Honour School of Mathematics Part C: Paper C7.7
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.7
Master of Science in Mathematical Sciences: Paper C7.7
Master of Science in Mathematical and Theoretical Physics: Paper C7.7

Random Matrix Theory
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1. Consider a $n \times n$ random symmetric matrix M where the entries M_{ij}, $1 \leqslant i \leqslant j \leqslant n$, are IID random variables with

$$\mathbf{P}(M_{ij} = 1) = \mathbf{P}(M_{ij} = -1) = 1/2.$$

We write $\mathbf{E}$ for the expectation over the entries.

(a) [7 marks] Let μ_n be the *empirical spectral measure* of $M/\sqrt{n}$, i.e., $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}}$, where $(\lambda_i)_{i \leqslant n}$ are the eigenvalues of M.

(i) Let $k \in \mathbb{N}$. Show that the *k-th moment of* μ_n, denoted by $m_n^{(k)}$, satisfies

$$m_n^{(k)} = \int_{\mathbb{R}} x^k \, \mathrm{d}\mu_n(x) = \frac{1}{n^{k/2+1}} \sum_{\mathbf{i}} M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_k i_1}, \tag{1}$$

where the sum is over all the k-tuples $\mathbf{i} = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$. In particular, show that $m_n^{(2)} = 1$ for any $n \geqslant 1$.

(ii) Prove that $\mathbf{E}[m_n^{(k)}] = 0$ for any $n \geqslant 1$ if k is odd.

(iii) Briefly show that $\mathbf{E}[m_n^{(2k)}] \leqslant \frac{(2k)!}{2^k k!}$, for any $k \in \mathbb{N}$, and $n \geqslant 1$.
[*Hint: Think in terms of pairings.*]

(b) [10 marks] For the even moments, evaluate $\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}]$ proceeding as follows:

(i) Establish a correspondence between the $2k$-tuples appearing in (1) and graphs with vertices labeled by elements of $\{1, \ldots, n\}$.

(ii) Show that only the $2k$-tuples corresponding to graphs with exactly $k+1$ distinct labels contribute to $\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}]$.

(iii) Define the notion of *Dyck path* of length $2k$. Prove that

$$\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}] = \#\{\text{Dyck paths of length } 2k\}.$$

[*Hint: The relation $\#V - \#E + \#F \leqslant 1$ for a connected graph with vertex set V, edge set E and faces F (or loops) might be useful.*]

(c) [8 marks] Consider now the *variance* of the k-th moment of the empirical spectral measure:

$$v_n^{(k)} = \mathbf{E}\left[\left(m_n^{(k)} - \mathbf{E}[m_n^{(k)}]\right)^2\right], \ m \in \mathbb{N}.$$

(i) Argue that $v_n^{(2)} = 0$ for all n.

(ii) Prove that $v_n^{(4)} \leqslant \frac{c}{n^2}$ for some constant c independent of n.

(iii) Conclude that $\lim_{n\to\infty} m_n^{(4)} = 2$ almost surely.
[*You can use the Borel-Cantelli lemma without proof.*]

Solution. (a) (i) This is simply by opening the trace

$$m_n^{(k)} = \frac{1}{n}\mathrm{Tr}\left(\frac{M}{\sqrt{n}}\right)^k = \frac{1}{n^{k/2+1}} \sum_{i_1,\ldots,i_k=1}^{n} M_{i_1 i_2} \ldots M_{i_k i_1}.$$

In the case $k = 2$, since M is symmetric, we get $m_n^{(2)} = \frac{1}{n^2}\sum_{i_1,i_2=1}^{n} M_{i_1 i_2}^2 = 1$.

(ii) If we take the expectation, we get

$$\mathbf{E}[m_n^{(k)}] = \frac{1}{n^{k/2+1}} \sum_{i_1,\ldots,i_k=1}^{n} \mathbf{E}[M_{i_1 i_2} \ldots M_{i_k i_1}].$$

The product inside the expectation is 1 or -1. If k is odd, then one pair $i_j i_{j+1}$ must appear an odd number of times. The conclusion follows from the fact the the odd moments of M_{ij} are 0.

