Problem Sheet 4

Section B

QUESTION 3. **Strongly monotone operator** Let $\Omega = (-1,1)$ and $X = H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the H^2-norm.

(a) Let $A\colon X \to X^*$ be defined via

$$\langle A(u), v\rangle := \int_\Omega u''v''dx.$$

Show that A is a strongly monotone operator, i.e. hemicontinuous and so that there exists some $c_0 > 0$ with

$$\langle A(u) - A(v), u - v\rangle \geq c_0\|u - v\|^2 \quad \text{for all } u, v \in M.$$

Hint: Use Poincaré's inequality, as well as Poincaré's inequality for functions with mean value zero.

(b) Let now $F_\mu(u) := A(u) + \mu B(u)$ where $B(u)(v) := u(0)\cdot v(0) + \int_\Omega x \cdot v(x)dx$.

Show that $F_\mu : X \to X^*$ is well defined for any $\mu \in \mathbb{R}$ and that there exists a number $\mu_0 > 0$ so that for each μ with $|\mu| \leq \mu_0$ there exists a unique solution of the equation

$$F_\mu(u) = 0.$$

(c) Let now $\mu \geq 0$. Determine a functional $I_\mu : X \to \mathbb{R}$ so that the following holds: $u \in X$ is a solution of $F_\mu(u) = 0$ if and only if u is a minimiser of I_μ on X

Solution. (a) (i) Proof of hemi-continuity: Note that A is linear and

$$|\langle A(u), v\rangle| \leq \|u\|\|v\|$$

which implies that $\|A(u)\|_* \leq \|u\|$, which in turns implies that A is bounded and so continuous. It follows that A is hemi-continuous.

(Alternatively, one can argue directly: Let $u, v, w \in X$ and consider

$$f(t) = \langle A(tu + (1-t)v), w\rangle = \int_{-1}^1 (tu + (1-t)v)''w''\,dx, \quad t \in [0,1]$$

Note that the integrand of $f(t)$ is bounded in absolute value by $(|u''| + |v''|)|w''|$, which, by Cauchy-Schwarz, belongs to $L^1((-1,1))$. A simple application of Lebesgues' dominated convergence theorem implies that f is continuous in $[0,1]$. Since u, v, w are arbitrary, we conclude that A is hemi-continuous.)

(ii) Proof of strong monotonicity: Let $u, v \in X$. We have

$$\langle A(u) - A(v), u - v\rangle = \int_{-1}^1 ((u-v)'')^2\,dx.$$

Applying Wirtinger's inequality for the function $(u-v)'$ which has zero average (since $\int_{-1}^1 (u-v)'\,dx = 0$ by Newton-Leibnitz), we have

$$\langle A(u) - A(v), u - v\rangle = \int_{-1}^1 ((u-v)'')^2\,dx \geq \pi^2 \int_{-1}^1 ((u-v)')^2\,dx.$$

Applying Wirtinger's inequality for the function $u - v$ which vanishes at ± 1, we then have

$$\langle A(u) - A(v), u - v\rangle \geq \pi^2 \int_{-1}^1 ((u-v)')^2\,dx \geq \frac{\pi^4}{4} \int_{-1}^1 (u-v)^2\,dx.$$

Combining these estimates, we obtain

$$(1 + \frac{1}{\pi^2} + \frac{4}{\pi^4})\langle A(u) - A(v), u - v\rangle \geq \int_{-1}^1 [((u-v)'')^2 + ((u-v)')^2 + (u-v)^2]\,dx = \|u - v\|^2,$$

which proves the stated inequality with $c_0 = (1 + \frac{1}{\pi^2} + \frac{4}{\pi^4})^{-1}$.

