Geometric Group Theory

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An inspirational quotation

Henri Poincaré argued that the understanding of a structure means the understanding of the group of transformations preserving it, and that the concept of group is innate, and key to reasoning itself.

Henri Poincaré: "The object of geometry is the study of a particular 'group'; but the general concept of group pre-exists in our minds, at least potentially. It is imposed on us not as a form of our senses, but as a form of our understanding.

Only, from among all the possible groups, that must be chosen which will be, so to speak, the standard to which we shall refer natural phenomena."

Reduced paths of graphs of groups

Definition

Let (G, Y) be a graph of groups. A path $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ is reduced if

1. $g_0 \neq 1$ if $n = 0$;
2. If $e_{i+1} = \bar{e}_i$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

We say that $g_0 e_1 ... e_n g_n$ is a reduced word.

Recall that $|c|$ is the element in $F(G, Y)$ represented by a path c.

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

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Proof

First assume that Y is finite. We will argue by induction on the number of edges in Y. If there are no edges, then the theorem holds. So assume the theorem is true for graphs with n edges, and suppose that Y has $n+1$ edges.

Reduced paths of graphs of groups

Case 1: $Y = Y' \cup \{e\}$, $o(e) \in V(Y')$, $v = t(e) \notin V(Y')$. Then

$$F(G, Y) = (F(G, Y') * G_v)*_{\alpha_e(G_e)}$$

with stable letter e. A reduced word containing e corresponds to a reduced word in the HNN extension that is $\neq 1$.

Case 2: $Y = Y' \cup \{e\}$, $\{o(e), t(e)\} \subseteq V(Y')$. Then

$$F(G, Y) = F(G, Y')*_{\alpha_e(G_e)}$$

and the comment above applies again.

Now suppose that Y is infinite. Any reduced path c involves finitely many orbits of vertices and edges and so c lies within a finite subgraph Y_1 of Y.

c is a reduced path in $F(G, Y_1)$ and so $c \neq 1$ in $F(G, Y_1)$. □

Reduced paths of graphs of groups

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

Corollary

For every $v \in V(Y)$, the homomorphism $G_v \to \pi_1(G, Y, T)$ is injective.

Proof.

$G_v \to F(G, Y)$ is injective and $\pi : \pi_1(G, Y, v) \to \pi_1(G, Y, T)$ is an isomorphism. $\square$

Graphs of groups

One can easily see that

1. If Y has 2 vertices and one edge then
$$\pi_1(G, Y, T) = G_u *_{G_e} G_v.$$

2. If Y has 1 vertex and 1 edge with stable letter 'e' then
$$\pi_1(G, Y, T) = G_v *_{\alpha_e(G_e)}$$
and $\theta : \alpha_e(G_e) \to \alpha_{\bar{e}}(G_e) \in G_v$, $\theta(g) = \alpha_{\bar{e}} \circ \alpha_e^{-1}$.

3. If $Y = Y' \cup \{e\}$ and $t(e) = v \notin Y'$ then
$$\pi_1(G, Y, T) = \pi_1(G, Y', T') *_{G_e} G_v.$$

4. If $Y = Y' \cup \{e\}$ and $v = t(e) \in Y'$ then
$$\pi_1(G, Y, T) = \pi_1(G, Y', T) *_{\alpha_e(G_e)}.$$

Reduced words of graphs of groups

We will find a choice of representatives for elements in $F(G, Y)$, where (G, Y) is a graph of groups. For each edge $e \in E(Y)$, pick a set S_e of left coset representatives of $\alpha_{\bar{e}}(G_e)$ in $G_{o(e)}$, with $1 \in S_e$.

Definition

An S-reduced path is a path $(s_1, e_1, ..., s_n, e_n, g)$ with

- $s_i \in S_{e_i} \ \forall i$;
- $s_i \neq 1$ if $e_i = \bar{e}_{i-1}$;
- $g \in G_{t(e_n)}$.

Lemma

Given $a, b \in V(Y)$, every element in $\pi[a, b]$ is represented by a unique S-reduced path.

Reduced words of graphs of groups

Lemma

Given $a, b \in V(Y)$, every element in $\pi[a, b]$ is represented by a unique S-reduced path.

Proof

Existence: Let $\gamma \in \pi[a, b]$ and consider the path $c = (g_0, e_1, g_1, e_2, \dots, g_{n-1}, e_n, g_n)$ such that $t(e_i) = o(e_{i+1})$, $g_i \in G_{t(e_i)} = G_{o(e_{i+1})}$ and $\gamma = |c|$.

We will prove by induction on n that γ can be represented by an S-reduced path. For $n = 0$ it is obvious. For $n = 1$,

$$\gamma = g_0 e_1 g_1 = s_0 \alpha_{\bar{e}_1}(h_0) e_1 g_1 = s_0 e_1 \alpha_{e_1}(h_0) g_1 = s_0 e_1 g_1'$$

A similar argument holds for the inductive step.

Reduced words of graphs of groups

Uniqueness: Consider two reduced paths

$$c = (s_1, e_1, ..., s_n, e_n, g)$$
$$c' = (\sigma_1, \eta_1, ..., \sigma_k, \eta_k, \gamma)$$

such that $|c| = |c'|$. Then

$$\gamma^{-1}\eta_k^{-1}\sigma_k^{-1}...\eta_1^{-1}\sigma_1^{-1}s_1e_1...s_ne_ng = 1$$

We will prove that $c = c'$ by induction on the length. The above word cannot be reduced hence $\eta_1^{-1} = e_1^{-1}$ and $\sigma_1^{-1}s_1 \in \alpha_{\bar{e}_1}(G_{e_1})$. So $\sigma_1 = s_1$. And so we can apply the inductive assumption. □

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions and such that

1. *The quotient graph $H\backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
   - a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
   - b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

Proof: For all $a \in V(Y)$, we define an equivalence relation on $\pi[a_0, a]$ by

$$|c_1| \sim |c_2| \iff |c_1| = |c_2|g \text{ for some } g \in G_a$$

Vertices of the tree:

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

Graphs of groups and actions on trees

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

Every element of $\pi[a_0, a]/\sim$ has a unique representative corresponding to an S-reduced path of the form $(s_1, e_1, \dots, s_n, e_n)$, $o(e_1) = a_0$, $t(e_n) = a$. Thus $V(T)$ can also be identified with S-reduced paths as above.