(iii) For $\mathbf{E}[m_n^{(2k)}]$, we note that the $\mathbf{E}[M_{i_1 i_2} \ldots M_{i_{2k} i_1}] = 0$ if one pair $i_j i_{j+1}$ appears only once. Therefore, each pair must appear at least twice. Since the joint moment of the entries is bounded by 1, we simply need to estimate the number of pairings of $2k$ objects which is $(2k-1)(2k-3)\ldots(5)(3)(1)$. Note that the number of labelings can be bounded by n^{k+1} canceling the prefactor.

Comments on marks (7 marks):

(i) 2=1B+1VB: is book work done in class several time, with a simple observation for this particular distribution.

(ii) 3=3VB: is a variation of book work for this distribution.

(iii) 2=2N: is new. For this distribution the moments of the entries is easily bounded by 1 and it's not too hard to estimate the pairings.

(b) (i) We associate to each $\mathbf{i}$ a graph $G(\mathbf{i})$ where the vertices $V(\mathbf{i})$ are labeled by the distinct values in $\{i_1, \ldots, i_{2k}\}$. An edge is placed between two labeled vertices i_j and i_{j+1} if i_j and i_{j+1} appear consecutively in $\mathbf{i}$. Note that the graph is connected by construction.

(ii) When taking the expectation, $\mathbf{E}[M_{i_1 i_2} \ldots M_{i_{2k} i_1}]$ is non-zero only if each edge appears at least twice. Therefore the number of edges in the graph $\#E(\mathbf{i}) \leqslant k$. Since the graph is connected, we have $\#V(\mathbf{i}) \leqslant k+1$. The number of possible labeling is $\sim n^{\#V(\mathbf{i})}$. In particular, if $\#V(\mathbf{i}) < k+1$, and we have

$$\lim_{n\to\infty} \mathbf{E}[m_n^{(2k)}] = \lim_{n\to\infty} \frac{1}{n^{k+1}} n^{\#V(\mathbf{i})} = 0,$$

since $|\mathbf{E}[M_{i_1 i_2} \ldots M_{i_{2k} i_1}]| \leqslant 1$.

(iii) A Dyck path of length $2k$ is a function $\pi : \{0, 1, \ldots, 2k\} \to \{0, 1, \ldots, k\}$ such that $\pi(0) = \pi(2k) = 0$ and the increments are ± 1, i.e., $|\pi(j+1) - \pi(j)| = 1$ for all $j \geqslant 0$. Note that π is positive. Going back to part (i), only the $\mathbf{i}$ with $\#V(\mathbf{i}) = k+1$ will contribute in the limit. But such $\mathbf{i}$'s must have k edges by the Euler relation. Moreover, there can't be no faces, so $G(\mathbf{i})$ must be a tree. Since every edge must then appear twice in $\mathbf{i}$, we have that the limit is exactly the number of excursions on a tree with k vertices from the root to the root, with the trees labeled by the order of visit. (There are $\sim n^{k+1}$ possible labeling of the vertices, canceling the pre-factor.) But such an excursion correspond to a Dyck path where $\pi(k)$ is the distance from the root at time k.

Comments on marks (12 marks):

(i) 3=3B: Done several in lectures and classes

(ii) 4=2B+2VB: Same. One difference is that it is easier here to bound the moment of the entries.

(iii) 3=3B: This sort of computation appeared in lectures and tutorials.

(c) (i) This is clear since by part (a)i., $m_n^{(2)} = 1$ for all n with probability 1.

(ii) If we write out the variance and the traces, one gets

$$v_n^{(4)} = \frac{1}{n^{4+2}} \sum_{\mathbf{i}\,\mathbf{j}} \mathbf{E}[M_{i_1 i_2} \dots M_{i_4 i_1} M_{j_1 j_2} \dots M_{j_4 j_1}] - \mathbf{E}[M_{i_1 i_2} \dots M_{i_4 i_1}]\mathbf{E}[M_{j_1 j_2} \dots M_{j_4 j_1}]$$

where the sum is over 4-tuples $\mathbf{i}$ and $\mathbf{j}$. We construct a graph $G(\mathbf{i}, \mathbf{j})$ as before. Each edge must appear at least twice, so there are at most 5 vertices. Moreover the graph must be connected between the $\mathbf{i}$ and the $\mathbf{j}$ vertices otherwise the first expectation factors yielding a zero contribution. In particular, the number of labelings of vertices is $\leqslant n^4$. This shows that the variance is $\mathrm{O}(1/n)$. To improve this to $1/n^2$, suppose that one has exactly 5 vertices. As before, this means that there are exactly 4 edges, and that the underlying graph $G(\mathbf{i}, \mathbf{j})$ is a tree. In particular, the subgraphs $G(\mathbf{i})$ is also a tree. This is a contradiction, since it is possible to start at i_1 and going back to i_1 by passing one edge only once (since there is one edge that must be in $G(\mathbf{i})$ and $G(\mathbf{j})$).