(b)(i) Well-definition of F_μ: Note that $X \subset H^2((-1,1)) \hookrightarrow C^1([-1,1]) \hookrightarrow C^0([-1,1]) \hookrightarrow L^1((-1,1))$. Hence, for every $u \in X$, it makes sense to speak of $u(0)$, and for every $v \in X$, it makes sense to speak of $v(0)$ and $\int_{-1}^1 xv(x)\,dx$. Thus, for $u, v \in X$, $\langle B(u), v\rangle$ is well-defined and

$$|\langle B(u), v\rangle| \leq \|u\|_{L^\infty}\|v\|_{L^\infty} + \|v\|_{L^1} \leq C(\|u\| + 1)\|v\|,$$

where here and below C denote a universal constant depending only on the constants in the embedding of X into $C^0([-1,1])$ and $L^1((-1,1))$. This shows that $B(u) \in X^*$, and so $F_\mu : X \to X^*$ is well-defined for every $\mu \in \mathbb{R}$.

(ii) Existence of zero of F_μ: We first show that F_μ satisfies (H1) and (H3).

We saw above that $\|A(u)\|_* \leq \|u\|$ and $\|B(u)\|_* \leq C(\|u\| + 1)$ and so

$$\|F_\mu(u)\|_* \leq \|u\| + \mu C(\|u\| + 1).$$

This implies that F_μ maps bounded sets into bounded sets, i.e. (H1) holds.

Before proving (H3), we note that A is a monotone operator (by (a)) and B is a compact operator. To see that B is compact, suppose that (w_n) is bounded in X. By reflexivity of X and compactness of the embedding $H^2((-1,1)) \hookrightarrow C^1([-1,1])$, there exists a subsequence (w_{n_k}) which converges weakly in X and uniformly to a limit $w \in X$. Then

$$\langle B(w_{n_k}) - B(w), v\rangle = |(w_{n_k}(0) - u(0))v(0)| \leq C\|w_{n_k} - w\|_{L^\infty}\|v\|$$

which implies $\|B(w_{n_k}) - B(w)\|_* \leq C\|w_{n_k} - w\|_{L^\infty} \to 0$, i.e. $B(w_{n_k}) \to B(w)$ in X^*. Thus B is compact.

We can now prove (H3). Assume that $u_n \rightharpoonup u$ in X and $F_\mu(u_n) \rightharpoonup \xi$ in X^*. By compactness, we have that $B(u_n) \to B(u)$ in X^*. This, on one hand, implies that

$$\langle B(u), u\rangle = \lim\langle B(u_n), u_n\rangle,$$

and, on the other hand, implies that $A(u_n) \rightharpoonup \xi - \mu B(u)$ in X^* and so (since monotone operator satisfies (H3)),

$$\langle \xi - \mu B(u), u\rangle \leq \liminf\langle A(u_n), u_n\rangle.$$

We thus have

$$\langle \xi, u\rangle \leq \liminf\langle F_\mu(u_n), u_n\rangle.$$

Moreover, if equality holds, then by (H3) for A, it must hold that $A(u) = \xi - \mu B(u)$ and so $F_\mu(u) = \xi$. We have thus shown that F_μ satisfies (H3).

[Note that, with a little bit of further effort, one can show that if $u_n \rightharpoonup u$, then $F_\mu(u_n) \rightharpoonup F_\mu(u)$.]

We next show that if $|\mu| \leq \mu_0$ for some threshold μ_0, then F_μ satisfies (H2), namely there exists $R > 0$ such that

$$\langle F_\mu(u), u\rangle > 0 \text{ for } \|u\| > R.$$

Indeed we have

$$\begin{aligned}\langle F_\mu(u), u\rangle &= \int_{-1}^1 (u'')^2\,dx + \mu u(0)^2 + \mu\int_{-1}^1 xu(x)\,dx \\ &\geq c_0\|u\|^2 - |\mu|\|u\|_{L^\infty}^2 - |\mu|\|u\|_{L^1} \\ &\geq c_0\|u\|^2 - C|\mu|\|u\|^2 - C|\mu|.\end{aligned}$$

Thus, if $|\mu\| \leq \mu_0 := \frac{c_0}{2C}$, then $\langle F_\mu(u), u\rangle > 0$ when $\|u\| > \frac{C\mu_0}{c_0 - C\mu_0}$. (H2) is established.

