A12204W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2
Master of Science in Mathematical Sciences: Paper C5.2

Elasticity and Plasticity

TRINITY TERM 2024

Thursday 06 June, 14:30pm to 16:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. An elastic beam of length L and bending stiffness EI undergoes two-dimensional deformations in the (x, z)-plane subject to negligible body force. The centre-line of the beam makes an angle $\theta(s)$ with the x-axis, where s is arc-length. A horizontal compressive force P is applied at either end of the beam. The end at $s = 0$ is simply supported, while a bending moment $M_\star$ about the y-axis is applied at the other end at $s = L$.

   (a) [8 marks] Show that in equilibrium the tension T, normal shear force N and bending moment M about the y-axis satisfy

   $$\frac{\mathrm{d}}{\mathrm{d}s}\big(T\cos\theta - N\sin\theta\big) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\big(T\sin\theta + N\cos\theta\big) = 0, \qquad \frac{\mathrm{d}M}{\mathrm{d}s} = N.$$

   Assuming the constitutive relation $M = -EI\mathrm{d}\theta/\mathrm{d}s$, derive the *beam equation*

   $$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P\sin\theta = 0.$$

   By scaling $s = L\xi$ derive the dimensionless model

   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \pi^2\lambda\sin(\theta) = 0, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\xi}(0) = 0, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\xi}(1) = -\gamma, \qquad (\star)$$

   where the normalised compressive load λ and the normalised moment γ should be defined.

   (b) [7 marks] Assuming that $|\gamma| \ll 1$ and that θ remains small enough for the problem $(\star)$ in part (a) to be linearised, obtain the approximate solution

   $$\theta(\xi) \sim A\cos\left(\pi\xi\sqrt{\lambda}\right),$$

   and derive an expression for the amplitude A in terms of λ and γ.
   Show that the linearisation fails as the applied load λ approaches n^2, where n is a positive integer.
   Explain why nonlinearity becomes important when $\lambda - n^2 = O\left(|\gamma|^{2/3}\right)$ and $\theta = O\left(|\gamma|^{1/3}\right)$.

   (c) [10 marks] Now suppose that λ is close to one of the critical values with

   $$\lambda = n^2 + \epsilon\lambda_1, \qquad \gamma = \epsilon^{3/2}\gamma_1, \qquad \theta = \epsilon^{1/2}u,$$

   where λ_1, γ_1 and u are of order unity as $\epsilon \to 0+$.
   Show that

   $$u(\xi) \sim A_1\cos(n\pi\xi)$$

   as $\epsilon \to 0+$, where

   $$A_1^3 - \frac{8\lambda_1}{n^2}A_1 = -\frac{16(-1)^n}{n^2\pi^2}\gamma_1.$$

   For $n = 1$ and $\gamma_1 > 0$, sketch a bifurcation diagram of A_1 versus λ_1, and hence describe qualitatively how the beam responds as λ is gradually increased through 1.

2. A uniform cylindrical bar of linear isotropic elastic material is subject to a steady torsional displacement of the form $\mathbf{u} = \Omega(-yz, xz, \psi(x, y))^T$ for $(x, y) \in D$ and $0 < z < L$, where the twist Ω is a positive constant, D is the simply connected cross-section of the bar in the (x, y)-plane and L is the length of the bar.

   (a) [10 marks] Write down the linear elastic constitutive relation and hence find the nonzero stress components in terms of $\psi(x, y)$. Assuming that there are no body forces, explain why there exists a stress potential $\phi(x, y)$ such that

   $$\tau_{xz} = \mu\Omega\frac{\partial\phi}{\partial y} \qquad \text{and} \qquad \tau_{yz} = -\mu\Omega\frac{\partial\phi}{\partial x},$$

   where μ is the shear modulus. Deduce that $\nabla^2\phi = -2$ in D.

   Assuming that the curved boundary of the bar is stress free, show that ϕ may be chosen so that $\phi = 0$ on the boundary ∂D of D.

   Show that the moment about the z-axis applied at the end of the bar at $z = L$ is given by $M = R\Omega$, where the *torsional rigidity* is

   $$R = 2\mu\iint_D \phi \, \mathrm{d}x\mathrm{d}y.$$

   (b) [15 marks] Suppose that the cross-section D of the bar is a disk of radius a and centre the origin, but with the negative x-axis removed, forming a Mode III crack. The surface of the crack is stress free and ϕ is bounded at the crack tip as $r \to 0$.

   (i) Formulate the boundary value problem for $\Phi(r, \theta) = y^2 + \phi(x, y)$ on the rectangle $0 < r < a$, $-\pi < \theta < \pi$, where (r, θ) denote plane polar coordinates. Use the method of separation of variables to derive the series solution for Φ.

