# 5 Water waves

## 5.1 Equations and boundary conditions

### 5.1.1 Setup

In this Section we will analyse so-called *Stokes waves*, namely small-amplitude waves on the free surface of an inviscid fluid, for example small ripples on a container of water. Consider fluid filling the half-space $y < 0$ with a free surface at $y = 0$, gravity acting in the $-y$-direction. Now suppose that the fluid is disturbed by small-amplitude waves, so that the free surface is displaced to $y = \eta(x, t)$, as shown schematically in Figure 5.1.

We assume that the flow is irrotational and incompressible, so that it may be described by a velocity potential $\phi$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ and $\phi$ satisfies Laplace's equation. We will restrict our attention to purely two-dimensional disturbances, so that $\phi$ is a function of $x$, $y$ and $t$ and hence

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{5.1}$$

### 5.1.2 Boundary conditions

Far from the free surface, as the depth tends to infinity, we expect the velocity to tend to zero, that is

$$\boldsymbol{\nabla}\phi \to \boldsymbol{0} \quad \text{as } y \to -\infty. \tag{5.2}$$

At the free surface, there are two boundary conditions, and we will treat each separately in detail.

**Dynamic boundary condition** A force balance on the interface $y = \eta(x, t)$ implies that the pressure must be continuous there; otherwise there would be a finite force acting on a surface with zero mass, which contradicts Newton's Second Law. We therefore impose the dynamic boundary condition

$$p = P_{\text{atm}} \quad \text{at } y = \eta, \tag{5.3}$$

where $P_{\text{atm}}$ denotes the atmospheric pressure above the fluid, which we assume to be constant.

We can write the boundary condition (5.3) in terms of the velocity potential by using Bernoulli's Theorem. For unsteady irrotational flow, we recall from Section 1 the equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \chi = F(t), \tag{5.4}$$

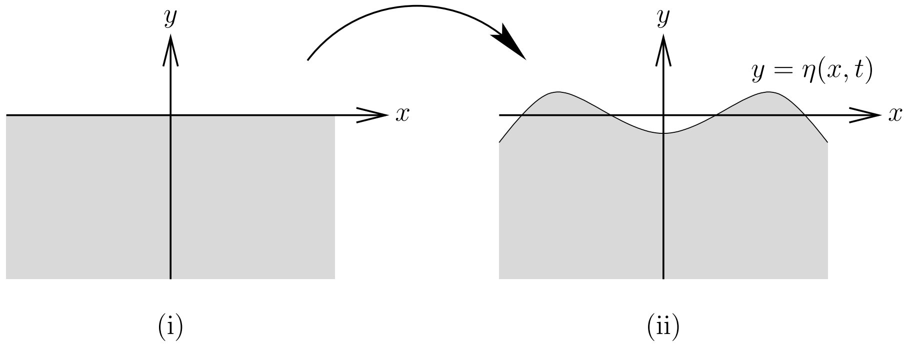


Figure 5.1: (i) Fluid at rest in the half-space $y < 0$. (ii) The fluid following a disturbance that displaces the free upper surface to $y = \eta(x, t)$.

where the gravitational potential $\chi = gy$ for gravity acting in the $-y$-direction. The integration function $F(t)$ may be chosen arbitrarily by absorbing a suitable function of $t$ into $\phi$.

Evaluating (5.4) at the free surface $y = \eta$ and using (5.3), we find that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{P_{\mathrm{atm}}}{\rho} + g\eta = F(t) \quad \text{on } y = \eta. \tag{5.5}$$

It is convenient to choose the arbitrary function $F(t) = P_{\mathrm{atm}}/\rho$ to cancel the constant term on the left-hand side of (5.5), and thus we obtain the dynamic boundary condition in the form

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + g\eta = 0 \quad \text{at } y = \eta. \tag{5.6}$$

**Kinematic boundary condition** We recall that the normal velocity of the fluid is required to be zero at a fixed impermeable wall. The corresponding condition at a moving boundary such as the free surface of a fluid is that *the velocity of the fluid normal to the boundary must equal the velocity of the boundary normal to itself*. If this were not true, the fluid would either be flowing through the boundary or separating from it, leaving behind a vacuum, neither of which is acceptable. It may be shown that this condition is equivalent to the requirement that *material fluid elements on the free surface must remain on the free surface*. Hence, if $y = \eta$ for some particular fluid particle at time $t$, then $y = \eta$ for the same particle for all time.