We may now apply the theorem in the lectures to conclude that F_μ has a zero for $|\mu| \leq \mu_0$.

(iii) Uniqueness of zero of F_μ: If u_1 and u_2 are two zeros of F_μ, then with $v = u_1 - u_2$,

$$0 = \langle F_\mu(u_1) - u_2, v\rangle = \int_{-1}^{1} (u'')^2\,dx + \mu u(0)^2 \geq (c_0 - C\mu)\|v\|^2,$$

Since $c_0 - C\mu > 0$ (by the very definition of μ_0), we deduce that $v = 0$, i.e. $u_1 = u_2$.

[In fact, the above prove the existence and uniqueness of zero of F_μ for $\mu \geq -\mu_0$.]

(c) The desired functional is

$$I_\mu[u] = \int_{-1}^{1} \frac{1}{2}(u'')^2\,dx + \frac{\mu}{2}(u(0))^2 + \int_{-1}^{1} \mu x u(x)\,dx.$$

When $\mu > 0$, it is routine to check that I_μ is strictly convex (using (a)) and Frechet differentiable with derivative

$$\langle DI_\mu[u], v\rangle = \int_{-1}^{1} u''v''\,dx + u(0)v(0) + \int_{-1}^{1} \mu x v(x)\,dx = \langle F_\mu(u), v\rangle.$$

Thus u is a zero of F_μ if and only if it is a critical point of I_μ, which must be unique because of the convexity of I_μ.

To conclude, we show that I_μ has a minimizer. By convexity, we know that I_μ is weakly sequentially lower semi-continuous. Thus, we only need to show that I_μ is coercive. Indeed, by (a) and noting that $\mu \geq 0$,

$$I_\mu[u] \geq \frac{\pi^4}{4}\|u\|_{L^2}^2 - \mu\sqrt{2}\|u\|_{L^2} \geq \frac{\pi^4}{8}\|u\|_{L^2}^2 - \frac{4\mu^2}{\pi^4}.$$

We are done.

QUESTION 4. Consider a domain $\Omega \subset \mathbb{R}^n$ which is smooth and bounded, and $g \in C^2(\mathbb{R}^n)$ such that $g \leq 0$ on $\partial\Omega$. Consider the energy I given by

$$I(v) = \int_\Omega |\Delta v|^2 + f v dx, \quad v \in H^2(\Omega) \cap H_0^1(\Omega)$$

for some $f \in L^2(\Omega)$.

(1) Find the Euler-Lagrange equation satisfied by the critical points of $I(v)$ and prove that every critical point of I is a minimiser.

(2) Consider the set M given by

$$M := \left\{ v \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, v \geq g \text{ a.e. on } \Omega \right\}.$$

Show that there exists a unique minimizer of I on M —check carefully that the assumptions of the Theorem(s) you use are satisfied. You may use without proof that for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\|u\|_{H_0^1(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)},$$

where the constant C is independent of u.

Solution. (1) Let $X = H^2(\Omega) \cap H_0^1(\Omega)$. It is routine to check that I is strictly convex and Frechet differentiable on X with derivative

$$\langle DI(u), v\rangle = \int_\Omega (2\Delta u \Delta v + f v)\,dx$$

The Euler-Lagrange equation is thus

$$2\Delta^2 u + f = 0.$$

The assertion that every criticial point is a minimizer of I follows from the same argument as in Q3(c).

(2) Since $g \leq 0$ on $\partial\Omega$, $0 \in M$ and so M is non-empty. M is clearly convex.

We have that I is convex and so is weakly sequentially lower semi-continuous.

