## Section B

3. Assume $r = |\mathbf{r}| \neq 0$ throughout this question. For $\mathbf{k} \in \mathbb{R}^3$ constant, define a vector field

$$\mathbf{A}(\mathbf{r}) \equiv \frac{\mathbf{k} \wedge \mathbf{r}}{r^3}.$$

(a) Show that $\mathbf{A} = \nabla\phi \wedge \mathbf{k}$ for $\phi = 1/r$, and therefore $\nabla \cdot \mathbf{A} = 0$.

**Solution:** Since $\nabla(1/r) = -\hat{\mathbf{r}}/r^2 = -\mathbf{r}/r^3$ we have

$$\mathbf{A} \equiv \frac{1}{r^3}\mathbf{k} \wedge \mathbf{r} = -\mathbf{k} \wedge \nabla(1/r) = \nabla(1/r) \wedge \mathbf{k} = \nabla\phi \wedge \mathbf{k}\,. \tag{12}$$

Now using identity (A.12) from lecture notes, and that curl of a gradient is zero,

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\nabla\phi \wedge \mathbf{k}) = \mathbf{k} \cdot (\nabla \wedge (\nabla\phi)) - \nabla\phi \cdot (\nabla \wedge \mathbf{k}) = 0. \tag{13}$$

(b) Defining the magnetic field $\mathbf{B} \equiv \nabla \wedge \mathbf{A}$, show that $\mathbf{B} = \nabla\big(\mathbf{k} \cdot \nabla(1/r)\big)$. Deduce that $\nabla \cdot \mathbf{B} = 0$ and $\nabla \wedge \mathbf{B} = \mathbf{0}$.

**Solution:** Using identity (A.11) in the lecture notes,

$$\mathbf{B} \equiv \nabla \wedge \mathbf{A} = \nabla \wedge (\nabla\phi \wedge \mathbf{k}) = \nabla\phi(\nabla \cdot \mathbf{k}) + (\mathbf{k} \cdot \nabla)\nabla\phi - \mathbf{k}(\nabla^2\phi) - (\nabla\phi \cdot \nabla)\mathbf{k}\,, \tag{14}$$

The first and last terms are zero, as $\mathbf{k}$ is constant. We also saw $\nabla^2\phi = \nabla^2(1/r) = 0$ for $r \neq 0$ many times before. Thus we are left with only the second term

$$\mathbf{B} = (\mathbf{k} \cdot \nabla)\nabla\phi = \nabla(\mathbf{k} \cdot \nabla(1/r))\ , \tag{15}$$

where we commute the directional derivative $\mathbf{k} \cdot \nabla$ through the gradient. It then follows that $\nabla \wedge \mathbf{B} = \mathbf{0}$ as $\mathbf{B}$ is a gradient, and $\nabla \cdot \mathbf{B} = \nabla^2(\mathbf{k} \cdot \nabla(1/r)) = \mathbf{k} \cdot \nabla(\nabla^2(1/r)) = 0$, again passing the directional derivative back through the Laplacian.

(c) Show that this describes the field of a *magnetic dipole*,

$$\mathbf{B} = \frac{3\,\mathbf{k} \cdot \mathbf{r}}{r^5}\mathbf{r} - \frac{\mathbf{k}}{r^3},$$

where $\mathbf{k} = (\mu_0/4\pi)\mathbf{m}$ in terms of the magnetic dipole moment $\mathbf{m}$. Hence show that

$$\lim_{r\to\infty} \int_{\Sigma_r} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0,$$

where $\Sigma_r$ is a sphere of radius $r$ centred at the origin. Explain why therefore the magnetic flux through *any* closed surface $\Sigma$ is zero. [ *This shows that the magnetic dipole field has zero magnetic charge, even at the origin where it is singular.* ]

**Solution:** Next we compute $\mathbf{B}$ explicitly:

$$\mathbf{B} \;=\; \mathbf{k}\cdot\nabla(\nabla(1/r)) \;=\; -\mathbf{k}\cdot\nabla\left(\frac{\mathbf{r}}{r^3}\right) \;. \tag{16}$$

Note $\mathbf{k}\cdot\nabla\,(x_i) = k_i$, so using also $\nabla\, r = \mathbf{r}/r$ this is

$$\mathbf{B} \;=\; -\frac{\mathbf{k}}{r^3} + \frac{3\,\mathbf{k}\cdot\mathbf{r}}{r^5}\,\mathbf{r} \;=\; \frac{1}{r^3}\left(3\,(\mathbf{k}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{k}\right) \;. \tag{17}$$

We recognize this as the *magnetic dipole* defined in lectures, with

$$\mathbf{k} \;=\; \frac{\mu_0}{4\pi}\,\mathbf{m} \;, \tag{18}$$

with $\mathbf{m}$ the magnetic dipole moment.