It follows that

$$\frac{\mathrm{D}}{\mathrm{D}t}(y - \eta) = 0 \quad \text{when } y - \eta = 0, \tag{5.7}$$

and, by expanding out the convective derivative, we obtain the kinematic boundary condition in the form

$$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \quad \text{at } y = \eta. \tag{5.8}$$

**Linearised boundary conditions** Although Laplace's equation is linear, the boundary conditions (5.8) and (5.6) on the free surface are nonlinear, and the problem is therefore difficult to solve in general. If the disturbances are small, then the boundary conditions can be simplified by *linearising*, that is neglecting terms of quadratic and higher order. For example, if we neglect the quadratic terms in (5.8), we find

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{at } y = \eta. \tag{5.9}$$

This can be simplified further by Taylor-expanding the left-hand side as follows:

$$\frac{\partial \phi}{\partial y}(x, \eta, t) \sim \frac{\partial \phi}{\partial y}(x, 0, t) + \frac{\partial^2 \phi}{\partial y^2}(x, 0, t)\eta + \cdots, \tag{5.10}$$

in which all terms except the first are nonlinear. When linearising the boundary conditions, it is thus consistent also to evaluate the left-hand side of (5.9) at $y = 0$ rather than at $y = \eta$. The same simplification applies when we linearise (5.6), so we end up with the boundary conditions

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } y = 0. \tag{5.11}$$

## 5.2 Harmonic waves

Now we look for solutions in which the free surface displacement $\eta$ takes the form of a sinusoidal travelling wave, that is

$$\eta(x, t) = A\cos(kx - \omega t - \beta), \tag{5.12}$$

where $A$, $k$, $\omega$ and $\beta$ are constants. The *amplitude* of the perturbations is measured by $A$, while $\omega$ represents the *frequency* at which the surface oscillates at any fixed position $x$. The *wavenumber* $k$ is $2\pi/\lambda$, where $\lambda$ is the wavelength; thus $k$ is small for long waves and large for short waves. The *wave-speed* at which the wave crests propagate is related to $\omega$ and $k$ by

$$c = \frac{\omega}{k}. \tag{5.13}$$

Finally, $\beta$ is an arbitrary phase shift, which may be set to zero without loss of generality by choosing the origin for $t$ appropriately. We show a typical harmonic travelling wave in Figure 5.2.

By substituting (5.12) into the boundary conditions(5.11), we infer that $\phi$ is out of phase with $\eta$, so that

$$\phi(x, y, t) = f(y)\sin(kx - \omega t - \beta) \tag{5.14}$$

for some function $f(y)$ still to be determined. By substituting (5.14) into Laplace's equation (5.1), we find that $f(y)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - k^2 f = 0. \tag{5.15}$$

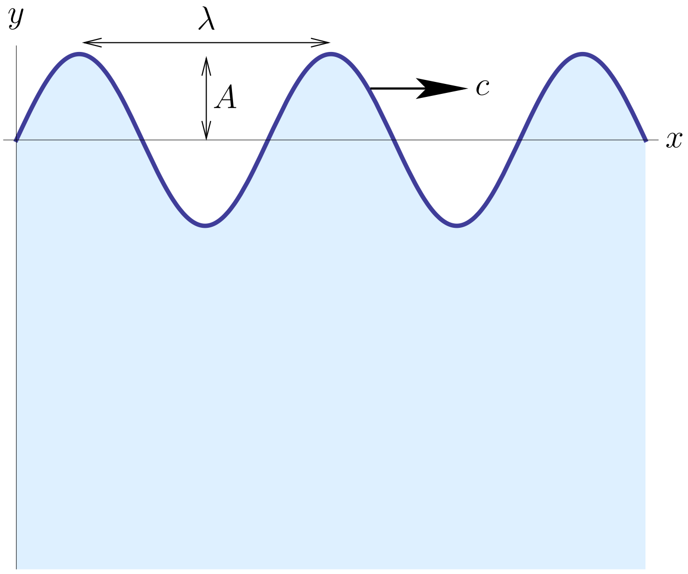


Figure 5.2: Schematic of a harmonic travelling wave, showing the amplitude $A$, wavelength $\lambda$ and wave-speed $c$.

The far-field condition (5.2) and the free-surface conditions (5.11) imply that $f(y)$ must satisfy the boundary conditions

$$f(y) \to 0 \quad \text{as } y \to -\infty, \tag{5.16}$$

$$f'(0) = \omega A, \quad -\omega f(0) + gA = 0. \tag{5.17}$$

Without loss of generality, we assume that $k$ is positive, so the solution of (5.15) that satisfies the far-field condition (5.16) is

$$f(y) = B\mathrm{e}^{ky} \tag{5.18}$$

for some constant $B$. The boundary conditions (5.17) at $y = 0$ thus give us a system of linear equations for the two constants $A$ and $B$, which may be written in the form

$$\begin{pmatrix} \omega & -k \\ g & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.19}$$

The homogeneous linear system (5.19) admits the trivial solution $A = B = 0$, corresponding to $\eta$ and $\phi$ both being identically zero. A nontrivial solution can only exist if the determinant of the left-hand side is zero, that is if

$$\omega^2 = gk. \tag{5.20}$$

This equation for the frequency in terms of the wavenumber is called the *dispersion relation*. The corresponding wave-speed $c$ satisfies

$$c^2 = \frac{g}{k}, \tag{5.21}$$

which depends on the wavenumber $k$, so that waves with different wavenumbers move at different speeds. Such waves are called *dispersive*, in contrast with waves on a string or sound waves, for example, which have a constant wave speed.