   (ii) Deduce that the torsional rigidity is given by

   $$R = \left(1 - \left(\frac{8}{3\pi}\right)^2\right)\pi\mu a^4.$$

   (iii) The crack propagates if $K_{III} > K^\star$, where $K^\star$ is a positive constant and the stress intensity factor is

   $$K_{III} = \lim_{x\to 0+} \left|\tau_{yz}(x, 0)\right|\sqrt{2\pi x}.$$

   Show that the crack propagates if Ω exceeds a critical value that should be determined in terms of $K^\star$, μ and a.

   [*In (b) you may assume without proof the identities*

   $$\begin{aligned}
   \nabla^2\Phi &= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \\
   \int_0^\pi \sin^2\theta\cos\left(\left(n + \frac{1}{2}\right)\theta\right)\mathrm{d}\theta &= \frac{16(-1)^{n+1}}{(2n-3)(2n+1)(2n+5)}, \\
   \sum_{n=0}^{\infty}\frac{1}{(2n-3)(2n+1)^2(2n+5)^2} &= \frac{128 - 27\pi^2}{9\cdot 32^2},
   \end{aligned}$$

   where n is a non-negative integer.]

3. (a) [10 marks] Show that the normal stress N and shear stress F on a line element with unit normal $\mathbf{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$ in a two-dimensional granular material lie on the *Mohr circle*

$$F^2 + \left(N - \frac{1}{2}(\tau_{xx} + \tau_{yy})\right)^2 = \frac{(\tau_{xx} - \tau_{yy})^2}{4} + \tau_{xy}^2,$$

where τ_{xx}, τ_{xy} and τ_{yy} denote the stress components.
Hence show that the *Coulomb yield condition* $|F| \leqslant -N \tan\phi$, where $\phi \in (0, \pi/2)$ is the angle of friction, leads to the condition

$$\sqrt{(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2} \leqslant -(\tau_{xx} + \tau_{yy}) \sin\phi.$$

State the conditions under which the material is either (i) elastic or (ii) plastic in the resulting *perfectly plastic* model.

(b) [15 marks] Granular material undergoes quasi-steady radially symmetric strain in the region $a < r < b$, with displacement field given by $\mathbf{u}(r) = u(r)\mathbf{e}_r$, where (r, θ) denote plane polar coordinates and $\mathbf{e}_r$ is a unit vector in the r-direction. The inner boundary at $r = a$ is subject to a non-negative pressure P_{in}, while the outer boundary at $r = b$ is subject to a non-negative pressure P_{out}. In plane polar coordinates the stress tensor is diagonal with entries τ_{rr} and $\tau_{\theta\theta}$ that satisfy the Navier equation

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0.$$

(i) Show that, while the material remains elastic, the stress components satisfy the *compatibility condition*

$$\frac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 0.$$

Explain why $\tau_{rr} = -P_{\mathrm{in}}$ on $r = a$ and $\tau_{rr} = -P_{\mathrm{out}}$ on $r = b$.
Hence show that, as P_{in} is increased gradually from a starting value of P_{out}, yield first occurs at $r = a$ when P_{in} reaches the critical value P_{c1} given by

$$P_{\mathrm{c1}} = \left(\frac{1 + \sin\phi}{1 + (a/b)^2 \sin\phi}\right) P_{\mathrm{out}}.$$

(ii) Show that, as P_{in} is increased further, the material yields in a region $a < r < s$ in which $\tau_{\theta\theta} = (1 - \gamma)\tau_{rr}$, where s is given by

$$\frac{\alpha s^{\gamma}}{1 + \beta s^2} = \frac{P_{\mathrm{in}}}{P_{\mathrm{out}}}$$

and the constants α, β and γ should be determined in terms of a, b and ϕ. Using a diagram explain why there is a unique solution for s until P_{in} reaches a second critical value P_{c2} that you should determine in terms of P_{out}, a, b and ϕ. Is the material under compression throughout? Justify your answer.

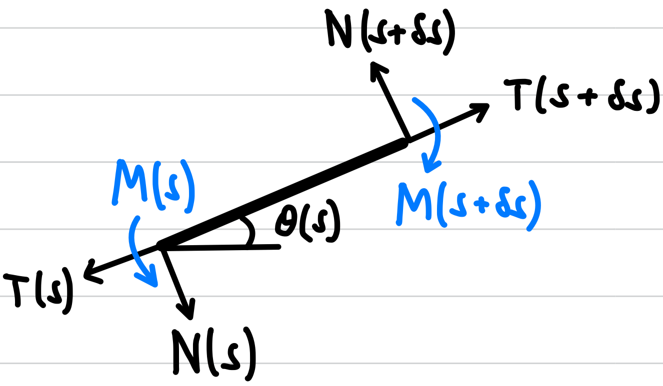
[*In (b) you may assume the linear elastic constitutive relations*

$$\tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda\frac{u}{r}, \qquad \tau_{\theta\theta} = \lambda\frac{\mathrm{d}u}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u}{r},$$

where λ and μ are the Lamé constants.]

