Additional Lecture Notes for W7

Theorem 1. *Let X be a reflexive separable Banach space, X^* its dual, and M a non-empty, convex closed subset of X. Let $F : M \to X^*$ satisfies*

(H1) F maps bounded sets in X into bounded sets in X^*.

(H2) (Coercivity) There exists $u_0 \in M$ such that

$$\lim_{\|u\| \to \infty} \frac{\langle F(u), u - u_0 \rangle}{\|u\|} = \infty.$$

(H3) Weak (sequential) lower semi-continuity relative to M: If $u_n \rightharpoonup u$ and $F(u_n) \rightharpoonup \xi$, then

$$\langle \xi, u \rangle \leq \liminf_{n \to \infty} \langle F(u_n), u_n \rangle,$$

and if equality holds then

$$\langle F(u) - \xi, u - v \rangle \leq 0 \text{ for all } v \in M.$$

Then there exists $u \in M$ such that

$$\langle F(u), u - v \rangle \leq 0 \text{ for all } v \in M.$$

We will only present the proof in the case M is a subspace of X. See the lecture notes for the proof in the full generality.

Theorem 2. *Let X be a reflexive separable Banach space, X^* its dual, and $M \subset X$ a closed subspace of X. Let $F : M \to X^*$ satisfies*

(H1) F maps bounded sets into bounded set.

(H2) (Star-like at infinity): There exists $u_0 \in M$ and $K > 0$ such that

$$\langle F(u), u - u_0 \rangle > 0 \text{ if } \|u\| \geq K.$$

(H3) Weak (sequential) lower semi-continuity relative to M: If $u_n \rightharpoonup u$ and $F(u_n) \rightharpoonup \xi$, then

$$\langle \xi, u \rangle \leq \liminf_{n \to \infty} \langle F(u_n), u_n \rangle,$$

and if equality holds then

$$F(u) - \xi \in M^\circ.$$

Then there exists $u \in M$ such that

$$F(u) \in M^\circ \text{ and } \|u\| \leq K.$$

Proof. Step 1: Finite dimensional reduction. Suppose that the theorem is established when $\dim X < \infty$. We show that the theorem remains true in the general case.

Since X is separable, there exist linearly independent vectors $e_1, e_2, \ldots$ such that X is the closed linear span of $\{e_1, e_2, \ldots\}$. Let $X_n = \text{Span}(\{e_1, \ldots, e_n\})$ and $M_n = M \cap X_n$.

For technical reason (see below), we assume that the vector u_0 in (H2) belongs to $Span(e_1) = X_1$.

Noting that, for every $u \in M$, $F(u)$ is a bounded linear functional on X and so is also a bounded linear function on X_n. Hence $F_n := F|_{M_n} : M_n \to X_n^*$. It's easy to see that F_n satisfies (H1) and (H3). As $u_0 \in X_1 \subset X_n$, F_n satisfies (H2). We may thus apply the theorem in the finite dimensional case to obtain $u_n \in M_n$ with $\|u_n\| \leq K$ such that

$$F(u_n) \in (M_n)^\circ \quad \text{that is } \langle F(u_n), v\rangle = 0 \text{ for all } v \in M_n. \tag{*}$$

Since (u_n) is bounded, so is $F(u_n)$ by (H1). Thus, by reflexivity and after passing to a subsequence, we have $u_n \rightharpoonup u$ (with $\|u\| \leq K$) and $A(u_n) \rightharpoonup \xi$. We would like to show that $F(u) \in M^\circ$.

By (*), we have that

$$\langle \xi, v\rangle = 0 \text{ for all } v \in \cup_n M_n \text{ and so all } v \in \overline{\cup_n M_n} = M,$$

that is $\xi \in M^\circ$. In particular, we have $\langle \xi, u\rangle = 0$. On the other hand, we have $\langle F(u_n), u_n\rangle = 0$ by (*), and so by (H3),

$$\langle \xi, u\rangle \leq \liminf \langle F(u_n), u_n\rangle = 0.$$

Hence

$$\langle \xi, u\rangle = \liminf \langle F(u_n), u_n\rangle = 0,$$

which together with (H3) implies $F(u) - \xi \in M^\circ$. Hence $F(u) = \xi + (F(u) - \xi) \in M^\circ$.

Step 2: Reduction to Hilbert setting when $\dim X < \infty$.

Note that $F : M \to X^* \hookrightarrow M^*$. Thus, we may replace X by M, i.e. we may assume $M = X$.

Note that when $\dim X$ is finite, weak and strong convergence are equivalent. Hence (H3) reads as follows: If $u_n \to u$ and $F(u_n) \to \xi$, then $F(u) = \xi$. Note also that, by (H1), for any bounded sequence u_n, $F(u_n)$ is bounded and hence every subsequence of $(F(u_n))$ has a subsequence which converges to $F(u)$. This implies that F is continuous.

Since $\dim X$ is finite dimensional, we can equip X with an inner product denoted by a dot, whose norm is denoted $|\cdot|$. By Riesz representation theorem, every linear function $T \in X^*$ can be identify with an element $t = J(T) \in X$ such that

$$\langle T, x\rangle = t \cdot x.$$

Let $\tilde{F} = J \circ F : X \to X$. Then

(H1) $\tilde{F}$ maps bounded sets into bounded set.

(H2) (Star-like at infinity): There exists $u_0 \in X$ and $K > 0$ such that

$$\tilde{F}(u) \cdot (u - u_0) > 0 \text{ if } \|u\| \geq K.$$

(H3) Continuity: F is continuous.

We aim to show that there exists $u \in X$ such that

$$\tilde{F}(u) = 0 \text{ and } \|u\| \leq K.$$

Note that if $\tilde{F}(u) = 0$, then by (H2), $\|u\| \leq K$. Thus, we only need to find $u \in X$ such that $\tilde{F}(u) = 0$.

Step 3: Proof of the theorem when X is a finite dimensional Hilbert space.

Consider $T = Id - \tilde{F}$ and we aim to show that T has a fixed point. Fix some $R > 0$ for the moment and let P_R denote the closest point projection onto $\bar{B}_R(0)$, that is

$$P_R(x) = \frac{Rx}{|x|}.$$

Then $P_R \circ T$ is a continuous maps from $\bar{B}_R$ into $\bar{B}_R$. By Brouwer's fixed point theorem, there exists $x_R \in \bar{B}_R$ which is fixed by $P_R \circ T$. We claim that, with $R \geq |u_0|$, then $|x_R| < R$, which implies that x_R is also a fixed point of T and we are done. Suppose for a contradiction that $|x_R| = R$ for some $R \geq |u_0|$. Then

$$x_R = P_R(x_R - \tilde{F}(x_R))$$

which implies that, for any $v \in \bar{B}_R$,

$$0 \geq [(x_R - \tilde{F}(x_R)) - x_R] \cdot (v - x_R) = -\tilde{F}(x_R) \cdot (v - x_R) = \tilde{F}(x_R) \cdot (x_R - v).$$

As $R \geq |u_0|$, we may take $v = u_0$ in the above inequality and obtain a contradiction to (H2). $\square$