We see from (5.21) that the wave-speed is a decreasing function of the wavenumber, so that longer waves propagate more quickly. In principle, the wave-speed may be arbitrarily large for very long waves. We will see below that this is an artefact of our assumption that the fluid has infinite depth.

## 5.3 Generalisations

### 5.3.1 Finite depth

The analysis performed above is easily generalised to describe waves on a fluid of finite depth $h$. Suppose fluid occupies the region $-h < y < \eta(x,t)$ between a rigid base at $y = -h$ and a free surface at $y = \eta(x,t)$. We recall that the normal velocity at the base must be zero, and hence $\phi$ must satisfy the boundary condition

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h. \tag{5.22}$$

This replaces the far-field condition (5.2); otherwise the problem is identical to that solved in §5.2.

We again seek a solution in the form of a harmonic travelling wave, so that

$$\eta(x,t) = A\cos(kx - \omega t - \beta), \qquad \phi(x,y,t) = f(y)\sin(kx - \omega t - \beta), \tag{5.23}$$

for some function $f(y)$. By substituting this expression for $\phi$ into Laplace's equation, we again find that $f(y)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - k^2 f = 0 \tag{5.24}$$

and the boundary conditions

$$f'(0) = \omega A, \quad -\omega f(0) + gA = 0. \tag{5.25}$$

However, the condition (5.22) on the base now leads to the boundary condition

$$f'(-h) = 0. \tag{5.26}$$

Clearly the general solution of (5.24) is a linear combination of $\mathrm{e}^{ky}$ and $\mathrm{e}^{-ky}$. Alternatively, we can write $f(y)$ as a combination of $\cosh(ky)$ and $\sinh(ky)$. However, the neatest approach is to note that the boundary condition (5.26) is satisfied identically by setting

$$f(y) = B\cosh\big(k(y+h)\big), \tag{5.27}$$

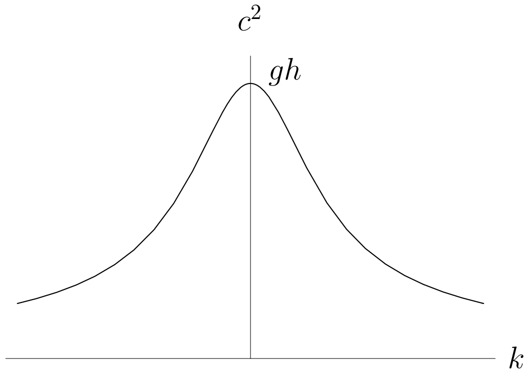


Figure 5.3: The squared wave-speed $c^2$ given by (5.30) versus wavenumber $k$.

for some constant $B$. Substitution into the free-surface conditions (5.25) again leads to a system of linear equations for $A$ and $B$, which now takes the form

$$\begin{pmatrix} \omega & -k\sinh(kh) \\ g & -\omega\cosh(kh) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.28}$$

For nontrivial solutions, the determinant of the system must be zero, and this now gives us the dispersion relation

$$\omega^2 = gk\tanh(kh). \tag{5.29}$$

The wave-speed $c$ is therefore given by

$$c^2 = \frac{g}{k}\tanh(kh). \tag{5.30}$$

As depicted in figure 5.3, for positive $k$, the right-hand side of (5.30) is a decreasing function, indicating that long waves travel faster than short waves. However, the wave-speed is now bounded, with a maximum achieved in the limit $k \to 0$, where we find that

$$c \to \sqrt{gh} \quad \text{as } k \to 0. \tag{5.31}$$

### 5.3.2 Flowing fluid

We can study waves on a flowing liquid by linearising about uniform flow, setting $\boldsymbol{u} = U\boldsymbol{i} + \boldsymbol{\nabla}\phi$, that is,

$$u = U + \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \tag{5.32}$$

where $\phi$ and its derivatives are again assumed to be small. It is clear that $\phi$ still satisfies Laplace's equation. Furthermore, if we consider fluid of finite depth $h$, with a rigid impermeable base at $y = -h$, then $\phi$ still satisfies the boundary condition