C5.2/2024/Q1

(a) The forces and moments act on a short section as indicated:



Force balance: $\left[T\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + N\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right]_s^{s+\delta s} = \begin{pmatrix}0\\0\end{pmatrix}$

Moment balance: $M(s) + \delta s\, N(s+\delta s) - M(s+\delta s) = 0$

$\delta s \to 0 \Rightarrow \frac{d}{ds}\left(T\cos\theta - N\sin\theta\right) = 0,\ \frac{d}{ds}\left(T\sin\theta + N\cos\theta\right) = 0,\ N = \frac{dM}{ds}$ B4

Integrate and apply boundary conditions

$$\Rightarrow \quad T\cos\theta - N\sin\theta = -P,\ T\sin\theta + N\cos\theta = 0$$

$$\Rightarrow \quad T = -P\cos\theta,\ N = P\sin\theta$$

Now use $M = -EI\frac{d\theta}{ds} \Rightarrow EI\frac{d^2\theta}{ds^2} + P\sin\theta = 0$

Boundary conditions $\Rightarrow \frac{d\theta}{ds}(0) = 0,\ -EI\frac{d\theta}{ds}(L) = M(L) = M_*$

Scale $s = L\xi \Rightarrow \frac{d^2\theta}{d\xi^2} + \pi^2\lambda\sin\theta$ with $\frac{d\theta}{d\xi}(0) = 0,\ \frac{d\theta}{d\xi}(1) = -\gamma$

where $\lambda = \frac{PL^2}{\pi^2 EI},\ \gamma = \frac{LM_*}{EI}$. B/S4

$\boxed{8}$

(b) Linearizing for $\theta \ll 1 \Rightarrow \quad \theta'' + \pi^2 \lambda \theta = 0$

$$\Rightarrow \quad \theta = A\cos(\pi\sqrt{\lambda}\,\xi) + B\sin(\pi\sqrt{\lambda}\,\xi), \quad \text{with } A, B \in \mathbb{R}$$

$$\theta'(0) = 0,\ \theta'(1) = -\gamma \Rightarrow B = 0, \quad -A\pi\sqrt{\lambda}\sin(\pi\sqrt{\lambda}) = -\gamma$$

i.e. $\theta \sim \gamma \dfrac{\cos(\pi\sqrt{\lambda}\,\xi)}{\pi\sqrt{\lambda}\sin(\pi\sqrt{\lambda})}$ for $\theta, \gamma \ll 1$ provided $\sqrt{\lambda} \notin \mathbb{Z}^+$

Amplitude $A \to \infty$ and linearization fails as $\lambda \to n^2$ for each $n \in \mathbb{Z}^+$

B|S3

$$\lambda = n^2 + \Lambda,\ \Lambda \ll 1 \Rightarrow \sin(\pi\sqrt{\lambda}) = \sin\left(n\pi\sqrt{1+\tfrac{\Lambda}{n^2}}\right) \sim \sin\left(n\pi + \frac{\pi\Lambda}{2n}\right) \sim (-1)^n \frac{\pi\Lambda}{2n}$$

$$\Rightarrow A \sim \frac{\gamma}{n\pi(-1)^n \frac{\pi\Lambda}{2n}} = O\left(\frac{|\gamma|}{\Lambda}\right)$$

Beam equation $\Rightarrow \quad \theta'' + \pi^2(n^2+\Lambda)\left(\theta - \tfrac{1}{6}\theta^3\right) = O(\theta^5)$ for $\theta \ll 1$

$$\Rightarrow \quad \theta'' + n^2\pi^2\theta \sim -\pi^2\Lambda\theta + \tfrac{1}{6}n^2\pi^2\theta^3$$

So nonlinearity balances excess load when $\Lambda A = O(A^3)$.

But $A = O\left(\frac{|\gamma|}{\Lambda}\right)$, so $\lambda - n^2 = \Lambda = O(|\gamma|^{2/3})$, $A = O(|\gamma|^{1/3})$, as required.

S4

$\boxed{7}$

(c) Plug in $\Rightarrow \quad u'' + \pi^2(n^2 + \varepsilon\lambda_1)\dfrac{\sin(\sqrt{\varepsilon}\,u)}{\sqrt{\varepsilon}} = 0$

$$\Rightarrow \quad u'' + \pi^2(n^2+\varepsilon\lambda_1)\left(u - \tfrac{\varepsilon}{6}u^3\right) = O(\varepsilon^2)$$

with $u'(0) = 0$, $u'(1) = -\varepsilon\gamma_1$

Now expand $u \sim u_0 + \varepsilon u_1$ as $\varepsilon \to 0$, substitute into BVP and equate powers of ε.

