A12165W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part B: Paper B5.4

WAVES AND COMPRESSIBLE FLOW

TRINITY TERM 2024

Monday 3 June, 9:30am to 11:15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. In the absence of any body force, the density ρ, velocity $\mathbf{u}$ and pressure p in a homentropic inviscid fluid satisfy the equations of motion

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p, \qquad p = k\rho^{\gamma},$$

where $k > 0$ and $\gamma > 1$ are constants.

(a) [7 marks] Define the *circulation* Γ around a material curve $C(t)$ that is convected by the fluid.

Prove *Kelvin's Circulation Theorem*, namely that Γ is independent of time.

(b) [7 marks] Define what it means for the flow to be *irrotational.* Deduce from part (a) that, if the flow is irrotational initially, then it remains irrotational for all time.

Suppose the flow is indeed irrotational and described by a velocity potential ϕ such that $\mathbf{u} = \boldsymbol{\nabla}\phi$. Define the *speed of sound* c for a homentropic gas, and derive *Bernoulli's Equation*, namely

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\boldsymbol{\nabla}\phi|^2 + \frac{c^2}{\gamma - 1} = F(t),$$

where F is an arbitrary function.

(c) [11 marks] Consider steady radially symmetric flow in two dimensions, with $\rho = \rho(r)$ and $\phi = \phi(r)$, where (r, θ) are plane polar coordinates. The fluid flows radially outwards such that mass q per unit time and per unit length in the z-direction crosses any circle enclosing the origin. As $r \to \infty$, the fluid velocity tends to zero while the density approaches a constant value ρ_0.

By evaluating $\phi'(r)$ and using Bernoulli's equation, obtain the relation

$$\frac{q^2}{4\pi^2 r^2} = \frac{2\rho_0^2 c_0^2}{(\gamma - 1)} \left(\frac{\rho}{\rho_0}\right)^2 \left[1 - \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}\right],$$

where c_0 is the speed of sound at infinity.

By considering the function $f(\xi) = 2\xi^2 \left(1 - \xi^{\gamma-1}\right)$ for $0 < \xi < 1$, show that such a solution can only exist for $r > r_*$, where

$$r_* = \left(\frac{\gamma + 1}{2}\right)^{(\gamma+1)/2(\gamma-1)} \frac{q}{2\pi\rho_0 c_0}.$$

2. Inviscid incompressible fluid of constant density ρ undergoes two-dimensional irrotational flow in the (x, z)-plane, with gravity g acting in the negative z-direction. The fluid lies between a rigid impermeable boundary at $z = -h < 0$ and a free surface at $z = \eta(x, t)$, where t is time and $|\eta| \ll h$. You may assume that the dynamic boundary condition at the free surface is given by $p - p_a = -\gamma\kappa$, where p_a and γ are the constant atmospheric pressure and surface tension, respectively, and

$$\kappa = \left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{-3/2} \frac{\partial^2 \eta}{\partial x^2}$$

is the curvature.

(a) [6 marks] Explain briefly why there exists a velocity potential ϕ which satisfies Laplace's equation.

Write down the boundary condition satisfied by ϕ at $z = -h$, and derive the linearised boundary conditions satisfied by η and ϕ at $z = 0$.

[*A suitable form of* Bernoulli's Theorem *may be quoted without proof.*]

(b) [7 marks] Suppose the free surface starts from rest at $t = 0$ with a given initial displacement $\eta(x, 0) = \eta_0(x) = a\mathrm{e}^{-\lambda|x|}$, where a and λ are given positive constants.