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h. \tag{5.33}$$

At the free surface, the kinematic boundary condition (5.8) now reads

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \left(U + \frac{\partial \phi}{\partial x}\right)\frac{\partial \eta}{\partial x} \quad \text{at } y = \eta(x,t). \tag{5.34}$$

When we linearise, as in §5.1.2, this is simplified to

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} \quad \text{at } y = 0. \tag{5.35}$$

Next we turn to the dynamic boundary condition. With the velocity given by (5.32), Bernoulli's equation (5.4) is modified to

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|U\boldsymbol{i} + \boldsymbol{\nabla}\phi|^2 + \frac{p}{\rho} + gy = F(t). \tag{5.36}$$

Setting $p$ equal to the atmospheric pressure $P_{\text{atm}}$ at the free surface $y = \eta$, we therefore obtain the boundary condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(U + \frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{P_{\text{atm}}}{\rho} + g\eta = F(t) \quad \text{at } y = \eta(x,t). \tag{5.37}$$

It is convenient to choose the arbitrary function $F(t)$ to cancel the constant terms on the left-hand side, that is

$$F(t) = \frac{1}{2}U^2 + \frac{P_{\text{atm}}}{\rho}. \tag{5.38}$$

Then linearisation of (5.37) leads to the condition

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = 0 \quad \text{at } y = 0. \tag{5.39}$$

Again, we can seek travelling-wave solutions of the form (5.23). The modified boundary conditions (5.35) and (5.39) imply that $f(y)$ must satisfy

$$f'(0) = (\omega - Uk)A, \quad -(\omega - Uk)f(0) + gA = 0. \tag{5.40}$$

The boundary condition (5.33) at the base again implies that $f(y)$ should take the form

$$f(y) = B\cosh\big(k(y+h)\big), \tag{5.41}$$

and substitution into (5.40) leads to the homogeneous linear system

$$\begin{pmatrix} \omega - Uk & -k\sinh(kh) \\ g & -(\omega - Uk)\cosh(kh) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.42}$$

For there to exist nontrivial solutions, $\omega$ must satisfy the dispersion

$$(\omega - Uk)^2 = gk\tanh(kh). \tag{5.43}$$

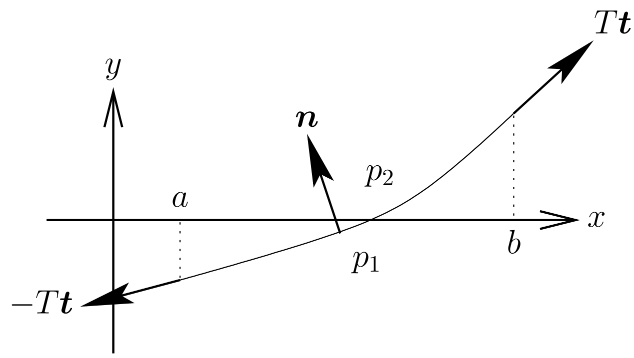


Figure 5.4: Schematic showing the surface tension $T$ acting at a fluid interface.

We deduce that there are two possible wave-speeds, namely

$$c_{\pm} = U \pm \left(\frac{g}{k}\tanh(kh)\right)^{1/2}. \tag{5.44}$$

We recall that the bracketed term in this equation is bounded by $gh$, as shown in Figure 5.3. Hence we can identify two possible cases in (5.44). If the flow speed $U$ is less than $\sqrt{gh}$, then waves may propagate both upstream and downstream. Such a flow is termed *subcritical*. On the other hand, if $U > \sqrt{gh}$, then all waves are carried downstream and the flow is said to be *supercritical*.

### 5.3.3 Two fluids

Now suppose the interface $y = \eta$ separates two fluids with different densities, say $\rho = \rho_1$ in $y < 0$ and $\rho = \rho_2$ in $y > 0$. We denote the velocity potentials and pressures on either side by $\phi_1$, $\phi_2$ and $p_1$, $p_2$ respectively. The kinematic condition (5.8) applies to both sides of the interface, and leads to the linearised boundary conditions

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \quad \text{at } y = 0. \tag{5.45}$$

The dynamic boundary condition (5.3) is replaced by the pressure continuity condition $p_1 = p_2$ at $y = \eta$. After use of Bernoulli's equation (5.4) and linearisation, this leads to the boundary condition

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\eta\right) \quad \text{at } y = 0. \tag{5.46}$$

Notice that (5.11) is recovered if we let the density ratio $\rho_2/\rho_1$ tend to zero.