At $O(\varepsilon^0)$: $\quad u_0'' + n^2\pi^2 u_0 = 0$ with $u_0'(0) = u_0'(1) = 0$

$$\Rightarrow \quad u_0 = A_1 \cos(n\pi\zeta), \text{ where } A_1 \in \mathbb{R}$$

At $O(\varepsilon^1)$: $\quad u_1'' + n^2\pi^2 u_1 = -\pi^2\lambda_1 u_0 + \frac{1}{6}n^2\pi^2 u_0^3$ with $u_1'(0) = 0, u_1'(1) = -\gamma_1$

B3

To derive the solvability condition for A_1, note that since $v(\zeta) = \cos(n\pi\zeta)$ is a solution of the homogeneous problem,

$$\int_0^1 (u_1'' + n^2\pi^2 u_1) v \, d\zeta = \int_0^1 \{(u_1'' + n^2\pi^2 u_1)v - (v'' + n^2\pi^2 v)u_1\} d\zeta = [u_1'v - u_1 v']_0^1$$

$$\Rightarrow \int_0^1 (-\pi^2\lambda_1 A_1 \cos^2(n\pi\zeta) + \tfrac{1}{6}n^2\pi^2 A_1^3 \cos^4(n\pi\zeta)) d\zeta = -(-1)^n \gamma_1$$

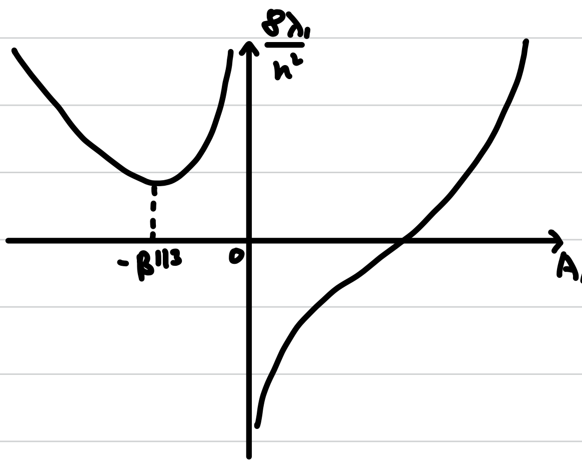
But $\int_0^1 \cos^2(n\pi\zeta) d\zeta = \int_0^1 \frac{1}{2}(1 + \cos(2n\pi\zeta)) d\zeta = \frac{1}{2}$

$$\int_0^1 \cos^4(n\pi\zeta) d\zeta = \int_0^1 \tfrac{1}{4}(1 + \cos(2n\pi\zeta))^2 d\zeta = \int_0^1 \tfrac{1}{4} + \tfrac{1}{4}\cos^2(2n\pi\zeta) d\zeta = \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{3}{8}$$

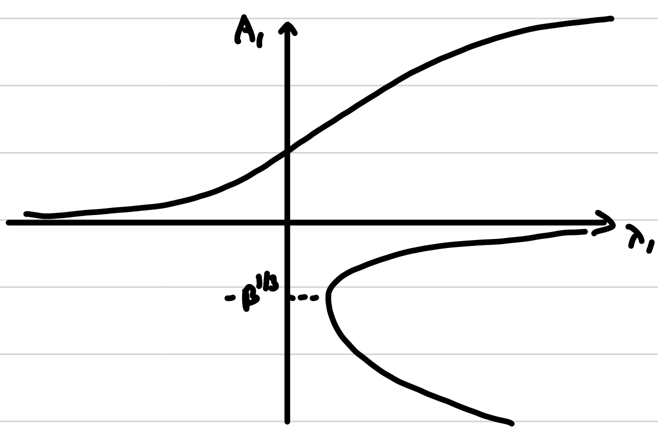
so $-\frac{1}{2}\pi^2\lambda_1 A_1 + \frac{3}{8}\frac{1}{6}n^2\pi^2 A_1^3 = -(-1)^n\gamma_1$ or $A_1^3 - \frac{8\lambda_1}{n^2} A_1 = -\frac{16(-1)^n\gamma_1}{n^2\pi^2} = 2\beta$ say

S/N4

To sketch note $\frac{8\lambda_1}{n^2} = A_1^2 - \frac{2\beta}{A_1} \Rightarrow \frac{\partial}{\partial A_1}\left(\frac{8\lambda_1}{n^2}\right) = 2A_1 + \frac{2\beta}{A_1^2} = 0$ for $A_1 = -\beta^{1/3} < 0$ as $n = 1$ and $\gamma_1 > 0$.



$\Rightarrow$



Hence beam buckles with $A_1 > 0$, i.e. upwards, as λ increases through 1.

N3

$\boxed{10}$

C5.2/2024/Q2

(a) Linear constitutive relation: $\tau_{ij} = \lambda(\operatorname{div}\underline{u}) I + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Plug in $\underline{u}$ $\Rightarrow$ nonzero stress components are $\tau_{xz} = \tau_{zx} = \mu(u_z + w_x)$ and and $\tau_{yz} = \tau_{zy} = \mu(v_z + w_y)$ $\Rightarrow$ $\tau_{xz} = \mu\Omega\left(-y + \frac{\partial \psi}{\partial x}\right)$, $\tau_{yz} = \mu\Omega\left(x + \frac{\partial \psi}{\partial y}\right)$.

