A1 Differential Equations I: MT 2024

Additional examples on maximal existence interval, blow-up, global existence

We consider the maximal existence interval (T_-, T_+) for initial value problems of the form

$$y'(x) = f(x, y(x)), y(0) = b.$$

The goal is to

- determine whether $T_- = -\infty$ or $T_- > \infty$ and in the later case find numbers $-\infty < c_3 < c_4 < 0$ so that $T_- \in (c_3, c_4)$
- determine whether $T_+ = \infty$ or $T_+ < \infty$ and in the later case find numbers $0 < c_1 < c_2 < \infty$ $T_+ \in (c_1, c_2)$

for the following problems

1. $$y'(x) = y^3(x) \sin(\frac{\pi}{1 + y(x)^2}) \text{ with } y(0) = a$$

2. $$y'(x) = e^{y(x)} + x \text{ for } y(0) = 1.$$

Sketches of solutions

Note that the function $f(x,y)$ in ① & ② is smooth on $\mathbb{R}^2$ so satisfies a Lip. cond. on every compact rectangle.

Thus Theorem on maximal existence interval implies that $T_+ < \infty$ is only possible if $y(x) \to +\infty$ or $y(x) \to -\infty$ in finite positive time & analogue for $T_- > -\infty$.

Also: All functions used in r.h.s. of comparison ODEs $z'(x) = g(x, z(x))$ are smooth so comparison principle is applicable.

1) $y'(x) = y^3(x) \sin\left(\frac{\pi}{1+y^2(x)}\right)$, $y(0) = b$

Note: If $y(x)$ solves this problem then $\tilde{y}(x) = -y(x)$ solves same ODE with $\tilde{y}(0) = \tilde{b} = -b$

→ enough to discuss behav. of $y(x)$ for $b \geq 0$

$b=0$: Solution $y_0(x) = 0$ so global existence

$b>0$: By uniqueness part of Picard or as special case of comparison principle: Know solutions cannot intersect so have $y(x) > y_0(x) = 0$ $\forall x \in I = (T_-, T_+)$

As $\frac{\pi}{1+y^2} \in (0, \pi]$ where $\sin \geq 0$ have thus $y'(x) \geq 0$ $\forall x \in I$

In particular: $0 \leq y(x) \leq y(0) = b$ $\forall x \leq 0$

so $T_- = \infty$ as $y(x)$ cannot blow up in negative direction.

Also: As $\sin(t) \leq t$ we have

$$0 \leq y'(x) \leq y^3(x) \cdot \frac{\pi}{1+y^2(x)} \leq \pi \cdot y(x)$$

so $(e^{-\pi x} y(x))' \leq 0$ so $y(x) \leq e^{\pi x} y(0)$ $\forall x \geq 0$

Thus $y(x)$ can't blow up at a finite time $T \geq 0$ so we have $T_+ = \infty$

i.e.

$\forall b \in \mathbb{R}$ $(T_-, T_+) = \mathbb{R}$.

2) $y'(x) = e^{y(x)} + x$, $y(0) = 1$

$x \geq 0$ Finite time blowup & upper bound on T_+:

For $x \geq 0$ have $y'(x) \geq e^{y(x)}$

so comparing with $(*) \begin{cases} z'(x) = e^{z(x)} \\ z(0) = 1 \end{cases}$

get $y(x) \geq z(x)$ for all $x \geq 0$ for which both sol. exist

Solving $(*)$ via sep. of var: $\int e^{-z} dz = \int 1 \, dx$

$-e^{-z(x)} = x + c$

At $x = 0$ $-e^{-1} = c$

so $z(x) = -\log(e^{-1} - x)$

i.e. $z(x) \to \infty$ as $x \nearrow e^{-1}$

so $y(x)$ blows up as $x \nearrow T_+$ for some $T_+ \leq e^{-1}$

Lower bound on T_+

As $y'(x) = e^{y(x)} + x \geq 0$ for $x \in [0, T_+) \subset [0, e^{-1})$

we have $y(x) \geq y(0) = 1$ so $e^{y(x)} \geq e$

For $0 \leq x < T_+ \leq e^{-1}$ thus $x \leq e^{-2} \cdot e \leq e^{-2} e^{y(x)}$

so get $y'(x) \leq (1 + e^{-2}) e^{y(x)}$ $\forall x \in (0, T_+)$.

Comparison principle applied with the solution $\tilde{z}(x)$ of $\begin{cases} \tilde{z}'(x) = (1+e^{-2}) e^{\tilde{z}(x)} \\ \tilde{z}(0) = 1 \end{cases}$

which is $\tilde{z}(x) = -\log(e^{-1} - (1+e^{-2})x)$

gives $1 \leq y(x) \leq \tilde{z}(x)$ which prevents a blow-up for as long as $\tilde{z}$ does not blow up, which happens as $x \nearrow \frac{e^{-1}}{1+e^{-2}}$.

Thus $T_+ \geq \frac{e^{-1}}{1+e^{-2}}$

$x \leq 0$

Switching orientation of time consider $\tilde{y}(x) = y(-x)$ which satisfies $\tilde{y}'(x) = -y'(-x) = -e^{\tilde{y}(x)} + x$, $\tilde{y}(0) = 1$.

Behaviour of $(\tilde{1}) \begin{cases} \tilde{y}'(x) = -e^{\tilde{y}(x)} + x \\ \tilde{y}(0) = 1 \end{cases}$ **on $x \geq 0$:**

We cannot have $\tilde{y}(x) \to +\infty$ in finite time since $\tilde{y}'(x) \leq x$ so $\tilde{y}(x) \leq 1 + \frac{1}{2}x^2$ which remains finite.

We cannot have $\tilde{y}(x) \to -\infty$ as $x \nearrow T_+$ for some $T_+ < \infty$ as in this case we'd find that $\forall M > 0$ $\exists \delta_M \geq 0$ s.t. $\tilde{y}(x) \leq -M$ on $[T_+ - \delta_M, T_+)$

wlog chose $\delta_M < T_+/2$. ⇒ get

$\tilde{y}'(x) = -e^{\tilde{y}(x)} + x \geq \frac{T_+}{2} - e^{-M} \geq 0$

↑ provided we initially choose M s.t. $e^{-M} < \frac{T_+}{2}$.

Thus $\tilde{y}(x) \geq \tilde{y}(T_+ - \delta_M)$ on $(T_+ - \delta, T_+)$

so can't have $\tilde{y}(t) \to -\infty$ as $x \nearrow T_+$.

By charact. of max existence interval thus: $\tilde{y}(x)$ exists $\forall x \geq 0$

so solution $y(x)$ of (1) exists $\forall x \leq 0$.