### 5.3.4 Surface tension

Real fluid interfaces exhibit a phenomenon called *surface tension*, which acts like a membrane stretched over the interface to a tension $T$. In Figure 5.4 we show the forces acting on small element of the interface between $x = a$ and $x = b$, namely the pressures on either side and the surface tension at the ends. These forces must sum to zero, that is

$$\int_{x=a}^{x=b} (p_1 - p_2)\boldsymbol{n}\, \mathrm{d}s + \big[T\boldsymbol{t}\big]_{x=a}^{x=b} = \mathbf{0}, \tag{5.47}$$

where d$s$ denotes integration with respect to arc length, $\boldsymbol{t}$ is the unit tangent and $\boldsymbol{n}$ is the unit normal to the interface, chosen to point from fluid 1 into fluid 2, as shown in Figure 5.4. These are given respectively by

$$\mathrm{d}s = \sqrt{1+\eta_x^2}\,\mathrm{d}x, \qquad \boldsymbol{t} = \frac{1}{\sqrt{1+\eta_x^2}}\begin{pmatrix} 1 \\ \eta_x \end{pmatrix}, \qquad \boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}}\begin{pmatrix} -\eta_x \\ 1 \end{pmatrix}. \tag{5.48}$$

By using the Fundamental Theorem of Calculus, we can write (5.47) in the form

$$\int_a^b \left( (p_1 - p_2)\boldsymbol{n}\sqrt{1+\eta_x^2} + \frac{\partial}{\partial x}(T\boldsymbol{t}) \right) \mathrm{d}x = \mathbf{0}. \tag{5.49}$$

This must be true for all intervals $[a, b]$ along the surface, and the integrand, if continuous, must therefore be zero. The surface tension $T$ is assumed to be constant, and we therefore obtain the boundary condition

$$(p_1 - p_2)\boldsymbol{n} + \frac{T}{\sqrt{1+\eta_x^2}}\frac{\partial \boldsymbol{t}}{\partial x} = \mathbf{0} \quad \text{at } y = \eta. \tag{5.50}$$

Direct differentiation of (5.48) reveals that

$$\frac{1}{\sqrt{1+\eta_x^2}}\frac{\partial \boldsymbol{t}}{\partial x} = \kappa \boldsymbol{n}, \tag{5.51}$$

where

$$\kappa = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \tag{5.52}$$

is the *curvature* of the interface. Hence we deduce from (5.50) that there is a pressure jump across the interface equal to

$$p_2 - p_1 = T\kappa \quad \text{at } y = \eta. \tag{5.53}$$

After linearisation, this reads

$$p_2 - p_1 = T\eta_{xx} \quad \text{at } y = 0, \tag{5.54}$$

and the dynamic boundary condition (5.46) is thus modified to

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\eta\right) = T\frac{\partial^2 \eta}{\partial x^2} \quad \text{at } y = 0 \tag{5.55}$$

to take account of surface tension. Note that (5.46) is recovered if $T$ is set to zero.

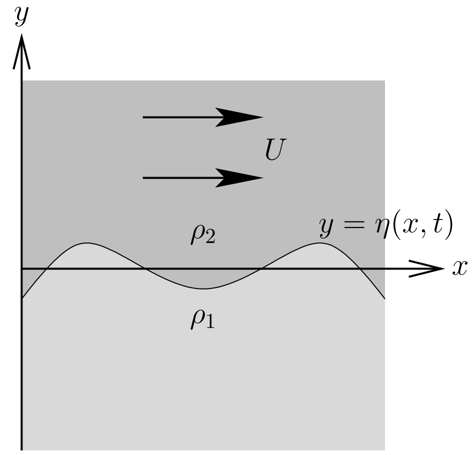


Figure 5.5: Schematic of a fluid of density $\rho_2$ flowing at speed $U$ over a fluid of density $\rho_1$.

**Example 5.1** ***One fluid flowing over another***
*We illustrate the effects outlined above by analysing the situation shown in* Figure 5.5, *where an infinite layer of fluid with density $\rho_2$ flows at speed U over an infinite layer of density $\rho_1$. We include a surface tension $T$ at the interface between the two fluids, so that the disturbance potentials $\phi_1$, $\phi_2$ and the free-surface deflection $\eta$ satisfy*

$$\nabla^2\phi_1 = 0 \qquad \text{in } y < 0, \qquad \nabla^2\phi_2 = 0 \qquad \text{in } y > 0, \tag{5.56}$$

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial y} = \frac{\partial\eta}{\partial t}, \qquad \frac{\partial\phi_2}{\partial y} = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x}, \\ \rho_1\left(\frac{\partial\phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial\phi_2}{\partial t} + U\frac{\partial\phi_2}{\partial x} + g\eta\right) = T\frac{\partial^2\eta}{\partial x^2} \end{aligned}\right\} \quad \text{at } y = 0. \tag{5.57}$$