Steady Cauchy equation with no body force is $\frac{\partial \tau_{ij}}{\partial x_j} = 0 \Rightarrow \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$
$\Rightarrow \exists \phi(x,y)$ s.t. $\tau_{xz} = \mu\Omega\frac{\partial \phi}{\partial y}$, $\tau_{yz} = -\mu\Omega\frac{\partial \phi}{\partial x}$.

Combo $\Rightarrow$ $-y + \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$ and $x + \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}$

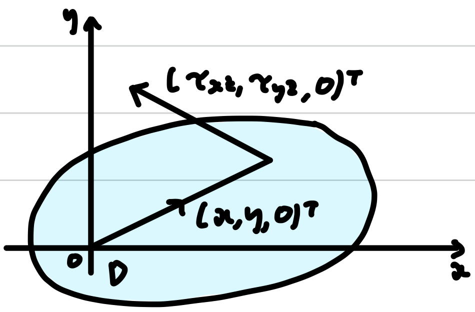
$$\Rightarrow \nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial x}\left(-x - \frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-y + \frac{\partial \psi}{\partial x}\right) = -2$$

Parametrize $\partial D = \{(x(s), y(s))\}$, where s is arclength in anticlockwise direction, then $\underline{n} = (y', -x', 0)^T$ is outward unit normal to curved boundary of bar.

Boundary stress free $\Rightarrow$ $(\tau_{ij})\underline{n} = \underline{0}$ on $\partial D \times [0, L]$
$\Rightarrow$ $\tau_{xz} y' - \tau_{yz} x' = 0$ on ∂D
$\Rightarrow$ $x' \frac{\partial \phi}{\partial x} + y' \frac{\partial \phi}{\partial y} = 0$ on ∂D
$\Rightarrow$ $\frac{d\phi}{ds} = 0$ on ∂D
$\Rightarrow$ $\phi = \text{constant} = 0$ wlog on ∂D
as ϕ is a potential

B6

The stress $(\tau_{ij})\underline{e}_z = (\tau_{xz}, \tau_{yz}, 0)^T$ on plane $z = \text{constant}$ generates a moment M about the z-axis:



$$M\underline{e}_z = \iint_D \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \\ 0 \end{pmatrix} dx\,dy$$

$$\Rightarrow M = \iint_D x\tau_{yz} - y\tau_{xz}\, dx\,dy$$

Plug in $\Rightarrow$ $M = R\Omega$, where torsional rigidity

$$R = -\mu \iint_D x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\, dx\,dy$$
$$= \mu \iint_D 2\phi - \frac{\partial}{\partial x}(x\phi) - \frac{\partial}{\partial y}(y\phi)\, dx\,dy$$
$$= 2\mu \iint_D \phi\, dx\,dy - \mu\int_{\partial D} (x\phi, y\phi)\cdot \underline{n}\, ds \quad \text{by div. thm}$$
$$= 2\mu \iint_D \phi\, dx\,dy \quad \text{as required as } \phi = 0 \text{ on } \partial D.$$

B4 $\boxed{10}$

(b)(i) $\phi = -y^2 + \Phi \Rightarrow \dfrac{\partial^2\Phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\Phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2} = 0$ for $0<r<a$, $|\theta|<\pi$

with $|\Phi(0,\theta)| < \infty$ for $|\theta| < \pi$

$\Phi(r, \pm\pi) = 0$ for $0<r<a$

$\Phi(a,\theta) = a^2\sin^2\theta$ for $|\theta|<\pi$

S|N4

Separate variables $\Phi = F(r)G(\theta) \Rightarrow \dfrac{r^2F''(r) + rF'(r)}{F(r)} = -\dfrac{G''(\theta)}{G(\theta)} = \omega^2 \in \mathbb{R}^+$

with $\omega > 0$ wlog for G nontrivial as $G(\pm\pi) = 0$.

General solution is $G(\theta) = A\cos\omega\theta + B\sin\omega\theta$, $A, B \in \mathbb{R}$.

$G(\pm\pi) = 0 \Rightarrow A\cos\omega\pi \pm B\sin\omega\pi = 0 \Rightarrow A\cos\omega\pi = 0$ and $B\sin\omega\pi = 0$

But $\Phi(a,\theta)$ is even, so only need the even modes. Setting $B=0$ to eliminate the odd modes, we need $A \neq 0$ for $G \neq 0 \Rightarrow \cos\omega\pi = 0 \Rightarrow \omega = n+\frac{1}{2}$, $n \in \mathbb{Z}_0^+$.

Then $F(r) = r^m \Rightarrow m(m-1) + m - (n+\frac{1}{2})^2 \Rightarrow m = \pm(n+\frac{1}{2})$

But Φ bounded at $r=0 \Rightarrow$ only $F(r) \propto r^{n+1/2}$ is admissible.