I is coercive because

$$I(u) \geq c_0\|u\|^2 - \|f\|_{L^2}\|u\| \geq \frac{c_0}{2}\|u\|^2 - \frac{\|f\|_{L^2}^2}{2c_0}.$$

To obtain a minimiser of I in M, we only need to show that M is closed (hence weakly closed by convexity of M). Indeed, if $(u_n) \subset M$ and $u_n \to u$ in X, then along a subsequence $u_{n_k} \to u$ a.e. which implies $u \geq g$ a.e. and so $u \in M$.

The uniqueness of the minimiser follows from the strict convexity of I.

QUESTION 5. **Three approaches to the same problem**. Consider a domain $\Omega = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } x^2 + y^2 \leq 1\}$ and the equation

$$-\Delta u + u^5 = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

- Show that this equation makes sense in $H_0^1(\Omega)$, that is, it has a legitimate weak variational formulation.
- Using the first part of the course, show that you can formulate it as a fixed point problem of the form $u = T(u)$ where T is a continuous compact map.
- Find a simple subsolution $\underline{u}$ and a simple supersolution $\bar{u}$. Show that the problem can be transformed into

$$-\Delta u + \lambda u = f_\lambda(u)$$

  for a constant $\lambda > 0$ chosen so that $f_\lambda(u)$ is increasing when $\underline{u} \leq u \leq \bar{u}$, and use the method of sub and super solutions to show that a solution u can be found by a constructive (iterative) method.
- Using Schauder's FPT and the above show that there exists a solution.
- Use the variational inequality approach to find a solution in $H_0^1(\Omega)$.
- What can you say about uniqueness?

Solution. (a) Note that if $u \in H_0^1(\Omega)$, then $u \in L^p(\Omega)$ for any $p < \infty$ by the Gagliardo-Nirenberg-Sobolev theorem. Thus, we can define a weak solution to the given problem as a function $u \in H_0^1(\Omega)$ such that

(WF)
$$\int_\Omega [\nabla u \cdot \nabla v + u^5 v - v]\, dx = 0 \text{ for all } v \in H_0^1(\Omega).$$

(b) We may recast this as a fixed point problem of the form $u = T(u)$ with T defined by

$$T(u) = (-\Delta)^{-1}N(u)$$

where $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ denotes the inverse of the operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $N : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by $N(u) = -u^5 + 1$.

By the result on the continuity of the Nemytsky map, N is continuous as a map from e.g. $L^{10}(\Omega)$ into $L^2(\Omega)$. Since $H_0^1(\Omega)$ embeds compactly into $L^{10}(\Omega)$ by Rellich-Kondrachov's theorem, and $L^2(\Omega)$ embeds continuously (in fact compactly, but this isn't needed) into $H^{-1}(\Omega)$, we have that N is a compact operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Since $(-\Delta)^{-1}$ is continuous, we deduce that $T = (-\Delta)^{-1}N$ is compact.

(c) One can simply takes $\underline{u} = 0$ and $\bar{u} = 1$.

Let $f(t) = -t^5 + 1$. Note that $f'(t) = -5t^4 \geq -5$ in $[0, 1]$. Hence $f(t) + 5t$ is increasing in $[0, 1]$. Define

$$g(t) = \begin{cases} f(t) + 5t & \text{if } t \in [0, 1], \\ f(0) & \text{if } t < 0, \\ f(1) + 5 & \text{if } t > 1. \end{cases}$$

Then g is increasing and the problem

$$-\Delta u + 5u = g(u) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

admits $\underline{u}$ as a subsolution and $\bar{u}$ as a super-solution. Hence it has a solution $u \in H_0^1(\Omega)$ satisfying $0 = \underline{u} \le u \le \bar{u} = 1$ a.e. In particular $g(u) = f(u) + 5u$ and so $-\Delta u = f(u)$ as required.