*As usual, we look for harmonic travelling waves with $\eta(x,t) = A\cos(kx - \omega t - \beta)$, where the wavenumber $k$ is assumed to be positive, without loss of generality. The corresponding solutions $\phi_1$, $\phi_2$ of Laplace's equation that decay as $y \to -\infty$ and $y \to +\infty$ respectively are then easily found to be*

$$\phi_1(x,y,t) = B\mathrm{e}^{ky}\sin(kx - \omega t - \beta), \qquad \phi_2(x,y,t) = C\mathrm{e}^{-ky}\sin(kx - \omega t - \beta), \tag{5.58}$$

*where $B$ and $C$ are arbitrary constants.*

*On substituting these into the boundary conditions* (5.57), *we obtain a system of three homogenous linear equations for $A$, $B$ and $C$, which can be written in the form*

$$\begin{pmatrix} \omega & -k & 0 \\ \omega - Uk & 0 & k \\ (\rho_1 - \rho_2)\,g + Tk^2 & -\rho_1\omega & \rho_2(\omega - Uk) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.59}$$

*For nontrivial solutions, the determinant of the system must be zero, and hence we obtain the dispersion relation*

$$(\rho_1 + \rho_2)\omega^2 - 2(\rho_2 Uk)\omega + \rho_2 U^2 k^2 - (\rho_1 - \rho_2)gk - Tk^3 = 0. \tag{5.60}$$

## 5.4 Instability

We have always assumed thus far that the dispersion relation gives rise to *real* values of the frequency $\omega$. However, it may well arise that $\omega$ is complex, for example when the dispersion relation is a quadratic equation such as (5.60). If we write the real and imaginary parts of $\omega$ as $\omega = \omega_{\rm R} \pm {\rm i}\omega_{\rm I}$, then a harmonic travelling wave like (5.12) becomes

$$\begin{aligned} \eta &= A\cos(kx - \omega t - \beta) \\ &= A\cos\left(kx - \omega_{\rm R} t - \beta\right)\cosh\left(\omega_{\rm I} t\right) + {\rm i}A\sin\left(kx - \omega_{\rm R} t - \beta\right)\sinh\left(\omega_{\rm I} t\right). \end{aligned} \tag{5.61}$$

We infer that a complex value of $\omega$ corresponds to an exponentially growing amplitude, and implies that the corresponding wave is *unstable*.

**Example 5.2** ***Rayleigh–Taylor instability***
*We return to the problem of one fluid flowing above another, analysed above in* Example 5.1. *If there is no relative flow, that is $U = 0$, then the dispersion relation* (5.60) *reduces to*

$$\omega^2 = \frac{\left((\rho_1 - \rho_2)g + Tk^2\right)k}{\rho_1 + \rho_2}. \tag{5.62}$$

*If $\rho_1 > \rho_2$ then the right-hand side of* (5.62) *is positive, so there are two equal and opposite values of $\omega$, corresponding to waves propagating at speed $c = \omega/k$ in either direction. However, if $\rho_1 < \rho_2$, $\omega^2$ is negative for some values of $k$, namely*

$$k < \sqrt{\frac{(\rho_2 - \rho_1)g}{T}}. \tag{5.63}$$

*For these wavenumbers, $\omega$ is pure imaginary, so the disturbance grows exponentially. Hence the situation with the denser fluid above the lighter fluid is (not surprisingly) unstable; this is known as the* Rayleigh–Taylor instability.

Example 5.2 reminds us that the frequency $\omega$ is a function of the wavenumber $k$ so that, in general, $\omega$ may be complex only for certain values of the wavenumber. This implies that the system is unstable only to waves of certain wavelengths. In Example 5.2, equation (5.63) implies that only waves with wavelength $\lambda$ such that

$$\lambda > 2\pi\sqrt{\frac{T}{(\rho_2 - \rho_1)g}} \tag{5.64}$$

are unstable. At a water-air interface, we would have $T \approx 0.07\,{\rm N\,m^{-1}}$, $\rho_{\rm air} \approx 1.2\,{\rm kg\,m^{-3}}$, $\rho_{\rm water} \approx 1000\,{\rm kg\,m^{-3}}$, $g \approx 9.8\,{\rm N\,kg^{-1}}$, so that only waves longer than roughly 1.7 cm are unstable. If the system is too narrow to allow waves this long, then the instability will be eliminated. This explains why a glass of water may be tipped upside-down without the water spilling out if a sufficiently fine mesh is stretched over the end.

