Honour School of Mathematics Part C: Paper C7.7
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.7
Master of Science in Mathematical Sciences: Paper C7.7
Master of Science in Mathematical and Theoretical Physics: Paper C7.7

Random Matrix Theory
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1. Consider a $n \times n$ random symmetric matrix M where the entries M_{ij}, $1 \leqslant i \leqslant j \leqslant n$, are IID random variables with

$$\mathbf{P}(M_{ij} = 1) = \mathbf{P}(M_{ij} = -1) = 1/2.$$

We write $\mathbf{E}$ for the expectation over the entries.

(a) [7 marks] Let μ_n be the *empirical spectral measure* of $M/\sqrt{n}$, i.e., $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}}$, where $(\lambda_i)_{i \leqslant n}$ are the eigenvalues of M.

(i) Let $k \in \mathbb{N}$. Show that the *k-th moment of* μ_n, denoted by $m_n^{(k)}$, satisfies

$$m_n^{(k)} = \int_{\mathbb{R}} x^k \,\mathrm{d}\mu_n(x) = \frac{1}{n^{k/2+1}} \sum_{\mathbf{i}} M_{i_1 i_2} M_{i_2 i_3} \dots M_{i_k i_1}, \tag{1}$$

where the sum is over all the k-tuples $\mathbf{i} = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$. In particular, show that $m_n^{(2)} = 1$ for any $n \geqslant 1$.

(ii) Prove that $\mathbf{E}[m_n^{(k)}] = 0$ for any $n \geqslant 1$ if k is odd.

(iii) Briefly show that $\mathbf{E}[m_n^{(2k)}] \leqslant \frac{(2k)!}{2^k k!}$, for any $k \in \mathbb{N}$, and $n \geqslant 1$.
[*Hint: Think in terms of pairings.*]

(b) [10 marks] For the even moments, evaluate $\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}]$ proceeding as follows:

(i) Establish a correspondence between the $2k$-tuples appearing in (1) and graphs with vertices labeled by elements of $\{1, \dots, n\}$.

(ii) Show that only the $2k$-tuples corresponding to graphs with exactly $k+1$ distinct labels contribute to $\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}]$.

(iii) Define the notion of *Dyck path* of length $2k$. Prove that

$$\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}] = \#\{\text{Dyck paths of length } 2k\}.$$

[*Hint: The relation* $\#V - \#E + \#F \leqslant 1$ *for a connected graph with vertex set* V*, edge set* E *and faces* F *(or loops) might be useful.*]

(c) [8 marks] Consider now the *variance* of the k-th moment of the empirical spectral measure:

$$v_n^{(k)} = \mathbf{E}\left[\left(m_n^{(k)} - \mathbf{E}[m_n^{(k)}]\right)^2\right], \; m \in \mathbb{N}.$$

(i) Argue that $v_n^{(2)} = 0$ for all n.

(ii) Prove that $v_n^{(4)} \leqslant \frac{c}{n^2}$ for some constant c independent of n.

(iii) Conclude that $\lim_{n\to\infty} m_n^{(4)} = 2$ almost surely.
[*You can use the Borel-Cantelli lemma without proof.*]

2. Consider the n eigenvalues $\{e^{\mathrm{i}\Theta_1}, \dots, e^{\mathrm{i}\Theta_n}\}$ of a $n \times n$ CUE matrix, i.e., sampled uniformly among the $n \times n$ unitary matrices. Recall that the joint probability density function of the random variables $(\Theta_1, \dots, \Theta_n)$ is given by

$$\rho_n(\theta_1, \dots, \theta_n)\, \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n = \frac{c_n}{(2\pi)^n} \prod_{1 \leqslant j < k \leqslant n} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^2 \; \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n,$$

for some normalization constant c_n.

(a) [12 marks] In this question, we find the marginal distributions of the eigenvalues proceeding as follows:

(i) Write ρ_n in terms of a *Vandermonde determinant.*

(ii) Consider the functions $\phi_j(\theta) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}j\theta}$, $0 \leqslant j \leqslant n-1$. Verify that $\{\phi_j\}_{0 \leqslant j \leqslant n-1}$ form an orthonormal set of functions for the Lebesgue measure $\mathrm{d}\theta$ on $[0, 2\pi]$. Show that ρ_n can be written as

$$\rho_n(\theta_1, \dots, \theta_n) = c_n \det\Big(\{K_n(\theta_i, \theta_j)\}_{i,j \leqslant n}\Big),$$

for the *projection kernel* $K_n(\theta, \theta') = \sum_{j=0}^{n-1} \phi_j(\theta)\phi_j(\theta')$.