For $\Sigma_r$ a sphere of radius $r$ centred on the origin we have $\mathrm{d}\mathbf{S} \;=\; \hat{\mathbf{r}}\, r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, and

$$\int_{\Sigma_r} \mathbf{B}\cdot\mathrm{d}\mathbf{S} \;=\; \int_{\theta,\varphi} \frac{1}{r^3}(3\,\mathbf{k}\cdot\hat{\mathbf{r}} - \mathbf{k}\cdot\hat{\mathbf{r}})\, r^2\, \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \;\to\; 0 \;, \qquad \text{as } r\to\infty \;. \tag{19}$$

Finally consider the flux of $\mathbf{B}$ through any closed surface $\Sigma$. If this does not contain the origin, we may use the divergence theorem together with $\nabla\cdot\mathbf{B} = 0$ to show the integral is zero. On the other hand if $\Sigma$ encloses the origin, consider the region $R$ bounded by $\Sigma$ and a very large sphere $\Sigma_\infty$ centred on the origin. Then

$$0 \;=\; \int_R \nabla\cdot\mathbf{B}\,\mathrm{d}V \;=\; \int_{\Sigma_\infty} \mathbf{B}\cdot\mathrm{d}\mathbf{S} - \int_{\Sigma} \mathbf{B}\cdot\mathrm{d}\mathbf{S} \;. \tag{20}$$

Here the left hand side is zero, as the region $R$ does not contain the origin, while the first term is also zero by the above computation. We deduce that $\int_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{S} = 0$ for any closed surface $\Sigma$.

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6. Consider two different homogeneous linear dielectric materials, with permittivities $\epsilon^+$ and $\epsilon^-$, filling the half-spaces $\{z > 0\} \subset \mathbb{R}^3$ and $\{z < 0\} \subset \mathbb{R}^3$, respectively. A point charge $q$ is placed at $\mathbf{r}_0 = (0, 0, d)$ with $d > 0$.

   (a) What are the boundary conditions for the electric field $\mathbf{E}$ and the displacement field $\mathbf{D}$? Show that there is a potential $\phi$ such that $\mathbf{E} = -\nabla\phi$ everywhere.

   **Solution:** We have $\nabla \times \mathbf{E} = \mathbf{0}$ everywhere, so $\mathbf{E}$ has a potential. At the boundary, the normal component of $\mathbf{D} = \epsilon\mathbf{E}$ and the tangential components of $\mathbf{E}$ are continuous:

$$E_1^+ = E_1^-, \qquad E_2^+ = E_2^-, \qquad \epsilon^+ E_3^+ = \epsilon^- E_3^-. \tag{43}$$

   (b) Use the method of images to determine $\phi$ and $\mathbf{E}$ in both regions.

   **Solution:** Without the boundary, the divergence $\nabla \cdot \mathbf{D}^+ = \epsilon^+ \nabla \cdot \mathbf{E}^+ = q\delta(\mathbf{r} - \mathbf{r}_0)$ would simply dampen the field of a point charge in vacuum by a factor $\epsilon_0/\epsilon^+$, and give the potential

$$\frac{q}{4\pi\epsilon^+}\frac{1}{|\mathbf{r}-\mathbf{r}_0|}. \tag{44}$$

To fulfil the boundary conditions, we need to superimpose a field with zero divergence in the region, like that of a mirror charge $q^-$ at $-\mathbf{r}_0$:

$$\phi^+(\mathbf{r}) = \frac{q}{4\pi\epsilon^+}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{q^-}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}+\mathbf{r}_0|} \quad \text{for} \quad z > 0. \tag{45}$$

In the region $\{z < 0\}$, the divergence of $\mathbf{E}$ is zero, hence the field could be that of a modified charge $q^+$ at $\mathbf{r}_0$:

$$\phi^-(\mathbf{r}) = \frac{q^+}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} \quad \text{for} \quad z < 0. \tag{46}$$