**Example 5.3** ***Kelvin–Helmholtz instability***
*When $U$ is nonzero, the solution of the quadratic equation* (5.60) *is given by*

$$\omega = \frac{\rho_2 U k \pm \sqrt{\Delta}}{\rho_1 + \rho_2}, \tag{5.65}$$

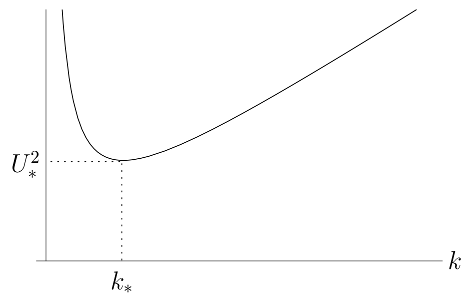


Figure 5.6: The right-hand side of (5.67) versus wavenumber $k$.

*where $\Delta$ is the discriminant*

$$\Delta = (\rho_1 + \rho_2)\, k \left((\rho_1 - \rho_2)g + Tk^2\right) - \rho_1\rho_2 U^2 k^2. \tag{5.66}$$

*We see that $\omega$ is complex (so the flow is unstable) when $\Delta$ is negative, that is when*

$$U^2 > \left(\frac{\rho_1 + \rho_2}{\rho_1\rho_2}\right)\left(\frac{(\rho_1 - \rho_2)g}{k} + Tk\right). \tag{5.67}$$

*Assuming $\rho_1 > \rho_2$ (so the lighter fluid is on top), the right-hand side of (5.67) tends to infinity as $k \to 0$ and as $k \to \infty$, with a minimum at $k = k_* = \sqrt{(\rho_1 - \rho_2)g/\gamma}$, as shown in* Figure 5.6. *This corresponds to a critical value of $U$, given by*

$$U_*^2 = \frac{2(\rho_1 + \rho_2)}{\rho_1\rho_2}\sqrt{\gamma(\rho_1 - \rho_2)g}. \tag{5.68}$$

*If $U > U_*$, then there is a band of values of $k$ for which* (5.67) *is satisfied and for which $\omega$ is therefore complex. In other words the flow is unstable if the velocity of the upper fluid exceeds this critical value. This* Kelvin–Helmholtz *instability is responsible for the formation of waves by wind blowing over the sea.*

## 5.5 Introduction to group velocity

We have seen that dispersive waves have the property that waves with different wavelengths propagate at different speeds. A localised disturbance, for example caused by dropping a pebble into a pond, will in general give rise to a spectrum of many different wavenumbers. As the waves spread out from the initial disturbance, the dispersion will cause them to be sorted according to their wavenumber. For deep water, we recall from equation (5.21) that the wave-speed $c$ is related to the wavenumber $k$ by $c^2 = g/k$, so that long waves travel more quickly than short waves. We would therefore expect the spreading disturbance to have longer ripples at the front and shorter ones at the back, as illustrated in Figure 5.7.

Now suppose at some time $t$ after the initial disturbance, we detect waves with wavenumber $k$ at a distance $x$ from the source. We would expect these to be related through the wave-speed $c$ by $x/t = c = \sqrt{g/k}$. However, it turns out that this prediction

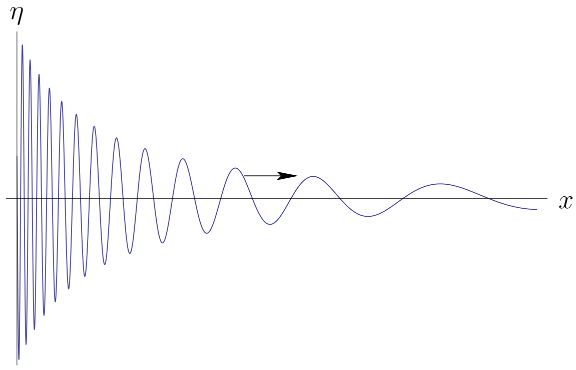


Figure 5.7: Schematic of a spreading train of ripples caused by a localised disturbance.

is out by a factor of 2. This occurs because the wave-speed $c$ is the speed at which the crests (or troughs) propagate in a pure sinusoidal wave, not in a realistic free surface profile containing a combination of many different wavenumbers.

