)\pi x)$ and integrating using the initial condition (20).

(d) The sample will melt if $u = 0$. The first place that this happens will be at $x = 0$. Here,

$$u = \frac{q}{2} - 1 - \sum_{n=0}^{\infty} \frac{2q(-1)^n}{(n+1/2)^3\pi^3} \exp(-(n+1/2)^2\pi^2 t/\mathrm{St}) \tag{21}$$

As $t \to \infty$, $u \to q/2-1$ so we certainly need $q > 2$. Setting $u = 0$ in (21) and rearranging gives (16).

(e) When $t_{\mathrm{m}}/\mathrm{St} \gg 1$ we retain only the first term in the exponential, which gives

$$\frac{1}{q} = \frac{1}{2} - \frac{16}{\pi^2}\exp\left(-\pi^2 t_{\mathrm{m}}/4\mathrm{St}\right), \tag{22}$$

which may be rearranged to give

$$\frac{t_{\mathrm{m}}}{\mathrm{St}} \sim \frac{4}{\pi^2}\log\left(\frac{32q}{\pi^3(q-2)}\right) \sim \frac{4}{\pi^2}\log\left(\frac{64}{\pi^3(q-2)}\right) \qquad \text{as} \quad t_{\mathrm{m}}/\mathrm{St} \to \infty \tag{23}$$

since $q \sim 2$ as $t_{\mathrm{m}}/\mathrm{St} \to \infty$. When $t_{\mathrm{m}}/\mathrm{St} \ll 1$ we split up the summation into $0 \le n \le m$ and $m \le n < \infty$ where $m^2 t_{\mathrm{m}}/St \ll 1$ and $m \gg 1$. Then in the first summation we can expand the exponential while we can neglect the second summation since it is $O(1/m^3)$. This gives

$$\frac{1}{q} = \frac{1}{2} - 2\sum_{n=0}^{m} \frac{(-1)^n}{(n+1/2)^3\pi^3} + 2\sum_{n=0}^{m} \frac{(-1)^n(n+1/2)^2\pi^2}{(n+1/2)^3\pi^3}\frac{t_{\mathrm{m}}}{\mathrm{St}}. \tag{24}$$

Taking the limit as $m \to \infty$ gives

$$\frac{1}{q} = \frac{1}{2} - 2\times\frac{1}{4} + 2\times\frac{1}{2}\frac{t_{\mathrm{m}}}{\mathrm{St}}, \tag{25}$$

and so

$$\frac{t_{\mathrm{m}}}{\mathrm{St}} \sim \frac{1}{q} \qquad \text{as} \quad t_{\mathrm{m}}/\mathrm{St} \to 0. \tag{26}$$

(f) In the mushy region, $\partial\theta/\partial t = q$ so θ takes a time $1/q$ to go from $\theta = 0$ to $\theta = 1$ when a purely liquid region exists.

(g) When all melting is done the mushy layer disappears and we are left with just solid and liquid and an interface $x = s$. In the solid we have

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= -q, && \text{in} \quad s(t) \le x \le 1,\\
u &= -1, && \text{on} \quad x = 1,\\
u &= 0, && \text{on} \quad x = s(t),\\
\frac{\partial u}{\partial x} &= 0, && \text{on} \quad x = s(t),
\end{aligned}$$

which gives $u = -q(x-s)^2/2$ and $s = 1 - \sqrt{2/q}$ as required.

(h) When all melting is done the mushy layer disappears and we are left with just solid and liquid and we have reached the previous state we are reduced to solving a regular Stefan problem again:

$$\frac{\partial^2 u}{\partial x^2} = -q \qquad 0 \le x \le s(t), \tag{27}$$

$$\frac{\partial^2 u}{\partial x^2} = -q \qquad s(t) \le x \le 1, \tag{28}$$

$$\frac{\partial u}{\partial x} = 0 \qquad x = 0, \tag{29}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x}, \qquad x = s(t) \tag{30}$$

$$u^+ = u^- = 0 \qquad x = s(t), \tag{31}$$

$$u = -1, \qquad x = 1. \tag{32}$$

This gives

$$u = \frac{q}{2}(s^2 - x^2) \qquad 0 \le x \le s(t), \tag{33}$$

$$u = (s - x)\left[\frac{1}{1-s} + \frac{q}{2}(x-1)\right], \qquad s(t) \le x \le 1 \tag{34}$$

and so

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{q}{2}(1+s) - \frac{1}{1-s}, \tag{35}$$

which finally gives

$$s = 1 - \sqrt{\frac{2}{q}} \qquad \text{at} \quad t = 0 \tag{36}$$

as required.