Superimposing $\Rightarrow$ $\Phi = \sum_{n=0}^{\infty} a_n \left(\frac{r}{a}\right)^{n+1/2} \cos(n+\frac{1}{2})\theta$, where $a_n \in \mathbb{R}$

BC on $r=a$ $\Rightarrow$ $\sum_{n=0}^{\infty} a_n \cos(n+\frac{1}{2})\theta = a^2 \sin^2\theta$ for $|\theta| < \pi$

But $\int_0^\pi \cos(n+\frac{1}{2})\theta \cos(m+\frac{1}{2})\theta \, d\theta = \frac{\pi}{2}\delta_{nm}$ for $n,m \in \mathbb{Z}_0^+$, so

$a_n = \frac{2a^2}{\pi}\int_0^\pi \sin^2\theta \cos(n+\frac{1}{2})\theta \, d\theta = \frac{2a^2}{\pi} \frac{16(-1)^{n+1}}{(2n-3)(2n+1)(2n+5)}$ by hint, giving solution S7

(b)(ii) $R = 2\mu \cdot 2 \cdot \int_0^\pi \int_0^a \left\{ -r^2\sin^2\theta + \sum_{n=0}^{\infty} a_n \left(\frac{r}{a}\right)^{n+1/2} \cos(n+\frac{1}{2})\theta \right\} r \, dr \, d\theta$

$$= 4\mu \left\{ -\frac{a^4}{4}\cdot\frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{a_n}{a^{n+1/2}} \cdot \frac{a^{n+5/2}}{n+5/2} \cdot \frac{\sin(n+\frac{1}{2})\pi}{n+1/2} \right\}$$

$$= 4\mu \left\{ -\frac{\pi}{8}a^4 + \sum_{n=0}^{\infty} \frac{2a^4}{\pi} \frac{4\cdot 16\cdot(-1)^{n+1}(-1)^n}{(2n-3)(2n+1)^2(2n+5)^2} \right\}$$

$$= \mu a^4 \left\{ -\frac{\pi}{2} - \frac{2\cdot 16^2}{\pi}\cdot\frac{128-27\pi^2}{9\cdot 32^2} \right\}$$

$$= \mu a^4 \left\{ -\frac{\pi}{2} - \frac{128-27\pi^2}{18\pi} \right\}$$

$$= \pi\mu a^4 \left\{ 1 - \left(\frac{8}{3\pi}\right)^2 \right\}$$ as required. N2

(b)(iii) $\phi \sim a_0 \left(\frac{r}{a}\right)^{1/2} \cos\frac{\theta}{2}$ as $r \to 0+$

$\Rightarrow$ $\frac{\partial\phi}{\partial x}(x,0) = \frac{a_0}{2}\frac{1}{\sqrt{ax}}$ as $x \to 0+$

$\Rightarrow$ $\tau_{yz}(x,0) = -\mu\Omega\frac{\partial\phi}{\partial x}(x,0) \sim -\frac{\mu\Omega a_0}{2\sqrt{ax}}$ as $x \to 0+$

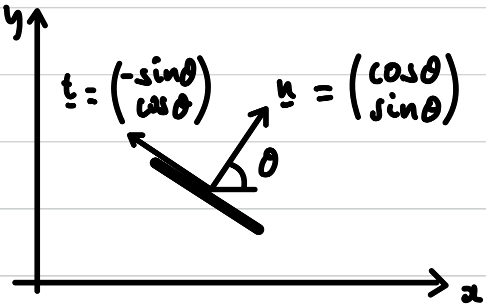
$\Rightarrow$ $K_{III} = \mu\Omega|a_0|\sqrt{\frac{\pi}{2a}} = \frac{16}{15}\sqrt{\frac{2}{\pi}}\mu\Omega a^{3/2}$

So $K_{III} > K_{3c} \Leftrightarrow \Omega > \frac{15}{16}\sqrt{\frac{\pi}{2}}\frac{K_{3c}}{\mu a^{3/2}}$. S/N2

$\boxed{15}$

C5.2/2024/Q3

(a)



Stress on line element is

$$\tau\underline{n} = \begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

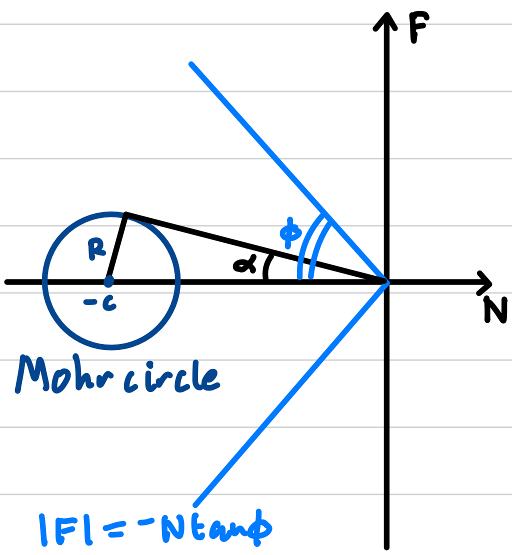
$$\Rightarrow \begin{cases} N = \underline{n}\cdot(\tau\underline{n}) = \tau_{xx}\cos^2\theta + 2\tau_{xy}\sin\theta\cos\theta + \tau_{yy}\sin^2\theta \\ F = \underline{t}\cdot(\tau\underline{n}) = (\tau_{yy}-\tau_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{cases}$$