(iii) Part (c)ii, implies that $m_n^{(4)}$ converges almost surely to its expectation by the Borel-Cantelli lemma, since by Chebyshev's inequality, $\mathbf{P}(|m_n^{(4)} - \mathbf{E}[m_n^{(4)}]| > \delta)$ are summable for every δ. The expectation converges by (b)(ii) to the number of Dyck paths of length 4. This is easy to check that there are only two such paths (or use the connection with the Catalan numbers).

Comments on marks (8 marks):

(i) 1=1VB: Simple understanding of variance using part a.

(ii) 5=5N: This was sketched in class for the general case. But coming up with the whole reasoning is new.

(iii) 2=2VB: Similar reasoning was done in class. Here they need to connect to the number of Dyck path of length 4 which they can explicitly count.

Marking of the question: 25=9B+9VB+7N $\square$

2. Consider the n eigenvalues $\{e^{\mathrm{i}\Theta_1}, \ldots, e^{\mathrm{i}\Theta_n}\}$ of a $n \times n$ CUE matrix, i.e., sampled uniformly among the $n \times n$ unitary matrices. Recall that the joint probability density function of the random variables $(\Theta_1, \ldots, \Theta_n)$ is given by

$$\rho_n(\theta_1, \ldots, \theta_n)\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n = \frac{c_n}{(2\pi)^n} \prod_{1 \leqslant j < k \leqslant n} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^2 \ \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n,$$

for some normalization constant c_n.

(a) [12 marks] In this question, we find the marginal distributions of the eigenvalues proceeding as follows:

(i) Write ρ_n in terms of a *Vandermonde determinant.*

(ii) Consider the functions $\phi_j(\theta) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}j\theta}$, $0 \leqslant j \leqslant n-1$. Verify that $\{\phi_j\}_{0 \leqslant j \leqslant n-1}$ form an orthonormal set of functions for the Lebesgue measure $\mathrm{d}\theta$ on $[0, 2\pi]$. Show that ρ_n can be written as

$$\rho_n(\theta_1, \ldots, \theta_n) = c_n \det\Big(\{K_n(\theta_i, \theta_j)\}_{i,j \leqslant n}\Big),$$

for the *projection kernel* $K_n(\theta, \theta') = \sum_{j=0}^{n-1} \phi_j(\theta)\phi_j(\theta')$.

(iii) State *Gaudin's Lemma.* Use it to prove that $c_n = n!$ and that the joint marginal distribution of $(\Theta_1, \ldots, \Theta_k)$, $1 \leqslant k \leqslant n$, is given by

$$\rho_n^{(k)}(\theta_1, \ldots, \theta_k) = \frac{(n-k)!}{n!} \det\Big(\{K_n(\theta_i, \theta_j)\}_{i,j \leqslant k}\Big).$$

(b) [5 marks] Write the *k-point correlation function* $R_n^{(k)}(\theta_1, \ldots, \theta_k)$ in terms of K_n. In particular, find a simple expression for $R_n^{(1)}(\theta, \theta)$. What is the expected number of eigenvalues in an interval $[a, b]$, $0 \leqslant a < b \leqslant 2\pi$, for fixed n?

(c) [8 marks] Compute the distribution of the spacings between non-consecutive eigenvalues as follows:

(i) Write an expression for $R_n^{(2)}(\theta, \theta')$. For an appropriate choice of scaling s_n, show that

$$\frac{4\pi^2}{n^2} R_n^{(2)}(s_n x, s_n y) \to 1 - \Big(\frac{\sin(\pi(x-y))}{\pi(x-y)}\Big)^2.$$

(ii) Prove that for $-1 \leqslant a < b \leqslant 1$

$$\lim_{n \to \infty} \frac{1}{n} \mathbf{E}\Big[\#\{j \neq k : a \leqslant \frac{n}{2\pi}(\Theta_k - \Theta_j) \leqslant b\}\Big] = \int_a^b \Big(1 - \big(\frac{\sin(\pi x)}{\pi x}\big)^2\Big)\, \mathrm{d}x.$$

How does the density in the integral above behave as $x \to 0$? Consider only the non-trivial leading order.