(iii) State *Gaudin's Lemma.* Use it to prove that $c_n = n!$ and that the joint marginal distribution of $(\Theta_1, \dots, \Theta_k)$, $1 \leqslant k \leqslant n$, is given by

$$\rho_n^{(k)}(\theta_1, \dots, \theta_k) = \frac{(n-k)!}{n!} \det\Big(\{K_n(\theta_i, \theta_j)\}_{i,j \leqslant k}\Big).$$

(b) [5 marks] Write the *k-point correlation function* $R_n^{(k)}(\theta_1, \dots, \theta_k)$ in terms of K_n. In particular, find a simple expression for $R_n^{(1)}(\theta, \theta)$. What is the expected number of eigenvalues in an interval $[a, b]$, $0 \leqslant a < b \leqslant 2\pi$, for fixed n?

(c) [8 marks] Compute the distribution of the spacings between non-consecutive eigenvalues as follows:

(i) Write an expression for $R_n^{(2)}(\theta, \theta')$. For an appropriate choice of scaling s_n, show that

$$\frac{4\pi^2}{n^2} R_n^{(2)}(s_n x, s_n y) \to 1 - \Big(\frac{\sin(\pi(x-y))}{\pi(x-y)}\Big)^2.$$

(ii) Prove that for $-1 \leqslant a < b \leqslant 1$

$$\lim_{n \to \infty} \frac{1}{n} \mathbf{E}\Big[\#\{j \neq k : a \leqslant \frac{n}{2\pi}(\Theta_k - \Theta_j) \leqslant b\}\Big] = \int_a^b \Big(1 - \big(\frac{\sin(\pi x)}{\pi x}\big)^2\Big)\, \mathrm{d}x.$$

How does the density in the integral above behave as $x \to 0$? Consider only the non-trivial leading order.

3. We consider the matrix-valued process $(M(t), t \geqslant 0)$ where $M(t)$ is a 2×2 symmetric matrix

$$M(t) = \begin{pmatrix} X(t) & Z(t) \\ Z(t) & Y(t) \end{pmatrix}.$$

The processes $X(t), Y(t)$ and $Z(t)$ have initial values $X(0) = x_0$, $Y(0) = y_0$, $Z(0) = z_0$. Their evolution satisfies the following *stochastic differential equations* (SDEs):

$$\mathrm{d}X(t) = \sqrt{2}\,\mathrm{d}B_1(t) - \frac{1}{2}X(t)\,\mathrm{d}t \qquad \mathrm{d}Y(t) = \sqrt{2}\,\mathrm{d}B_2(t) - \frac{1}{2}Y(t)\,\mathrm{d}t \qquad \mathrm{d}Z(t) = \mathrm{d}B_3(t) - \frac{1}{2}Z(t)\,\mathrm{d}t.$$

Here $(B_1(t), B_2(t), B_3(t))$ are IID standard Brownian motions starting at 0.

(a) [5 marks] Let $\Lambda_1(t)$ and $\Lambda_2(t)$, $t \geqslant 0$, be the largest and smallest eigenvalues of $M(t)$. Write $\Lambda_1(t)$ and $\Lambda_2(t)$ as a function of $X(t)$, $Y(t)$ and $Z(t)$.

(b) [10 marks] (i) Use Itô's formula and the rules of stochastic calculus to derive an SDE for the gap process $G(t) = \Lambda_1(t) - \Lambda_2(t)$.

(ii) Is the SDE well-defined for any initial value x_0, y_0, z_0? Discuss.

(c) [10 marks] (i) Compute the expected gap $\mathbf{E}[G(t)]$ at all time $t > 0$ for the initial condition $x_0 = y_0 = 1$ and $z_0 = 0$.

(ii) Find the probability density function (PDF) of $G(t)$ for any fixed time $t > 0$ for the same initial conditions.

(iii) Discuss the results obtained in c(i) and c(ii) when $t \to \infty$ and when t is close to 0.

[*Hint: If $(O(t), t \geqslant 0)$ is an Ornstein-Uhlenbeck process with SDE*

$$\mathrm{d}O(t) = \sigma\,\mathrm{d}B(t) - kO(t)\,\mathrm{d}t$$

starting at $O(0)$, then $O(t)$ is distributed like a Gaussian random variable of mean $O(0)e^{-kt}$ and variance $\frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)$.]