$$\Rightarrow \begin{cases} N - \tfrac{1}{2}(\tau_{xx}+\tau_{yy}) = \tfrac{1}{2}(\tau_{xx}-\tau_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \\ F = \tau_{xy}\cos 2\theta - \tfrac{1}{2}(\tau_{xx}-\tau_{yy})\sin 2\theta \end{cases}$$

$$\Rightarrow \left(N - \tfrac{1}{2}(\tau_{xx}+\tau_{yy})\right)^2 + F^2 = \tfrac{1}{4}(\tau_{xx}-\tau_{yy})^2 + \tau_{xy}^2$$

as required, the cross terms cancelling

So (N,F) lie on the Mohr circle with centre $-c$ and radius R, where $c = -\frac{1}{2}(\tau_{xx}+\tau_{yy})$ and $R = \sqrt{\frac{1}{4}(\tau_{xx}-\tau_{yy})^2 + \tau_{xy}^2}$



By sketch, $|F| \le -N\tan\phi \;\; \forall\theta$

$\Leftrightarrow$ Mohr circle lies in sector $|F| \le -N\tan\phi$

$\Leftrightarrow$ $c \geq 0$ and $R = c\sin\alpha \le c\sin\phi$

$\Leftrightarrow$ $\sqrt{(\tau_{xx}-\tau_{yy})^2 + 4\tau_{xy}^2} \le -(\tau_{xx}+\tau_{yy})\sin\phi$ ⊛

B8

Finally, (i) strict inequality in ⊛ $\Rightarrow$ elastic, while (ii) equality $\Rightarrow$ plastic.

B2

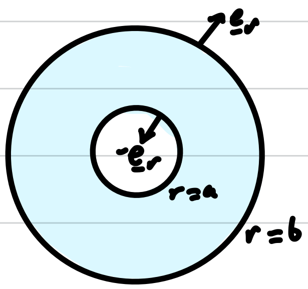
$\boxed{10}$

(b) (i) While material is elastic, plug the constitutive relations into the Navier equation to obtain

$$0 = \frac{d}{dr}\left((\lambda+2\mu)\frac{du}{dr} + \lambda\frac{u}{r}\right) + 2\mu\underbrace{\left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right)}_{=\frac{d}{dr}\left(\frac{u}{r}\right)}$$

$$\Rightarrow 0 = \frac{d}{dr}\left((\lambda+2\mu)\left(\frac{du}{dr} + \frac{u}{r}\right)\right)$$

So $\frac{d}{dr}\left(\tau_{rr}+\tau_{\theta\theta}\right) = \frac{d}{dr}\left((2\lambda+2\mu)\left(\frac{du}{dr}+\frac{u}{r}\right)\right) = 0$ as required. B2



The outward unit normal is $-e_r$ on $r=a$ & e_r on $r=b$ so

$\tau(-e_r) = P_{in}\, e_r$ on $r=a \Rightarrow \tau_{rr} = -P_{in}$ on $r=a$

$\tau\, e_r = -P_{out}\, e_r$ on $r=b \Rightarrow \tau_{rr} = -P_{out}$ on $r=b$

As P_{in} increases gradually from P_{out}, the material is initially elastic everywhere and the compatibility condition and Navier equation give

$$\tau_{rr} + \tau_{\theta\theta} = \text{constant} = 2A \text{ (say) and } \frac{d\tau_{rr}}{dr} + \frac{2\tau_{rr}}{r} = \frac{2A}{r}$$

So $\frac{d}{dr}\left(r^2\tau_{rr}\right) = 2Ar \Rightarrow \tau_{rr} = A + \frac{B}{r^2},\ \tau_{\theta\theta} = A - \frac{B}{r^2}\quad (A, B \in \mathbb{R})$ (†)

Then BCs $\Rightarrow \begin{pmatrix} 1 & a^{-2} \\ 1 & b^{-2}\end{pmatrix}\begin{pmatrix} A \\ B\end{pmatrix} = \begin{pmatrix} -P_{in} \\ -P_{out}\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} A \\ B\end{pmatrix} = \frac{1}{b^{-2}-a^{-2}}\begin{pmatrix} b^{-2} & -a^{-2} \\ -1 & 1\end{pmatrix}\begin{pmatrix} -P_{in} \\ -P_{out}\end{pmatrix}$$

$$\Rightarrow A = \frac{a^2P_{in} - b^2P_{out}}{b^2-a^2},\quad B = \frac{a^2b^2(P_{out}-P_{in})}{b^2-a^2}$$