Solution. (a) (i) By definition of the Vandermonde determinant $\prod_{1\leqslant j<k\leqslant n}(e^{\mathrm{i}\theta_k}-e^{\mathrm{i}\theta_j}) = \det(\{e^{\mathrm{i}k\theta_j}, 1\leqslant j\leqslant n, 0\leqslant k\leqslant n-1\})$.

(ii) We have

$$\int_0^{2\pi}\overline{\phi_j}(\theta)\phi_k(\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(k-j)\theta}\,\mathrm{d}\theta = \delta_{jk}$$

as claimed. Using the fact that $\det(A^\dagger A) = |\det A|^2$, we get that

$$|e^{\mathrm{i}\theta_j}-e^{\mathrm{i}\theta_k}|^2 = \det(\{K_n(\theta_j,\theta_k)\}_{j,k\leqslant n}),$$

where $K_n(\theta,\theta') = \sum_{k=0}^{n-1}\overline{\phi_k}(\theta)\phi_k(\theta')$. One can note (though it is not necessary at this point) that K_n depends only on $\theta-\theta'$.

(iii) Gaudin's lemma states that if a kernel K is such that $\int K(x,y)(y,z)\,\mathrm{d}y = K(x,z)$ then

$$\int \det(\{K(x_i,x_j)\}_{i,j\leqslant n})\,\mathrm{d}x_n = (r-n+1)\det(\{K(x_i,x_j)\}_{i,j\leqslant n-1}),$$

where $r=\int K(x,x)\,\mathrm{d}x$. In the setting of the problem $r=\int_0^{2\pi}K_n(x,x)\,\mathrm{d}x = n$. By integrating $\rho_n(x_1,\dots,x_n)$ over all x's, we get $n!c_n=1$ giving the claim. If we integrate x_n up to x_{n-k+1}, we get by Gaudin's lemma

$$\rho_n^{(k)}(\theta_1,\dots,\theta_k) = \frac{(n-k)!}{n!}\det\Big(\{K_n(\theta_i,\theta_j)\}_{i,j\leqslant k}\Big).$$

Comments on marks (12 marks):

(i) 2=2B: Done in lectures.

(ii) 5=5B: Done in lectures.

(iii) 5=5B: Done in lectures.

(b) The k-point correlation function is defined by $\frac{n!}{(n-k)!}\rho_n^{(k)}(\theta_1,\dots,\theta_k)$ so it is exactly $R_n^{(k)}(\theta_1,\theta_k) = \det\Big(\{K_n(\theta_i,\theta_j)\}_{i,j\leqslant k}\Big)$. In the case $k=1$, it is simply

$$R_n^{(1)}(\theta) = K_n(\theta,\theta) = \frac{n}{2\pi}.$$

It is independent of θ. So the eigenvalues are uniformly distributed on the unit circle as expected. In particular, we get

$$\int_a^b R_n^{(1)}(\theta)\,\mathrm{d}\theta = \mathbf{E}[\#\{j:\Theta_j\in[a,b]\}] = \frac{n}{2\pi}(b-a).$$

Comments on marks (5 marks):

(i) 3=3B: Done in lectures.

(ii) 2=2VB: The second part is a variant of an example done in class, but good students should see it as a simple application from the understanding of $R_n^{(1)}$.

