Geometric Group Theory

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Quotations

Ralph Waldo Emerson: “Life is a journey, not a destination.”

Donald Knuth: “It would be nice if we could design a virtual reality in Hyperbolic space, and meet each other there.”

Quasi-isometry

Definition

Let $f : X \to Y$ be a map between metric spaces.

1. We say that f is an (L, A)-quasi-isometric embedding if for some constants $L \geq 1$, $A \geq 0$ and for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A$$

   It is called a quasi-isometry if moreover we have that for all $y \in Y$, there exists some $x \in X$ such that $d(y, f(x)) \leq A$.

2. If $I \subseteq \mathbb{R}$ is an interval, then an (L, A)-quasi-isometric embedding $\gamma : I \to X$ is called an (L, A)-quasi-geodesic.

3. If there exists a quasi-isometry $f : X \to Y$ between two metric spaces then we say that X and Y are quasi-isometric.

Quasi-isometry

The following theorem is our main source of quasi-isometries.

Theorem (Milnor–Švarc)

Suppose G acts by isometries on a metric space X such that

1. a. *X is geodesic;*
   b. *X is proper (closed balls are compact);*
2. *the action is*
   a. *properly discontinuous: i.e. given a compact $K \subseteq X$, the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite;*
   b. *cocompact: i.e. there exists a compact $K' \subseteq X$ such that $GK' = X$;*

then G is finitely generated and every orbit map $G \to X$, $g \mapsto g \cdot x_0$ is a quasi-isometry when G is endowed with a word metric.

Proof is non-examinable.

Quasi-isometry

Corollary

Suppose G is a finitely generated group with some word metric.

1. *If $H \leq G$ is a finite index subgroup then H is quasi-isometric to G.*
2. *If $N \lhd G$ is a finite normal subgroup then G is quasi-isometric to G/N.*
3. *Suppose M is a compact Riemannian manifold. Then $\pi_1(M)$ is quasi-isometric to the universal cover $\tilde{M}$.*

Exercise: Suppose a group G is quasi-isometric to a finitely presented group H. Then G is finitely presented.

Hyperbolic space

Definition

Let X be a geodesic metric space. Given $A \subseteq X$ and $r > 0$, the *r-neighbourhood of A in X* is the subset

$$\mathcal{N}_r(A) = \{x \in X : d(x, A) < r\} \subseteq X.$$

Let $x, y, z \in X$. A *geodesic triangle* $[x, y, z]$ in X is the union of three geodesic paths $[x, y], [y, z], [z, x]$:

$$[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$$

We say that a geodesic triangle $[x, y, z]$ is δ-*slim* for some $\delta \geq 0$ if each side is within a δ-neighbourhood of the other two sides: for example $[x, y] \subseteq \mathcal{N}_\delta([y, z] \cup [z, x])$.

Hyperbolic space

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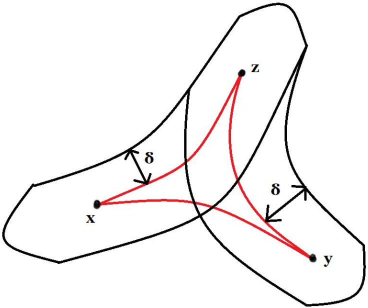
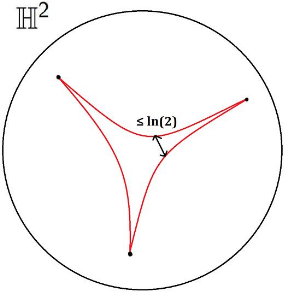


We say that X is δ-hyperbolic if every geodesic triangle is δ-slim.

Examples of δ-hyperbolic spaces

Examples

1. *Any tree is* 0*-hyperbolic.*
2. *Any metric space* X *with finite diameter is* δ*-hyperbolic (for example take* δ *to be the diameter of* X*).*
3. $\mathbb{R}^2$ *is not hyperbolic.*
4. $\mathbb{H}^2$ *is* $\ln(2)$*-hyperbolic:*



δ-thin geodesic triangles

Let $\Delta = [x, y, z]$ be a geodesic triangle in X. There is a unique metric tree T_Δ (a 'tripod' if x, y, z pairwise distinct) with endpoints x', y', z' (corresponding to x, y, z respectively) s. t. there exists an onto map $f_\Delta : \Delta \to T_\Delta$ which restricts to an isometry from each side $[x, y], [y, z], [z, x]$ to the corresponding side $[x', y'], [y', z'], [z', x']$.

Definition

A geodesic triangle $\Delta = [x, y, z]$ in X is δ-thin if for every $t \in T_\Delta$, $diam(f_\Delta^{-1}(t)) \leq \delta$.

Theorem

Let X be a geodesic metric space. X is δ-hyperbolic if and only if there exists some $\delta' \geq 0$ such that every geodesic triangle in X is δ'-thin.

Proof: Exercise 5, Ex Sheet 4.

