Gödel Incompleteness Theorems: Solutions to sheet 4

1. (i) Prove that for any sentence X, $\mathrm{PA} \vdash (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}))$.

Let $L = (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}))$.

We assume $\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner L\urcorner})$.

Using the assumption, the third provability rule (Theorem 5.1.3), the second rule, and MP, we obtain

$$(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner})\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner})\urcorner})).$$

$$(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner}) \to (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner})\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner})))$$

is an instance of the second provability rule (Theorem 5.1.2.).

We now use propositional logic to deduce from the formulae in the last two paragraphs the formula

$$(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner}) \to (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner}))\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner})))).$$

By the Third Provability Rule,

$$\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner})\urcorner}\urcorner}).$$

Now use more propositional logic to deduce

$$(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner}) \to X)\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner X\urcorner})),$$

which is L.

Hence $\mathrm{PA} \vdash (\mathrm{Pr}(\overline{\ulcorner L\urcorner}) \to L)$.

Now by Löb's Theorem, $\mathrm{PA} \vdash L$*, which is the required result.*

(ii) Show that $\mathrm{PA} \vdash (\mathrm{Con}_{\mathrm{PA}} \to \neg\, \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Con}_{\mathrm{PA}}\urcorner}))$.

The given formula is the contrapositive of $(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner(\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \bot\urcorner}) \to \bot)\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \bot\urcorner}))$*, where* $\bot$ *is* $\neg(\overline{\ulcorner 0\urcorner} = \overline{\ulcorner 0\urcorner})$*, and we can deduce this statement from the first part.*

(iii) Show that for X any Π_1 sentence, if $\mathrm{PA} \cup \{\neg\, \mathrm{Con}_{\mathrm{PA}}\} \vdash X$, then $\mathrm{PA} \vdash X$.

By the deduction theorem, $\mathrm{PA} \vdash (\neg\, \mathrm{Con}_{\mathrm{PA}} \to X)$.

Thus $\mathrm{PA} \vdash (\neg X \to \mathrm{Con}_{\mathrm{PA}})$.

So, using provability rules, $\mathrm{PA} \vdash (\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \neg X\urcorner}) \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \mathrm{Con}_{\mathrm{PA}}\urcorner}))$.

Now since $\neg X$ *is* Σ_1, $\mathrm{PA} \vdash (\neg X \to \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \neg X\urcorner}))$.

So we have $\mathrm{PA} \vdash (\neg X \to \mathrm{Pr}_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}))$.

However from $\mathrm{PA} \vdash (\neg\, \mathrm{Con}_{\mathrm{PA}} \to X)$*, we can deduce that* $\mathrm{PA} \vdash (\neg X \to \mathrm{Con}_{\mathrm{PA}})$*, and then from the previous part that* $\mathrm{PA} \vdash (\neg X \to \neg\, \mathrm{Pr}_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}))$.

So from $\neg X$ *we get a contradiction.*

So $\mathrm{PA} \vdash X$.

2. Show that $\mathrm{PA} \vdash (\mathrm{Con}_{\mathrm{PA}} \to \mathrm{Con}_{\mathrm{PA} \cup \neg \mathrm{Con}_{\mathrm{PA}}})$.

$(\mathrm{Con}_{\mathrm{PA}} \to \mathrm{Con}_{\mathrm{PA} \cup \{\mathrm{Con}_{\mathrm{PA}}\}})$ *is* $(\neg \mathrm{Pr}_{\mathrm{PA}}(\bot) \to \neg \mathrm{Pr}_{\mathrm{PA}}(\neg \mathrm{Con}_{\mathrm{PA}} \to \bot))$ *for some contradiction* $\bot$*, which is equivalent to* $(\neg \mathrm{Pr}_{\mathrm{PA}}(\bot) \to \neg \mathrm{Pr}_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}))$*, which is equivalent to* $(\neg \mathrm{Pr}_{\mathrm{PA}}(\bot) \to \neg \mathrm{Pr}_{\mathrm{PA}}(\neg \mathrm{Pr}_{\mathrm{PA}}(\bot)))$*, which is equivalent to* $(\mathrm{Pr}_{\mathrm{PA}}(\neg \mathrm{Pr}_{\mathrm{PA}}(\bot)) \to \mathrm{Pr}_{\mathrm{PA}}(\bot))$*, which follows from the Second Incompleteness Theorem.*

3. Verify that the following formulae are fixed points for the operators $p \mapsto A(p)$ given.

You could solve these by showing that the formulae given are provably equivalent to the fixed points you would derive using the Fixed Point Theorem. I will attempt to prove the statements directly.