Edges of the tree: $\{(s_1, e_1, \dots, s_n, e_n), (s_1, e_1, \dots, s_n, e_n, s_{n+1}, e_{n+1})\}$. Connectedness is obvious.

By our definition of edges, a cycle/circuit gives an S-reduced path with corresponding element $1 \in \pi[a_0, a]$ contradicting the uniqueness of the representation of a reduced path.

Graphs of groups and actions on trees

Action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a_0]$ on T: For all $h \in \pi[a_0, a_0]$ and for all $[g] \in V(T)$ (equivalence classes of $\pi[a_0, a]/\sim$) define the action

$$h \cdot [g] = [hg]$$

- $g_1 \sim g_2 \Rightarrow hg_1 \sim hg_2$ and $\{[g_1], [g_2]\}$ edge $\Rightarrow \{[hg_1], [hg_2]\}$ edge.
- If $[g_1], [g_2]$ are such that $h \cdot [g_1] = [g_2]$ then $a_1 = a_2$ where $g_i \in \pi[a_0, a_i]$.
- Conversely, if $[g_1], [g_2] \in \pi[a_0, a]$ then $h = g_2 g_1^{-1} \in \pi[a_0, a_0]$ and $h[g_1] = [g_2]$.

Thus $H \backslash V(T)$ can be identified with $V(Y)$. And likewise $H \backslash E(T)$ can be identified with $E(Y)$.

Stabilisers of vertices: For all $[v] \in V(T)$ with $v \in \pi[a_0, b]$, where $b \in V(Y)$,

$$h \in \mathrm{Stab}([v]) \iff hv \sim v \iff hv = vg_b \text{ for some } g_b \in G_b$$
$$\iff h = vg_b v^{-1} \text{ for some } g_b \in G_b$$

Thus $\mathrm{Stab}([v]) = vG_b v^{-1}$. This relation is in $F(G, Y)$.

Recall that each G_b was embedded in $H = \pi_1(G, Y, a_0)$ as follows:

- for a maximal subtree $T_Y \subset Y$, set $g_b = e_1 \dots e_n$ the unique geodesic path in T_Y from a_0 to b.
- $\forall g \in G_b$, identify it with $\hat{g} = g_b g g_b^{-1}$. Let $\hat{G}_b$ be the image of G_b.

The equality $\mathrm{Stab}([v]) = vG_b v^{-1}$ becomes

$\mathrm{Stab}([v]) = vg_b^{-1} \hat{G}_b g_b g = h\hat{G}_b h^{-1}$, where $h = vg_b^{-1} \in H = \pi_1(G, Y, a_0)$.

Stabilisers of edges: Every edge in $E(T)$ is of the form $\delta = [[v], [vge]]$, $v \in \pi[a_0, a], g \in G_a, \delta = [a, b]$. Then

$$\mathrm{Stab}(\delta) = \mathrm{Stab}(v) \cap \mathrm{Stab}(vge) = vG_av^{-1} \cap (vge)G_b(vge)^{-1}$$
$$= vg(G_a \cap eG_be^{-1})g^{-1}v^{-1} = vg(\alpha_{\bar{e}}(G_e))g^{-1}v^{-1}$$

As before, the equality above is in $F(G, Y)$.

The subgroup $\alpha_{\bar{e}}(G_e)$ of G_a appears as a subgroup $\hat{G}_e$ of H via the map $g \mapsto \hat{g} = g_agg_a^{-1}$. Thus

$$\mathrm{Stab}(\delta) = vgg_a^{-1}\hat{G}_eg_ag^{-1}v^{-1} = h\hat{G}_eh^{-1}, \text{ with } h = vgg_a^{-1} \in H.$$

We denote the tree thus obtained $\mathcal{T}(G, Y, a_0)$ and we call it the universal covering tree or the Bass–Serre tree of the graph of groups (G, Y). □

Graphs of groups and actions on trees

Conversely, if a group Γ acts on a tree T with quotient Y then there exists a graph of groups (G, Y) such that $\Gamma \simeq \pi_1(G, Y, a_0)$.

Indeed, suppose $\Gamma \curvearrowright T$, $Y = T/\Gamma$ and $p : T \to Y$.

Let $X \subset S \subset T$ be such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and $p\big|_{\text{edges of } S}$ is 1-to-1.

Notation: If v is a vertex of Y and e is an edge of Y, let

- v^X be the vertex of X such that $p(v^X) = v$;
- e^S be the edge of S such that $p(e^S) = e$.

A graph of groups with graph Y:

1. The map G:
    - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
    - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.

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   - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
   - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.
2. For each edge e, we define $\alpha_e : G_e \to G_{t(e)}$: For all $x \in V(S)$, define

$$g_x = \begin{cases} 1 & \text{if } x \in V(X) \\ \text{some } g_x \text{ such that } g_x x \in V(X) & \textit{otherwise}. \end{cases}$$

Define $\alpha_e : G_e \to G_{t(e)}, \alpha_e(g) = g_{t(e)} g g_{t(e)}^{-1}$.