δ-hyperbolic spaces

Lemma

Suppose X is a geodesic δ-hyperbolic space (i.e. all geodesic triangles are δ-slim) and let $x_0, \dots, x_n \in X$. Then

$$[x_0, x_n] \subseteq \mathcal{N}_{(\log_2(n)+1)\delta}([x_0, x_1] \cup \dots \cup [x_{n-1}, x_n])$$

Proof: Choose k such that $2^{k-1} < n \leq 2^k$. We will prove by induction on k that

$$[x_0, x_n] \subseteq \mathcal{N}_{k\delta}([x_0, x_1] \cup \dots \cup [x_{n-1}, x_n])$$

For $k = 1$ this is obvious. Assume true for $k-1$ and let $p \in [x_0, x_n]$. Let $m = 2^{k-1}$. By hyperbolicity, there exists $p_1 \in [x_0, x_m] \cup [x_m, x_n]$ such that $d(p_1, p) \leq \delta$. By the inductive hypothesis

$$d(p_1, [x_0, x_1] \cup \dots \cup [x_{n-1}, x_n]) \leq (k-1)\delta$$

and the result follows. □

δ-hyperbolic spaces

Proposition (Morse lemma)

Let X be a δ-hyperbolic metric space. For any $\lambda \geq 1$ and $\mu \geq 0$, there exists some $M = M(\lambda, \mu)$ such that if

- *$\alpha : [u, v] \to X$ is a (λ, μ)-quasi-geodesic with endpoints $x = \alpha(u)$, $y = \alpha(v)$;*
- *$\gamma = [x, y]$ is a geodesic with the same endpoints as α;*

then $\alpha \subseteq \mathcal{N}_M(\gamma)$ and $\gamma \subseteq \mathcal{N}_M(\alpha)$.

Proof

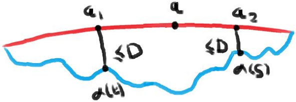
Without loss of generality we can assume α is continuous and such that

$$\mathrm{length}(\alpha([t, s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu$$

for every $t, s \in [u, v]$ (see Exercise 3 on Ex. Sheet 4).

δ-hyperbolic spaces

Step 1: Let $a \in \gamma$ be such that $d(a, \alpha) = D$ is maximal. Let $a_1 \neq a_2$ be points on γ such that, for $i \in \{1, 2\}$, either $d(a, a_i) = 2D$ or a_i is one of the endpoints of γ and $a_i \in B(a, 2D)$. Also let $\alpha(t)$ and $\alpha(s)$ be points in α realising $d(a_1, \alpha)$, $d(a_2, \alpha)$ respectively.



Consider the path $\beta = [a_1, \alpha(t)] \cup \alpha[t, s] \cup [\alpha(s), a_2]$. Then $d(a, \beta) \geq D$. Pick $x_0 = \alpha(t), x_1, ..., x_n = \alpha(s)$ such that $d(x_i, x_{i+1}) = 1$ for each $0 \leq i \leq n - 2$ and $d(x_{n-1}, x_n) \leq 1$. Then

$$a \in \mathcal{N}_{(\log_2(n+2)+1)\delta}([a_1, \alpha(t)] \cup ... \cup [a_2, \alpha(s)])$$

and so $(\log_2(n + 2) + 1)\delta \geq D - 1$.

$$(\log_2(n+2)+1)\delta \geq D-1$$

Also,

$$n-1 \leq \mathrm{length}(\alpha([t,s])) \leq 6\lambda D + \mu$$

as $d(\alpha(t), \alpha(s)) \leq 6D$. Hence,

$$\frac{D-1}{\delta} \leq \log_2(6\lambda D + \mu + 3) + 1$$

and therefore $D \leq L$.

Step 2: Let $b = \alpha(t) \in \alpha(I)$.

General fact: If K is compact, then $d(p, K)$ is continuous in p.

δ-hyperbolic spaces

General fact: If K is compact, then $d(p, K)$ is continuous in p.

Let $a \in \gamma$ be at the maximal distance from $x = \alpha(u)$ such that

$$d(a, \alpha[u, v]) \geq d(a, \alpha([u, t]))$$

Then

$$d(a, \alpha[u, v]) = d(a, \alpha([u, t])) = d(a, \alpha([t, v]))$$

and so there exists $b_1 \in \alpha([u, t])$ such that $d(a, b_1) \leq L$ and there exists $b_2 \in \alpha([t, v])$ such that $d(a, b_2) \leq L$. Let s_i be such that $b_i = \alpha(s_i)$. Then $s_1 \leq t \leq s_2$. We have $|s_1 - s_2| \leq 2\lambda L + \mu$ and so

$$s_2 - t \leq 2\lambda L + \mu$$

Hence, $d(b, b_2) \leq 2\lambda L + \mu$. And so $d(b, a) \leq 2\lambda L + \mu + L$ which concludes the proof. □