(i) $(\Box q \to q)$ is a fixed point for $A(p) = (\Box p \to q)$.

The question here is of proving that $(\Box q \to q)$ *is* $\Box$*-equivalent to* $(\Box(\Box q \to q) \to q)$*.*

So, first, let us prove that

$$\vdash \Box((\Box q \to q) \to (\Box(\Box q \to q) \to q))$$

in GL logic.

To begin with,

$$(\Box(\Box q \to q) \to \Box q)$$

is an axiom and therefore a theorem.

Then, using MP, we obtain

$$\vdash ((\Box q \to q) \to (\Box(\Box q \to q) \to q)),$$

as required.

Now secondly let us prove that

$$\vdash \Box((\Box(\Box q \to q) \to q) \to (\Box q \to q)).$$

The formula

$$q \to (\Box q \to q)$$

is an instance of a propositional tautology.

Using necessitation, and using an axiom and a rule to push the $\Box$ *operator through a* $\to$*, we have*

$$\vdash \Box q \to \Box(\neq q \to q).$$

So using propositional calculus

$$\vdash \big(\Box(\neq q \to q) \to q\big) \to (\Box q \to q).$$

(ii) $\Box q$ is a fixed point for $A(p) = \Box(p \leftrightarrow (\Box p \to q))$.

First, we show that $\vdash (\Box q \to \Box((\Box \Box q \to q) \to \Box q))$*.*

The following formula is a propositional tautology:

$$\vdash (\Box q \to ((\Box \Box q \to q) \to \Box q)).$$

Then by Necessitation,

$$\vdash \Box(\Box q \to ((\Box\Box q \to q) \to \Box q)).$$

Pushing the box through the arrow using the appropriate axiom scheme and MP,

Theorem 7.2.1. (the Solovay completeness theorem, though I didn't give it that name) tells us that

$$\vdash (\Box q \to \Box\Box q).$$

So by propositional logic,

$$\vdash (\Box q \to \Box((\Box\Box q \to q) \to \Box q)).$$

For the other half of the forward direction, we begin with a propositional tautology:

$$\vdash (q \to (\Box q \to (\Box\Box q \to q))).$$

Now we apply necessitation, push the box through an arrow and use MP, to get

$$\vdash (\Box q \to \Box(\Box q \to (\Box\Box q \to q))).$$

Now for the reverse direction.

We have

$$\vdash (\Box q \to \Box\Box q)$$

by Solovay completeness.

Propositional calculus then gives us that

$$\vdash ((\Box q \leftrightarrow (\Box\Box q \to q)) \to (\Box q \to q)).$$

Using necessitation, and using the appropriate axiom scheme and MP to push the resulting box through an arrow,

$$\vdash (\Box(\Box q \leftrightarrow (\Box\Box q \to q)) \to \Box(\Box q \to q)).$$

We quote an axiom:

$$\vdash (\Box(\Box q \to q) \to \Box q).$$

Now by propositional logic,

$$\vdash (\Box(\Box q \leftrightarrow (\Box\Box q \to q)) \to \Box q).$$

(iii) $\Box(\Box q \wedge \Box r)$ is a fixed point for $A(p) = \Box(\Box(p \wedge q) \wedge \Box(p \wedge r))$.

In this case it's much easier to work through the proof of the Fixed Point Theorem.

Let $B(p) = (\Box(p \wedge q) \wedge \Box(p \wedge r))$.

Then $\Box B(\top)$ *is a fixed point for the given operator.*

$\Box B(\top)$ *is* $\Box(\Box(\top \wedge q) \wedge \Box(\top \wedge r))$.

