Geometric Group Theory

Cornelia Druţu

University of Oxford

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δ-hyperbolic spaces

Proposition (Morse lemma)

Let X be a δ-hyperbolic metric space. For any $\lambda \geq 1$ and $\mu \geq 0$, there exists some $M = M(\lambda, \mu)$ such that if

- *$\alpha : [u, v] \to X$ is a (λ, μ)-quasi-geodesic with endpoints $x = \alpha(u)$, $y = \alpha(v)$;*
- *$\gamma = [x, y]$ is a geodesic with the same endpoints as α;*

then $\alpha \subseteq \mathcal{N}_M(\gamma)$ and $\gamma \subseteq \mathcal{N}_M(\alpha)$.

Corollary

Let X, Y be geodesic metric spaces. If X is δ-hyperbolic and Y is quasi-isometric to X then Y is δ' hyperbolic for some $\delta' \geq 0$.

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Proof.

Let $f : Y \to X$ be a (L, A)-quasi-isometry. For all geodesic triangles Δ in Y, $f(\Delta)$ is a triangle in X with quasi-geodesic edges. Hence, there exists a geodesic triangle Δ' such that

$$f(\Delta) \subseteq \mathcal{N}_M(\Delta')$$

Since Δ' is δ-slim, $f(\Delta)$ is $(\delta + 2M)$-slim and so Δ is δ'-slim where $\delta' = \delta'(\delta, M, L, A)$. □

Hyperbolic groups

Definition

A finitely generated group G is hyperbolic if some (equivalently, every) Cayley graph is hyperbolic.

Examples

1. *F_k is hyperbolic.*
2. *If $G \curvearrowright \mathbb{H}^2$ by isometries properly discontinuously and cocompactly, then G is hyperbolic.*
3. *Random groups (among finitely presented groups).*

Definition

A group G has a Dehn presentation if there exists a finite presentation $G = \langle S | R \rangle$ such that every $w \in F(S)$ with $w =_G 1$ contains more than half of a word in R.

Hyperbolic groups

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Lemma

Groups with Dehn presentations have solvable word problem.

Procedure: Check if $w \in F(S)$ contains more than half of a word in R.

- If the answer is no, then $w \neq 1$ in G.
- If the answer is yes, then $w = aub$ where $r = uv$ and $|u| > \frac{1}{2}|r| > |v|$. So in G, $w = \underbrace{av^{-1}b}_{w'}$ and $|w'| < |w|$.

The procedure terminates after finitely many steps. □

Hyperbolic groups

Theorem

A hyperbolic group has a Dehn presentation. Hence, it is finitely presented and has solvable word problem.

Proof

There exists some $\delta \geq 0$ such that $\Gamma(G, S)$ has δ-thin geodesic triangles. WLOG assume that $\delta \in \mathbb{N}$. Consider

$$R = \{w \in F(S) : |w| \leq 10\delta, w =_G 1\}$$

Claim: $\langle S|R \rangle$ is a Dehn presentation.

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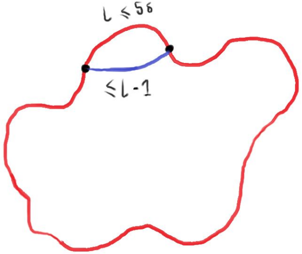
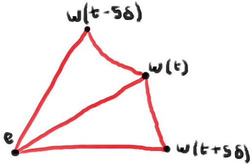
Take $w = 1$ in G. It labels a closed path in $\Gamma(G, S)$ of length n. Let $w(0) = e, w(1), ..., w(n-1)$ be the vertices of this path. If there exists a subpath of length $\leq 5\delta$ which is not geodesic then we are done.



Hyperbolic groups

Otherwise, take $w(t)$ such that $d(e, w(t))$ is maximal. Consider the geodesic triangles of vertices $[e, w(t), w(t-5\delta)]$ and $[e, w(t), w(t+5\delta)]$.



We have that $d(w(t \pm 5\delta), e) \leq d(w(t), e)$. Therefore, since both the triangles are δ-thin,

$$d(w(t-2\delta), w(t+2\delta)) \leq 2\delta$$

and so $w\big|_{[t-2\delta, t+2\delta]}$ is not geodesic. Contradiction. □

Hyperbolic groups

Proposition

A hyperbolic group G contains finitely many conjugacy classes of elements of finite order.