Now imposing the boundary conditions to glue these two solutions together along $\{z = 0\}$. In cylindrical coordinates $\mathbf{r} = (\rho\cos\phi, \rho\sin\phi, z)$, the continuity of the tangential components of $\mathbf{E}$ amounts to

$$0 = \left.\frac{\partial\phi^+}{\partial\rho}\right|_{z=0} - \left.\frac{\partial\phi^-}{\partial\rho}\right|_{z=0} = -\frac{\rho}{(\rho^2+d^2)^{3/2}}\left(\frac{q}{4\pi\epsilon^+} + \frac{q^-}{4\pi\epsilon_0} - \frac{q^+}{4\pi\epsilon_0}\right) \tag{47}$$

and therefore $q\epsilon_0 = (q^+ - q^-)\epsilon^+$. The normal components of $\mathbf{D}$ give

$$0 = \left.\frac{\partial\epsilon^+\phi^+}{\partial z}\right|_{z=0} - \left.\frac{\partial\epsilon^-\phi^-}{\partial z}\right|_{z=0} = -\frac{1}{(\rho^2+z^2)^{3/2}}\left(\frac{(-d)q}{4\pi} + \frac{dq^-\epsilon^+}{4\pi\epsilon_0} - \frac{(-d)q^+\epsilon^-}{4\pi\epsilon_0}\right) \tag{48}$$

and therefore $q\epsilon_0 = q^-\epsilon^+ + q^+\epsilon^-$. In conclusion, we find that with the values

$$q^+ = q\frac{2\epsilon_0}{\epsilon^+ + \epsilon^-} \quad \text{and} \quad q^- = q\frac{\epsilon_0(\epsilon^+ - \epsilon^-)}{\epsilon^+(\epsilon^+ + \epsilon^-)} \tag{49}$$

the image ansatz does indeed solve the boundary conditions. We conclude that

$$\phi(\mathbf{r}) = \begin{cases} \frac{q}{4\pi\epsilon^+}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{\epsilon^+-\epsilon^-}{\epsilon^++\epsilon^-}\frac{1}{|\mathbf{r}+\mathbf{r}_0|}\right) & \text{if } z \geq 0 \text{ and} \\ \frac{2q}{4\pi(\epsilon^++\epsilon^-)}\frac{1}{|\mathbf{r}-\mathbf{r}_0|} & \text{if } z \leq 0. \end{cases} \tag{50}$$

Note that $\phi$ is continuous on all of $\mathbb{R}^3$. We can obtain $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ from $\phi$:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon^+}\begin{cases} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\epsilon^+-\epsilon^-}{\epsilon^++\epsilon^-}\frac{\mathbf{r}+\mathbf{r}_0}{|\mathbf{r}+\mathbf{r}_0|^3} & \text{if } z \geq 0 \text{ and} \\ \frac{2\epsilon^+}{\epsilon^++\epsilon^-}\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} & \text{if } z \leq 0. \end{cases} \tag{51}$$

(c) Consider a surface $\Sigma = \partial R$ that bounds a dielectric region $R \subset \mathbb{R}^3$ with electric polarization density $\mathbf{P}(\mathbf{r})$. Let $\mathbf{n}$ denote the outward normal vector field on $\Sigma$.


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Show that the bound surface charge density on $\mathbf{r} \in \Sigma$ is equal to

$$\sigma(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \mathbf{n}. \tag{52}$$

Thus compute the bound surface charge densities $\sigma^+$ and $\sigma^-$ of both materials (permittivities $\epsilon^\pm$ in half-spaces $\{\pm z > 0\}$) on the boundary surface $\{z = 0\}$.

**Solution:** Consider a small cylinder $C$ with axis $\mathbf{n}$ and height $h$, top and bottom area $A$, with the top outside $R$ and the bottom inside. By the divergence theorem, the bound charge in the cylinder is

$$\text{sigma} = \int_C \rho_{\text{bound}} \mathrm{d}V = \int_C (-\nabla \cdot \mathbf{P}) \mathrm{d}V = -\int_{\partial C} \mathbf{P} \cdot \mathrm{d}\mathbf{S}. \tag{53}$$

In the limit $h \to 0$ only the top and bottom discs of $\partial C$ contribute; in fact the top gives zero because it is outside the medium $R$, hence its polarization there is $\mathbf{P} = 0$. At the remaining base of the cylinder, $\mathrm{d}\mathbf{S} = -\mathrm{d}A\mathbf{n}$, and the claim follows.