A profile like that illustrated in Figure 5.7 can be represented as

$$\eta(x,t) = A(x,t)\cos\big(\alpha(x,t)\big), \tag{5.69}$$

in terms of a rapidly-varying *phase* $\alpha$ and a slowly-varying *amplitude* $A$. In the vicinity of a fixed position $x = x_0$ and time $t = t_0$, we can Taylor-expand the phase to get

$$\alpha(x,t) \approx \alpha(x_0,t_0) + (x - x_0)\frac{\partial \alpha}{\partial x}(x_0,t_0) + (t - t_0)\frac{\partial \alpha}{\partial t}(x_0,t_0). \tag{5.70}$$

Hence, the profile (5.69) is locally approximated by the harmonic travelling wave

$$\eta(x,t) \approx A_0\cos\big(k_0 x - \omega_0 t - \beta_0\big), \tag{5.71}$$

where

$$A_0 = A(x_0,t_0), \qquad \beta_0 = \alpha(x_0,t_0) - k_0 x_0 + \omega_0 t_0, \tag{5.72a}$$

$$k_0 = \frac{\partial \alpha}{\partial x}(x_0,t_0), \qquad \omega_0 = -\frac{\partial \alpha}{\partial t}(x_0,t_0). \tag{5.72b}$$

It is therefore natural to define the local wavenumber and frequency at position $x$ and time $t$ by

$$k(x,t) = \frac{\partial \alpha}{\partial x}, \qquad \omega(x,t) = -\frac{\partial \alpha}{\partial t}. \tag{5.73}$$

It immediately follows from (5.73) that $k$ and $\omega$ satisfy the equation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{5.74}$$

For a pure harmonic wave, there is a dispersion relation $\omega = \omega(k)$ specifying the frequency as a function of the wavenumber. We assume that the same relation holds here,

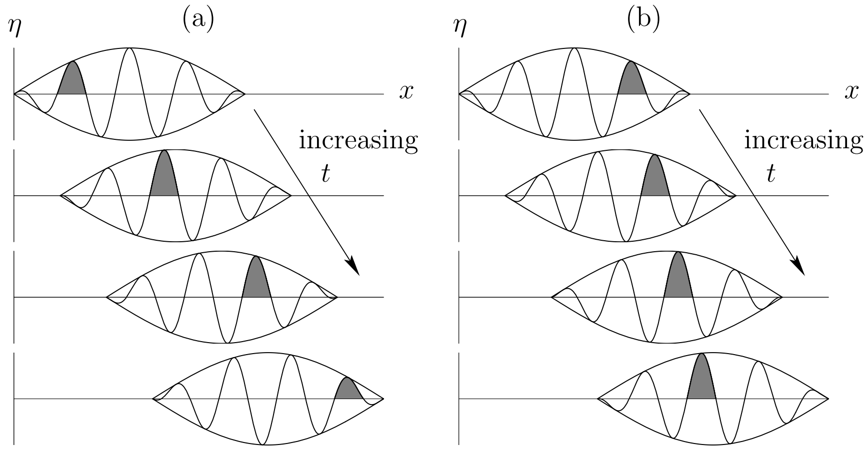


Figure 5.8: Schematic of a moving wave packet with (a) $c_g < c$, (b) $c_g > c$. One wave crest is highlighted to illustrate how it moves relative to the packet.

since the free surface profile is locally approximately sinusoidal. Thus (5.74) becomes a partial differential equation for the wavenumber $k$, namely

$$\frac{\partial k}{\partial t} + c_g(k)\frac{\partial k}{\partial x} = 0, \tag{5.75}$$

where

$$c_g(k) = \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{5.76}$$

is called the *group velocity.*

We deduce from equation (5.75) that $k$ is constant along straight lines in the $(x, t)$-plane satisfying $\mathrm{d}x/\mathrm{d}t = c_g$. Indeed, the general solution of (5.75) is

$$k = F\big(x - c_g(k)t\big). \tag{5.77}$$

It follows that waves with wavenumber $k$ propagate with speed $c_g(k)$, and not at the wave-speed $c(k)$ as might have been expected.

For waves on deep water, with the dispersion relation (5.21), the wave-speed and group velocity are given respectively by

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \qquad c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{c}{2}. \tag{5.78}$$

At first glance, this may appear to be a contradiction: how can the wave crests propagate twice as quickly as the waves themselves? The answer is that the waves separate into *wave packets* corresponding to different wavenumbers. Within each wave packet, the waves move at speed $c$, but the packet as a whole moves at speed $c_g$. This phenomenon is illustrated in Figure 5.8(a) for a single wave packet travelling from left to right at speed $c_g$. The wave crests move through the packet at speed $c = 2c_g$, seeming to appear at the back and disappear at the front. This behaviour can be observed in the radiating ripples caused by throwing a stone into a pond.

It is also possible for the group velocity to exceed the wave-speed; for example, it can be shown that $c_{\mathrm{g}} \approx 2c$ for short capillary waves on very shallow water. If this happens, then the wave crests appear to move *backwards* relative to a radiating wave packet, as illustrated in Figure 5.8(b). This counterintuitive behaviour can sometimes be observed in small ripples on a puddle.