Proof.

Let $G = \langle S | R \rangle$ be a Dehn presentation. Let w be a word of minimal length in the conjugacy class of a finite order element. This implies that w is cyclically reduced. Since $w^n = 1$, w^n contains more than half of a word $r \in R$.

Claim: $|w| \leq \frac{|r|}{2} + 2$.

Suppose otherwise that $|w| > \frac{|r|}{2} + 2$. Then $r = r_1 r_2$ for some $r_1 r_2$ with $|r_1| > |r_2|$ and $|r_1| \leq \frac{|r|}{2} + 2$. Also, $w = tr_1$ up to conjugation. So $w = tr_1 = tr_2^{-1}$. However, $|tr_2^{-1}| < |w|$. This is a contradiction. So we have proved the claim and hence the proposition. □

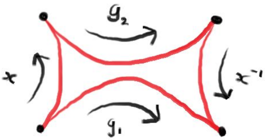
Hyperbolic groups

Lemma

Let $G = \langle S|R \rangle$ be a δ-hyperbolic group. If $g_1, g_2 \in G$ are conjugate then $g_1 = xg_2x^{-1}$ for some x with $|x| \leq (2|S|)^{2\delta+|g_1|} + |g_2|$.

Proof

Let x be of minimal length such that $g_1 = xg_2x^{-1}$.



Say $x = x_1 ... x_n$ for $x_i \in S \cup S^{-1}$. For all $i \leq n - |g_2|$ we have

$$|(x_1 ... x_i)^{-1} g_1 (x_1 ... x_i)| \leq 2\delta + |g_1|$$

Say $x = x_1 \dots x_n$ for $x_i \in S \cup S^{-1}$. For all $i \leq n - |g_2|$ we have

$$|(x_1 \dots x_i)^{-1} g_1 (x_1 \dots x_i)| \leq 2\delta + |g_1|$$

If $n - |g_2| \geq (2|S|)^{2\delta + |g_1|} + 1$ then there exists $i < j \leq n - |g_2|$ yielding equal elements:

$$(x_1 \dots x_i)^{-1} g_1 (x_1 .. x_i) = (x_1 \dots x_j)^{-1} g_1 (x_1 \dots x_j)$$

and so

$$(x_1 \dots x_i x_{j+1} \dots x_n)^{-1} g_1 (x_1 \dots x_i x_{j+1} \dots x_n) = g_2$$

which contradicts the minimality of $|x|$. □

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Corollary

The conjugacy problem is solvable for hyperbolic groups.

Proof.

Given $w_1, w_2 \in F(S)$, check whether $w_2 = xw_1x^{-1}$ for all $x \in F(S)$ with $|x| \leq (2|S|)^{2\delta+|w_1|} + |w_2|$. □

Theorem (Sela–Guirardel–Dahmani)

The isomorphism problem is solvable for hyperbolic groups.

More results and open questions

Theorem

Let G be an infinite hyperbolic group which is not virtually $\mathbb{Z}$. Then G contains a free subgroup of rank 2.

Theorem

Let G be a hyperbolic group and let $g_1, \dots, g_n \in G$. Then there is some $N > 0$ such that the group $\langle g_1^N, \dots, g_n^N \rangle$ is free.

Theorem (Sela)

Torsion-free hyperbolic groups are Hopf.

More results and open questions

Definition

Given a graph Γ, define

$$e(\Gamma) = \sup\{\text{number of connected components of } \Gamma - K : K \subseteq \Gamma \text{ compact}\}$$

$e(\Gamma)$ is said to be the number of ends of the graph Γ.

Exercise: $e(\Gamma)$ is invariant under quasi-isometry.

Exercise: If G is a finitely generated group then $e(\Gamma(G, S)) \in \{0, 1, 2, \infty\}$.

Theorem (Stallings)

G splits over a finite subgroup $\iff$ G has more than one end.

More results and open questions

Theorem (Gromov–Delzant)

Let G be a hyperbolic group and let H be a fixed one-ended group. Then G contains at most finitely many conjugacy classes of subgroups isomorphic to H.

There are a number of open questions about hyperbolic groups:

- Are hyperbolic groups residually finite?
- Let G be hyperbolic. Does G have a torsion-free subgroup of finite index?
- Gromov has conjectured that if G is torsion-free hyperbolic then G has finitely many torsion-free finite extensions.