It looks pretty clear that this is provably equivalent to the given formula. But let's check.

The following is a propositional tautology:

$$\vdash (q \leftrightarrow (\top \wedge q)).$$

Doing standard stuff with $\Box$, *we get*

$$\vdash (\Box q \leftrightarrow \Box(\top \wedge q)).$$

Similarly,

$$\vdash (\Box r \leftrightarrow \Box(\top \wedge r)).$$

Doing propositional calculus,

$$((\Box q \wedge \Box r) \leftrightarrow (\Box(\top \wedge q) \wedge (\top \wedge r))).$$

Doing more standard stuff with $\Box$,

$$(\Box(\Box q \wedge \Box r) \leftrightarrow \Box(\Box(\top \wedge q) \wedge (\top \wedge r))).$$

4. Find fixed points for

(i) $A(p) = (\Box p \to \Box \neg p)$,
Write $A(p)$ *in the form* $D(\Box C_1(p), \Box C_2(p), \ldots)$ *where* D *contains no instances of* $\Box$.
Then $D(x_1, x_2) = (x_1 \to x_2)$, $C_1(x) = x$, *and* $C_2(x) = \neg x$.
Now look for F_1 *and* F_2 *such that* $\vdash (F_1 \leftrightarrow \Box C_1(D(F_1, F_2)))$, *and* $\vdash (F_2 \leftrightarrow \Box C_2(D(F_1, F_2)))$.
First we find $G_1(q)$ *such that* $\vdash (G_1(q) \leftrightarrow \Box C_1(D(G_1(q)), q))$.
The solution is $\Box C_1(D(\top, q))$, *that is,* $\Box(\top \to q)$.
Now we look for F_2 *such that* $\vdash (F_2 \leftrightarrow \Box C_2(D(G_1(F_2), F_2)))$.
The solution is $\Box C_2(D(G_1(\top), \top))$, *that is,* $\Box\neg(\Box(\top \to \top) \to \top)$.
Now put $F_1 = G_1(F_2)$, *that is,* $F_1 = \Box(\top \to \Box\neg(\Box(\top \to \top) \to \top))$.
Now the fixed point we're looking for for $A(p)$ *is* $D(F_1, F_2)$, *that is,*

$$X = (\Box(\top \to \Box\neg(\Box(\top \to \top) \to \top)) \to \Box\neg(\Box(\top \to \top) \to \top)).$$

Of course, any other such formula X *is also correct.*

(ii) $A(p) = (\Box p \wedge \neg \Box \neg p)$.
Any contradiction is a fixed point.
Working through the method from the proof of Theorem 7.2.1., we put $D(x_1, x_2) = (x_1 \wedge \neg x_2)$, $C_1(x) = x$, *and* $C_2(x) = \neg x$.
We look for F_1 *and* F_2 *such that* $\vdash (F_1 \leftrightarrow \Box C_1(D(F_1, F_2)))$, *and* $\vdash (F_2 \leftrightarrow \Box C_2(D(F_1, F_2)))$.
First we find $G_1(q)$ *such that* $\vdash (G_1(q) \leftrightarrow \Box C_1(D(G_1(q), q)))$.
The solution is $G_1(q) = \Box C_1(D(\top, q)) = \Box(\top \wedge \neg q)$.
Now look for F_2 *such that* $\vdash (F_2 \leftrightarrow \Box C_2(D(G_1(F_2), F_2)))$.
The solution is $F_2 = \Box\neg(\Box(\top \wedge \neg\top) \wedge \neg\top)$.
Now put $F_1 = G_1(F_2)$, *that is,*

$$F_1 = \Box(\top \wedge \neg\Box\neg(\Box(\top \wedge \neg\top) \wedge \neg\top)).$$

Then the fixed point is $D(F_1, F_2) = (F_1 \wedge \neg F_2)$, *that is,*

$$(\Box(\top \wedge \neg\Box\neg(\Box(\top \wedge \neg\top) \wedge \neg\top)) \wedge \neg\Box\neg(\Box(\top \wedge \neg\top) \wedge \neg\top)).$$

This is indeed a contradiction.