For the positive region $\{z > 0\}$, the outward normal is $-\mathbf{e}_3$, and the polarization is $\mathbf{P}^+ = (\epsilon^+ - \epsilon_0)\mathbf{E}^+$. Therefore,

$$\begin{aligned}
\sigma^+ = \mathbf{P}^+(\mathbf{r}) \cdot (-\mathbf{e}_3) &= (\epsilon^+ - \epsilon_0) \left.\frac{\partial \phi^+}{\partial z}\right|_{z=0} && (54)\\
&= \frac{q(\epsilon^+ - \epsilon_0)}{4\pi\epsilon^+} \left(-\frac{z-d}{|\mathbf{r} - \mathbf{r}_0|^3}\right)_{z=0} + \frac{q(\epsilon^+ - \epsilon_0)(\epsilon^+ - \epsilon^-)}{4\pi\epsilon^+(\epsilon^+ + \epsilon^-)} \left(-\frac{z+d}{|\mathbf{r} + \mathbf{r}_0|^3}\right)_{z=0} && (55)\\
&= \frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} \frac{\epsilon^-(\epsilon^+ - \epsilon_0)}{\epsilon^+(\epsilon^+ + \epsilon^-)}. && (56)
\end{aligned}$$

Similarly, in the negative region, $\mathbf{P}^- = (\epsilon^- - \epsilon_0)\mathbf{E}^-$ and the outward normal is $\mathbf{n} = +\mathbf{e}_3$, so

$$\begin{aligned}
\sigma^- = -(\epsilon^- - \epsilon_0) \left.\frac{\partial \phi^-}{\partial z}\right|_{z=0} &= \frac{q}{4\pi} \frac{2(\epsilon^- - \epsilon_0)}{\epsilon^+ + \epsilon^-} \left(\frac{z-d}{|\mathbf{r} - \mathbf{r}_0|^3}\right)_{z=0} && (57)\\
&= \frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}} \frac{\epsilon_0 - \epsilon^-}{\epsilon^+ + \epsilon^-}. && (58)
\end{aligned}$$

Note that $\sigma^\pm = 0$ if $\epsilon^\pm = \epsilon_0$. This must be the case, since when say $\epsilon^+ = \epsilon_0$, then the medium in $\{z > 0\}$ is not polarizable, hence there are no bound charges.

(d) Determine the force on the point charge $q$.

**Solution:** The electric field generated by the image charges (which is the field that is actually generated by the bound surface charges) at $\mathbf{r}_0$ is

$$\mathbf{E}(\mathbf{r}_0) = \frac{q}{4\pi\epsilon^+} \frac{\epsilon^+ - \epsilon^-}{\epsilon^+ + \epsilon^-} \left(\frac{\mathbf{r} + \mathbf{r}_0}{|\mathbf{r} + \mathbf{r}_0|^3}\right)_{\mathbf{r}=\mathbf{r}_0} = \frac{q}{4\pi\epsilon^+} \frac{\epsilon^+ - \epsilon^-}{\epsilon^+ + \epsilon^-} \frac{\mathbf{e}_3}{(2d)^2}. \tag{59}$$

Note that the sign depends on the permittivities. If $\epsilon^- > \epsilon^+$, then $q$ is attracted towards the boundary $\{z = 0\}$; but if $\epsilon^- < \epsilon^+$, then $q$ is pushed away from the boundary.

(e) What happens in the limits $\epsilon^- \to \infty$ and $\epsilon^- \to \epsilon^+$? Interpret these configurations.

**Solution:** In the limit $\epsilon^- = \epsilon^+$, we obtain

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon^+}\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \tag{60}$$

in all of $\mathbb{R}^3$. This makes sense, because the distinction between the two half-spaces has disappeared, so it's just a point charge in a homogenous medium. Note also that the total bound surface charge computed in part (c) becomes $\sigma^+ + \sigma^- = 0$ as expected.

In the limit $\epsilon^- \to \infty$, we obtain

$$\phi(\mathbf{r}) = \begin{cases} \frac{q}{4\pi\epsilon^+}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|} - \frac{1}{|\mathbf{r}+\mathbf{r}_0|}\right) & \text{if } z \geq 0 \text{ and} \\ 0 & \text{if } z \leq 0. \end{cases} \tag{61}$$

This is exactly the situation as for a conducting material in the region $\{z < 0\}$. Also note that the surface charge $\sigma$ in part $(c)$ agrees with the one derived in the lecture notes for this conducting case.

In an infinitely polarizable medium ($\epsilon^- \to \infty$), an electric field will be cancelled completely by the dipoles and $\mathbf{E}^- = 1/\epsilon^- \mathbf{D}^-$ goes to zero. The high polarizability allows for an arbitrarily large bound surface charge, which completely cancels the field in $\{z < 0\}$.