By taking a Fourier transform in x, show that the Fourier transform of η in the subsequent motion is given by

$$\widehat{\eta}(k, t) = \frac{2a\lambda}{\lambda^2 + k^2} \cos\big(\omega(k)t\big),$$

where

$$\omega(k)^2 = \left(g + \frac{\gamma}{\rho} k^2\right) k \tanh(kh).$$

[*You may quote without proof the Fourier transform of η_0, namely $\widehat{\eta_0}(k) = \dfrac{2a\lambda}{\lambda^2 + k^2}$.*]

(c) [8 marks] Consider the limit where the layer is shallow and surface tension dominates gravity, so that $\omega(k) \approx (\gamma h/\rho)^{1/2} k^2$. In the limit where $x = O(t)$ and $t \to \infty$, use the method of stationary phase to obtain the leading-order approximation

$$\eta(x, t) \sim \frac{A t^{3/2}}{x^2 + (4\lambda^2 \gamma h/\rho)\, t^2} \cos\left(\sqrt{\frac{\rho}{\gamma h}} \frac{x^2}{4t} - \frac{\pi}{4}\right), \tag{1}$$

where the constant prefactor A is to be determined.

[*You may assume that, for sufficiently smooth f and ψ, such that ψ is real-valued and has a simple turning point at k_*,*

$$\int_{-\infty}^{\infty} f(k) \mathrm{e}^{\mathrm{i}\psi(k)t}\, \mathrm{d}k \sim f(k_*) \mathrm{e}^{\mathrm{i}\psi(k_*)t \pm \mathrm{i}\pi/4} \sqrt{\frac{2\pi}{|\psi''(k_*)|\, t}} \qquad \text{as} \quad t \to \infty,$$

where $\pm$ takes the sign of $\psi''(k_*)$.]

(d) [4 marks] Explain why the approximation (1) is valid only if

$$\sqrt{gh} \ll \frac{x}{t} \ll \sqrt{\frac{\gamma}{\rho h}}.$$

3. In one-dimensional unsteady flow of an ideal inviscid gas, with ratio of specific heats $\gamma > 1$, the density $\rho(x,t)$, velocity $u(x,t)$ and speed of sound $c(x,t)$ satisfy the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0,$$
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\rho c^2}{\gamma}\right) = 0,$$
$$\frac{\partial}{\partial t}\left(\frac{c^2}{\rho^{\gamma-1}}\right) + u\frac{\partial}{\partial x}\left(\frac{c^2}{\rho^{\gamma-1}}\right) = 0.$$

Gas with constant density ρ_0 and speed of sound c_0 is initially at rest in $x > 0$ with a stationary piston at $x = 0$. For $t > 0$, the piston is gradually withdrawn to a position $x = -s(t)$ at time t, where $s(0) = \dot{s}(0) = 0$ and $\ddot{s}(t) > 0$ for $t > 0$.

(a) [8 marks] Show that the *Riemann invariants* $u \pm 2c/(\gamma - 1)$ are constant along the *characteristics* satisfying $\mathrm{d}x/\mathrm{d}t = u \pm c$.

(b) [8 marks] Sketch the piston position and the characteristics in the (x,t)-plane.
Explain why the gas remains undisturbed in $x > c_0 t$.
For $-s(t) < x < c_0 t$, obtain the parametric solution

$$u = -\dot{s}(\tau), \qquad c = c_0 - \frac{\gamma-1}{2}\,\dot{s}(\tau), \qquad x = -s(\tau) + (t-\tau)\left(c_0 - \frac{\gamma+1}{2}\,\dot{s}(\tau)\right),$$

where $\tau > 0$.

(c) [5 marks] Let $p = \rho c^2/\gamma$ denote the pressure in the gas and $p_0 = \rho_0 c_0^2/\gamma$ its initial value. Suppose that the piston has mass m and cross-sectional area A, and is subject to a constant externally applied pressure $P \in (0, p_0)$ acting in the positive x-direction. Explain why

$$m\ddot{s}(t) = A(p - P) \quad \text{at } x = -s(t)$$

and thus derive the equation of motion

$$\mu\ddot{s} = \left(c_0 - \frac{\gamma-1}{2}\,\dot{s}\right)^{2\gamma/(\gamma-1)} - c_*^{2\gamma/(\gamma-1)},$$

giving expressions for the positive constants μ and $c_* < c_0$.

(d) [4 marks] Sketch a graph of the piston position $s(t)$ as a function of t. What happens to the speed of the piston in the limit as $t \to \infty$?