[In fact, one can show that all solutions must satisfy $0 \le u \le 1$:

(i) Proof that $u \ge 0$: Suppose u solves (WF). Using $v = u^-$ in (WF), we get

$$0 = \int_\Omega [\nabla u \cdot \nabla u^- + u^5 u^- - u^-]\,dx = \int_{\{u<0\}} [-|\nabla u|^2 - u^6 + u]\,dx \le 0$$

which implies that $u = 0$ a.e. in $\{u < 0\}$, which means $u \ge 0$ a.e.

(ii) Proof that $u \le 1$: Suppose u solves (WF). Using $v = (u-1)^+$ in (WF), we get

$$0 = \int_\Omega [\nabla u \cdot \nabla (u-1)^+ + u^5 (u-1)^+ - (u-1)^+]\,dx = \int_{\{u>1\}} [|\nabla u|^2 + (u^5 - 1)(u-1)]\,dx \ge 0$$

which implies that $u = 1$ a.e. in $\{u > 1\}$, which means $u \ge 1$ a.e.]

(d) Remark: If one follows the lecture notes, one will have a difficulty in finding $R > 0$ such that T maps $\bar{B}_R$ into $\bar{B}_R$ because T have super-linear growth at infinity.

Here we use the intuition from (c). One recast the problem in the fixed point form $u = \tilde{T}(u)$ where

$$\tilde{T}(u) = (-\Delta + 5)^{-1} \tilde{N}(u)$$

with $N(u) = f(u) + 5u =: h(u)$.

We knew from the discussion at the end of (c) that any solution if exists with be trapped between 0 and 1. We thus consider the non-empty closed convex set

$$M = \{u \in H_0^1(\Omega) : 0 \le u \le 1\}.$$

Claim: $\tilde{T}$ maps M into itself. Once this is done we can apply Schauder's fixed point theorem to get a fixed point of $\tilde{T}$ in M, which will conclude this part.

Let $u \in M$. We would like to show that $0 \le v := \tilde{T}(u) \le 1$. Note $v := T(u)$ satisfies

$$-\Delta v + 5v = h(u) \text{ in } \Omega, v = 0 \text{ on } \partial\Omega,$$

(i) Proof that $v \ge 0$: As $0 \ge u \le 1$ and h is increasing in $[0,1]$,

$$-\Delta v + 5v \ge h(0) = 0 \text{ in } \Omega, v = 0 \text{ on } \partial\Omega,$$

By the weak maximum principle, $v \ge 0$ a.e.

(ii) Proof that $v \le 1$: Similarly, we have

$$-\Delta(v-1) + 5(v-1) = h(v) - h(1) \le 0 \text{ in } \Omega, v - 1 = -1 \le 0 \text{ on } \partial\Omega,$$

By the weak maximum principle, $v - 1 \le 0$ a.e., i.e. $v \le 1$ a.e.

We can now apply Schauder's fixed point theorem to the compact map $\tilde{T} : M \to M$ to obtain the desired solution.

[Alternatively, one can work with $\hat{T} = (-\Delta + 5)^{-1}\hat{N}$ where $\hat{N}(u) = g(u)$, and using the fact that g is bounded to obtain a fixed point of $\hat{T}$. After this is achieved, use the weak maximum principle to show that the obtained solution satisfies $0 \le u \le 1$ and so satisfies the original equation.]

(e) The given equation is recognised as the Euler-Lagrange equation for critical points of the functional

$$J[u] = \int_\Omega (\frac{1}{2}|\nabla u|^2 + \frac{1}{6}u^6 - u)\,dx.$$

This is a convex functional, and we could use the argument in Q3, Q4 to deduce the existence of a minimiser, which is the unique critical point of J.

Alternatively, one uses $A(u) = -\Delta u + u^5 - 1$ as a map from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ and verify the condition (H1), (H2), (H3) as in Q1, noting that A is a monotone operator.

(f) As seen in (e), uniqueness of solution holds.