(c) (i) For the 2-point correlation function, note first that

$$K_n(\theta,\theta') = \frac{1}{2\pi}\sum_{j=0}^{n-1}e^{\mathrm{i}j(\theta-\theta')} = \frac{1}{2\pi}\frac{e^{\mathrm{i}n(\theta-\theta')}-1}{e^{\mathrm{i}(\theta-\theta')}-1} = \frac{e^{(n-1)(\theta-\theta')/2}}{2\pi}\frac{\sin(n(\theta-\theta')/2)}{n\sin((\theta-\theta')/2)}$$

Therefore, we get

$$R_n^{(2)}(\theta,\theta') = \frac{n^2}{4\pi^2}\begin{vmatrix} 1 & \frac{\sin(n(\theta-\theta')/2)}{n\sin(\theta-\theta')/2)} \\ \frac{\sin(n(\theta-\theta')/2)}{n\sin((\theta-\theta')/2)} & 1\end{vmatrix} = \frac{n^2}{4\pi^2}\left(1-\Big(\frac{\sin(n(\theta-\theta')/2)}{n\sin((\theta-\theta')/2)}\Big)^2\right)$$

Part (b) suggests the scaling $\theta \to \frac{2\pi}{n}x$. Under that scaling, we have

$$\lim_{n\to\infty} \frac{4\pi^2}{n^2} R_n^{(2)}(\frac{2\pi}{n}x, \frac{2\pi}{n}y) = 1 - \Big(\frac{\sin(\pi(x-y))}{\pi(x-y)}\Big)^2$$

(ii) By definition of the two-point correlation function we have for any function f on $[0, 2\pi]^2$,

$$\mathbf{E}\Big[\sum_{j\neq k} f(\Theta_j, \Theta_k)\Big] = \int_0^{2\pi}\int_0^{2\pi} f(\theta, \theta')R_n^{(2)}(\theta, \theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'.$$

If the function f depends only on the difference, since it is also the case for the correlation function, the above reduces to

$$\mathbf{E}\Big[\sum_{j\neq k} f(\Theta_j - \Theta_k)\Big] = 2\pi\int_{-2\pi}^{2\pi} f(u)R_n^{(2)}(u)\,\mathrm{d}u$$

Applying the change of variable $u = \frac{2\pi}{n}x$ as in (ii), one gets

$$\frac{4\pi^2}{n}\int_{-n}^{n} f(2\pi x/n)R_n^{(2)}(2\pi x/n)\,\mathrm{d}x$$

We apply the above with the indicator function $1_{[a,b]}(ny/2\pi)$ to get

$$\lim_{n\to\infty}\frac{1}{n}\mathbf{E}\Big[\#\{j\neq k : a \leqslant \frac{n}{2\pi}(\Theta_k - \Theta_j) \leqslant b\}\Big] = \int_a^b \Big(1 - \Big(\frac{\sin(\pi x)}{\pi x}\Big)^2\Big)\,\mathrm{d}x,$$

where we used the asymptotics of (ii). For small x, the density is $\frac{\pi^2 x^2}{3}$.

Comments on marks (8 marks):

(i) 4=4 VB: The first part was done in detail for GUE.

(ii) 4=4N: The second part is new, where they have to apply the appropriate scaling using part (i) and understand the meaning of the 2-point correlation function.

□

Marking of the question: 25=15B+6VB+4N

3. We consider the matrix-valued process $(M(t), t \geqslant 0)$ where $M(t)$ is a 2×2 symmetric matrix

$$M(t) = \begin{pmatrix} X(t) & Z(t) \\ Z(t) & Y(t) \end{pmatrix}.$$

The processes $X(t), Y(t)$ and $Z(t)$ have initial values $X(0) = x_0$, $Y(0) = y_0$, $Z(0) = z_0$ and their evolution satisfies the following *stochastic differential equations* (SDEs):

$$\mathrm{d}X(t) = \sqrt{2}\,\mathrm{d}B_1(t) - \frac{1}{2}X(t)\,\mathrm{d}t \qquad \mathrm{d}Y(t) = \sqrt{2}\,\mathrm{d}B_2(t) - \frac{1}{2}Y(t)\,\mathrm{d}t \qquad \mathrm{d}Z(t) = \mathrm{d}B_3(t) - \frac{1}{2}Z(t)\,\mathrm{d}t.$$

Here $(B_1(t), B_2(t), B_3(t))$ are IID standard Brownian motions starting at 0.

(a) [5 marks] Let $\Lambda_1(t)$ and $\Lambda_2(t)$, $t \geqslant 0$, be the largest and smallest eigenvalues of $M(t)$. Write $\Lambda_1(t)$ and $\Lambda_2(t)$ as a function of $X(t)$, $Y(t)$ and $Z(t)$.