In polars, (✱) with $\tau_{r\theta} = 0 \Rightarrow |\tau_{rr} - \tau_{\theta\theta}| \le -(\tau_{rr}+\tau_{\theta\theta})\sin\phi$

Note $\tau_{rr} + \tau_{\theta\theta} = 2A \le 0 \Leftrightarrow P_{in} \le \frac{b^2}{a^2} P_{out}$, while $B < 0 \Rightarrow$
$|\tau_{rr} - \tau_{\theta\theta}| = \tau_{\theta\theta} - \tau_{rr} = -\frac{2B}{r^2}$, so that ⊛ $\Leftrightarrow -\frac{B}{r^2} \le -A\sin\phi$

LHS maximal at $r=a$, so yield occurs there first when $\frac{B}{a^2} = A\sin\phi$
$\Rightarrow b^2(P_{out} - P_{in}) = (a^2 P_{in} - b^2 P_{out})\sin\phi \Rightarrow P_{in} = P_{c1} = \frac{1+\sin\phi}{1+\frac{a^2}{b^2}\sin\phi} P_{out}$ S6

(b)(ii) For $P_{in} > P_{c1}$, material must yield in a neighborhood of $r=a$ by continuity, say $a < r < s < b$, with s TBD.

In $r > s$, still have elastic solution given by (+) with A, B TBD.

In plastic region in $r < s$, ⊛ $\Rightarrow \tau_{\theta\theta} - \tau_{rr} = -(\tau_{rr} + \tau_{\theta\theta})\sin\phi$ with the sign determined by how yield condition was satisfied initially.

So $\tau_{\theta\theta} = (1-\gamma)\tau_{rr}$, where $\gamma = \frac{2\sin\phi}{1+\sin\phi}$.
Then Navier $\Rightarrow \frac{d\tau_{rr}}{dr} + \frac{\gamma}{r}\tau_{rr} = 0 \Rightarrow \tau_{rr} = -P_{in}\left(\frac{a}{r}\right)^\gamma$ as $\tau_{rr} = -P_{in}$ at $r=a$.

Continuity of stress at $r=s \Rightarrow [\tau_{rr}]_{r=s-}^{r=s+} = 0$, so yield condition at $r=s$
$\Rightarrow \tau_{rr}$ & $\tau_{\theta\theta}$ continuous across $r=s \Rightarrow A + Bs^{-2} = -P_{in}\left(\frac{a}{s}\right)^\gamma$, $A - Bs^{-2} = -(1-\gamma)P_{in}\left(\frac{a}{s}\right)^\gamma$
$\Rightarrow A = -\frac{1}{2}(2-\gamma)P_{in}\left(\frac{a}{s}\right)^\gamma$, $B = -\frac{\gamma s^2}{2}P_{in}\left(\frac{a}{s}\right)^\gamma$

Then $\tau_{rr} = -P_{out}$ on $r=b \Rightarrow A + Bb^{-2} = -P_{out} \Rightarrow \frac{P_{in}}{P_{out}} = G(s) = \frac{\alpha s^\gamma}{1+\beta s^2}$
where $\alpha = (1+\sin\phi)a^{-\gamma}$, $\beta = b^{-2}\sin\phi$, as required.

Calculate $G'(s) = \frac{\alpha\gamma s^{\gamma-1} - \alpha\beta(2-\gamma)s^{\gamma+1}}{(1+\beta s^2)^2} \begin{cases} > 0 \text{ for } 0 < s < s_c \\ = 0 \text{ for } s = s_c \end{cases}$, $s_c = \sqrt{\frac{\gamma}{(2-\gamma)\beta}} = b$ S/N5

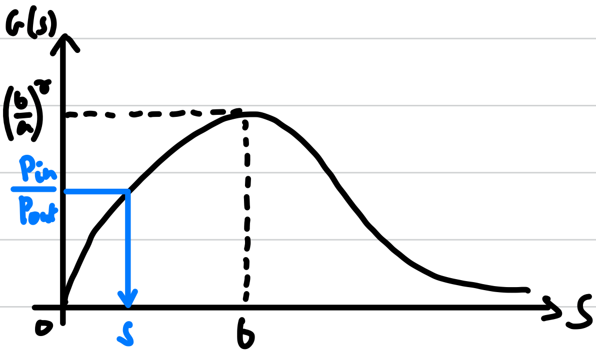


Diagram $\Rightarrow \exists! s \in [a,b]$ s.t. $G(s) = \frac{P_{in}}{P_{out}}$ for $P_{in} \in [P_{c1} = G(a)P_{out}, P_{c2} = G(b)P_{out}]$, so valid until $P_{in} = P_{c2} = \left(\frac{b}{a}\right)^\gamma P_{out}$ as $\gamma < 2 \Rightarrow A < 0$
$\Rightarrow$ under compression throughout.

N2

$\boxed{15}$