(i) The system is superheated if $\partial u^+/\partial x > 0$ at $x = s^+$. Now

$$\left.\frac{\partial u^+}{\partial x}\right|_{x=s^+} = -\frac{1}{1-s} + \frac{1}{2}q(1-s) \tag{37}$$

$$= (1-s)\left[\frac{q}{2} - \frac{1}{(1-s)^2}\right]. \tag{38}$$

Now $s > 1 - \sqrt{2/q}$ for all time, so $q/2 - 1/(1-s)^2 < 0$ and $1 - s^2 > 0$ and therefore $\partial u^+/\partial x < 0$ and the system is not superheated.

As $t \to \infty$, $\mathrm{d}s/\mathrm{d}t \to 0$ so

$$\frac{q}{2}(1+s) = \frac{1}{1-s} \qquad \Rightarrow \qquad s = \sqrt{1 - \frac{2}{q}} \tag{39}$$

as required.

6. **A solid–liquid interface with a density change**

Consider the one-dimensional Stefan problem for melting of a solid considered in lectures. The full system behaviour may be described by equations expressing conservation of mass, momentum and total energy, which are given respectively by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{40}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0, \tag{41}$$

$$\frac{\partial}{\partial t}\left(\rho h + \frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left(pv - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right) = 0, \tag{42}$$

where ρ is the density, v the velocity, p the pressure, T the temperature and

$$h = \begin{cases} c(T - T_{\mathrm{m}}) + L & T > T_{\mathrm{m}} \\ c(T - T_{\mathrm{m}}) & T < T_{\mathrm{m}}. \end{cases}$$

is the *enthalpy* of the system, which is the total energy per unit mass, including heat. Here, c is the specific heat and L the latent heat.

Suppose that liquid occupies a region $0 \le x \le s(t)$ and solid occupies a region $x > s(t)$.

(a) Show that when the density of the fluid and the solid are the same then $v = 0$ and the temperature in the liquid and the solid is described by the one-dimensional heat equation

$$\frac{\partial}{\partial t}(\rho c T) - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = 0. \tag{43}$$

(b) Now suppose that the densities in the solid and the liquid phases are different. Integrate (40) over a domain $x_1 < x < x_2$ that contains the interface (so $x_1 < s(t)$ and $x_2 > s(t)$). Divide the integral into $x_1 \le x \le s(t)$ and $s(t) \le x \le x_2$ and take the limit as $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ to show that the following jump condition is satisfied by the density:

$$[\rho]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = [\rho v]_-^+ . \tag{44}$$

(c) By performing an identical process for (41) and (42) obtain the jump conditions

$$[\rho v]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[\rho v^2 + p\right]_-^+, \tag{45}$$

$$\left[\rho h + \frac{1}{2}\rho v^2\right]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[pv - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right]_-^+ . \tag{46}$$

(d) Explain how these reduce to the Stefan condition presented in lectures when the solid and liquid densities are equal.

Solution

(a) Substitution of constant ρ into (40) gives v as an arbitrary function of time. Since the liquid occupies the region $0 \leq x \leq s(t)$, the boundary $x = 0$ is fixed and so $v = 0$ here and hence $v = 0$ everywhere. Substitution into (41) gives constant pressure gradient p. Substitution into (42) gives the required heat equation.

(b) Equation (40) only applies provided the variables are continuous, and so does not hold across jumps. We thus consider the integrated conservative version,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho \, \mathrm{d}x = [\rho v]_{x_1}^{x_2},$$

where $x_1 < s(t) < x_2$. We divide the integral into parts to the left and right of the jump,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{s(t)} \rho \, \mathrm{d}x + \int_{s(t)}^{x_2} \rho \, \mathrm{d}x = [\rho v]_{x_1}^{x_2} \int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t}\mathrm{d}x + \rho|_{x_1}\frac{\mathrm{d}s}{\mathrm{d}t} + \int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t}\,\mathrm{d}x - \rho|_{x_2}\frac{\mathrm{d}s}{\mathrm{d}t} \quad = [\rho v]_{x_1}^{x_2}$$

using Leibniz' rule. Then, taking the limit $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ and recognizing that

$$\lim_{x_1 \to s(t)^-} \int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t}\,\mathrm{d}x = 0, \qquad \lim_{x_2 \to s(t)^+} \int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t}\,\mathrm{d}x = 0,$$

we obtain the required result,

$$[\rho]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = [\rho v]_-^+ .$$

(c) This may be found easily by following the same steps as above.

(d) When the solid and liquid densities are equal, (45) gives $[p]_-^+ = 0$, so the pressure is continuous across the interface, and

$$\rho L \frac{\mathrm{d}s}{\mathrm{d}t} = -\left[k\frac{\partial T}{\partial x}\right]_-^+ \qquad (47)$$

if we assume that the temperature is continuous across the interface. This is precisely the Stefan condition from the lectures.