(b) [10 marks] (i) Use Itô's formula and the rules of stochastic calculus to derive an SDE for the gap process $G(t) = \Lambda_1(t) - \Lambda_2(t)$.

(ii) Is the SDE well-defined for any initial value x_0, y_0, z_0? Discuss.

(c) [10 marks] (i) Compute the expected gap $\mathbf{E}[G(t)]$ at all time $t > 0$ for the initial condition $x_0 = y_0 = 1$ and $z_0 = 0$.

(ii) Find the probability density function (PDF) of $G(t)$ for any fixed time $t > 0$ for the same initial conditions.

(iii) Discuss the results obtained in c(i) and c(ii) when $t \to \infty$ and when t is close to 0.

[*Hint: If $(O(t), t \geqslant 0)$ is an Ornstein-Uhlenbeck process with SDE*

$$\mathrm{d}O(t) = \sigma\,\mathrm{d}B(t) - kO(t)\,\mathrm{d}t$$

starting at $O(0)$, then $O(t)$ is distributed like a Gaussian random variable of mean $O(0)e^{-kt}$ and variance $\frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)$.]

Solutions. (a) This is straightforward linear algebra. The characteristic polynomial of $M(t)$ is $\lambda^2 - (X(t)+Y(t))\lambda + XY - Z^2$. Therefore the eigenvalues are

$$\begin{aligned}\Lambda_{1,2}(t) &= \frac{X(t)+Y(t) \pm \sqrt{(X(t)+Y(t))^2 - 4(X(t)Y(t)-Z(t)^2)}}{2}\\ &= \frac{X(t)+Y(t) \pm \sqrt{(X(t)-Y(t))^2 + 4Z(t)^2)}}{2}\end{aligned}$$

Comments on marks (5 marks): 5=5B: Basic linear algebra. Also reviewed in PS1.

(b) (i) The gap process from part (a) is

$$G(t) = 2\sqrt{\Big(\frac{X(t)-Y(t)}{2}\Big)^2 + Z(t)^2}.$$

Write $U(t)$ for the process $\frac{X(t)-Y(t)}{2}$. From the SDE of $X(t)$ and $Y(t)$ we get

$$\mathrm{d}U(t) = \frac{1}{\sqrt{2}}(\,\mathrm{d}B_1(t) - \,\mathrm{d}B_2(t)) - \frac{1}{2}U(t)\,\mathrm{d}t.$$

Note that $W(t) = \frac{1}{\sqrt{2}}(B_1(t) - B_2(t))$ is a standard Brownian motion independent of $(B_3(t), t \geqslant 0)$. Therefore, $U(t)$ is independent of $Z(t)$ and has the same distribution. We have $G(t) = F(U(t), Z(t))$ where $F(x,y) = 2\sqrt{x^2+y^2}$. We apply Itô's formula to F. The derivatives are

$$\begin{aligned}\nabla F(x,y) &= \Big(\frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}}\Big) \quad \partial_1^2 F(x,y) = \frac{2y^2}{(x^2+y^2)^{3/2}}\\ \partial_2^2 F(x,y) &= \frac{2x^2}{(x^2+y^2)^{3/2}} \qquad \partial_1\partial_2 F(x,y) = \frac{-2xy}{(x^2+y^2)^{3/2}}.\end{aligned}$$

Itô's formula then gives

$$\mathrm{d}F(U(t),Z(t)) = \partial_1 F(U(t),Z(t))\,\mathrm{d}U(t) + \partial_2 F(U(t),Z(t))\,\mathrm{d}Z(t) + \frac{1}{2}\Delta F(U(t),Z(t))\,\mathrm{d}t,$$

by the rules of stochastic calculus since $\mathrm{d}U(t)\,\mathrm{d}Z(t) = 0$, $(\,\mathrm{d}U(t))^2 = \,\mathrm{d}t$ and $(\,\mathrm{d}Z(t))^2 = \mathrm{d}t$. Therefore we get

$$\mathrm{d}G(t) = \frac{U(t)}{\sqrt{U(t)^2+Z(t)^2}}\,\mathrm{d}W(t) + \frac{Z(t)}{\sqrt{U(t)^2+Z(t)^2}}\,\mathrm{d}B_3(t) + \left(\frac{2}{G(t)} - \frac{G(t)}{2}\right)\,\mathrm{d}t.$$

Defining the process $W'(t)$ by

$$\mathrm{d}W'(t) = \frac{U(t)}{\sqrt{U(t)^2+Z(t)^2}}\,\mathrm{d}W(t) + \frac{Z(t)}{\sqrt{U(t)^2+Z(t)^2}}\,\mathrm{d}B_3(t).$$

We get by the rules of stochastic calculus that $(\,\mathrm{d}W'(t))^2 = \,\mathrm{d}t$. By Lévy's theorem (seen in lectures), it is also a standard Brownian motion. We finally get

$$\mathrm{d}G(t) = \,\mathrm{d}W'(t) + \left(\frac{2}{G(t)} - \frac{G(t)}{2}\right)\,\mathrm{d}t.$$

where $(W'(t), t \geqslant 0)$ is a standard Brownian motion.

(ii) There is an issue if $G(t) = 0$. This could only happen if $U(t) = 0$ and $Z(t) = 0$ for the same time t. This does not happen with probability 1 since they are independent. The only issue could occur if $G(0) = 0$, i.e., $x_0 = y_0$ and $z_0 = 0$. However, the process $G(t) = 2\sqrt{U(t)^2 + Z(t)^2}$ is well-defined for any t including $t = 0$. The SDE makes sense then for $t > 0$.

Comments on marks (10 marks):

(i) 8=4B+4VB: Similar computations were done in PS1 (for fixed t) and also in PS4 for other diffusions.

(ii) 2=2N: The student here has to reason a little bit and deduce from the properties of Brownian motion seen in class that the singularity is never hit when $z_0 \neq 0$ and $x_0 \neq y_0$.

(c) (i) We know that $U(t)$ and $Z(t)$ are IID Ornstein-Uhlenbeck process with $\sigma = 1$ and $k = 1/2$. With the given initial conditions, we have $U(0) = Z(0) = 0$. Since $G(t) = 2\sqrt{(U(t)^2 + Z(t)^2}$, we get

$$\mathbf{E}[G(t)] = \int_{\mathbb{R}} \int_{\mathbb{R}} 2\sqrt{u^2 + v^2} \frac{e^{-\frac{u^2+v^2}{2(1-e^{-t})}}}{2\pi(1 - e^{-t})} \, du \, dv,$$

using the hint. With polar coordinates, we get

$$\mathbf{E}[G(t)] = \frac{2}{1 - e^{-t}} \int_0^{\infty} r^2 e^{-\frac{r^2}{2(1-e^{-t})}} \, dr = \frac{1}{1 - e^{-t}} \int_{-\infty}^{\infty} r^2 e^{-\frac{r^2}{2(1-e^{-t})}} \, dr = \sqrt{2\pi}\sqrt{1 - e^{-t}}.$$

(ii) Proceeding similarly as in (i), we can write the CDF of $G(t)$ as

$$\mathbf{P}(G(t) \leqslant x) = \frac{1}{1 - e^{-t}} \int_0^{x/2} r e^{-\frac{r^2}{2(1-e^{-t})}} \, dr.$$

Taking the derivative, we get $f_{G(t)}(x) = \frac{1}{1-e^{-t}} \frac{x}{4} e^{-\frac{x^2}{8(1-e^{-t})}}$ for $x \geqslant 0$.

(iii) We have

$$\mathbf{E}[G(t)] = \sqrt{2\pi}\sqrt{1 - e^{-t}},$$

For $t \to \infty$, we get $\mathbf{E}[G(t)] \to \sqrt{2\pi}$. For t close to 0, we get by expanding $\mathbf{E}[G(t)] = \sqrt{2\pi}\sqrt{1 - e^{-t}} \sim \sqrt{2\pi}\sqrt{t}$.

As $t \to \infty$, the PDF becomes $\frac{x}{4} e^{-\frac{x^2}{8}}$ and as $t \to 0$, $f_{G(t)}(x) \sim \frac{x}{4t} e^{-\frac{x^2}{8t}}$.

Comments on marks (8 marks):

(i) 4=4B: Similar computation to PS1 using hint.

(ii) 4=4VB: Same

(iii) 2=2N: The good students should get that the square root behavior for small t is related to the singularity noticed in part (b).

Marking of the question: 25=13B+8VB+4N

□