GR 2 Revision Lecture

Linearised Gravity

Consider a perturbation of the metric around Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

small symmetric matrix

to first order the inverse is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad \underbrace{h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}}_{\text{Need to define this}} h_{\rho\sigma}$$

Want to compute Einstein's equations: we need the Ricci tensor and Ricci scalar.

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

note that the term $h^{\rho\sigma}$ would necessarly be $\mathcal{O}(h^2)$ so we can immediately forget it.

$$= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) + \mathcal{O}(h^2)$$

Similar computation for $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$, R

A bit of a mess but we can use our diffeomorphism invariance to simplify the problem.

$x^\mu \to x^\mu - \xi^\mu$

$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$

$\Rightarrow h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

Choose a gauge in which

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0$$

$$\boxed{h = h_{\mu\nu}\eta^{\mu\nu}}$$

"De Donder" gauge

Let $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

De Donder gauge $\Rightarrow \partial^\mu \bar{h}_{\mu\nu} = 0$

Einstein's equations become

$$\boxed{\Box \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}}$$

Plane waves

Simplest solutions are sourceless plane waves

$T_{\mu\nu} = 0$ $\quad\quad$ $\Box \bar{h}_{\mu\nu} = 0 \Rightarrow \bar{h}_{\mu\nu} = \mathrm{Re}\left[\underset{\uparrow}{H_{\mu\nu}} e^{i k_\sigma x^\sigma}\right]$

symmetric, constant matrix.

wave eq $\Rightarrow k^2 = 0$ ~ massless propagating at speed of light

Gauge condition $\Rightarrow k^\mu H_{\mu\nu} = 0$

Still some residual gauge freedom.

Go to transverse traceless gauge : $h_{0\mu} = 0$

Take $K^\mu = (\omega, 0, 0, \omega)$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 d.o.f the polarisations

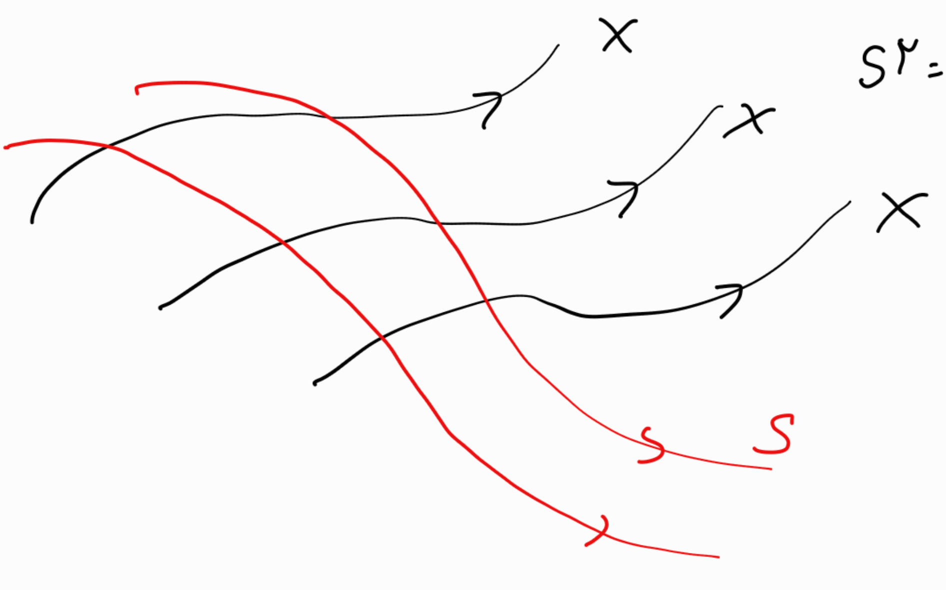
$$\begin{aligned} ds^2 = -dt^2 + dz^2 &+ \left(1 + \mathrm{Re}\left[h_+ e^{i\omega(z-t)}\right]\right) dx^2 \\ &+ \left(1 - \mathrm{Re}\left[h_+ e^{i\omega(z-t)}\right]\right) dy^2 \\ &+ 2\,\mathrm{Re}\left[h_\times e^{i\omega(z-t)}\right] dx\,dy \end{aligned}$$

Studied how to detect grav waves using geodesic deviation

Family of geodesics for a vector field X

$$X^\mu = \left.\frac{\partial x^\mu}{\partial \tau}\right|_\sigma$$

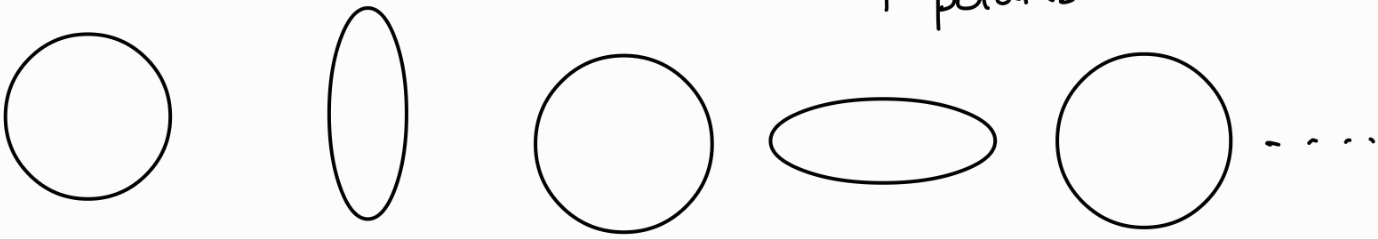
$$S^\mu = \left.\frac{\partial x^\mu}{\partial \sigma}\right|_\tau$$



Study how separation changes in GW background.

$$\nabla_X \nabla_X S^\mu = R^\mu{}_{\nu\rho\sigma} X^\nu X^\rho X^\sigma$$

+ polarisation



Energy of GW

$$\frac{dP}{d\Omega} = \frac{r^2}{32\pi G_N} \cdot \langle \underbrace{\dot{h}^{TT}_{ij} \dot{h}^{TT\,ij}}_{2(\dot{h}_+^2 + \dot{h}_\times^2)} \rangle$$

average over period of oscillation

Sources

Use Greens function

Define second moment of energy density

$$M^{ij}(t) = \int d^3y \; y^i y^j \, T_{00}(t, \vec{y})$$

then

$$h^{TT}_{ij}(t, \vec{x}) = \frac{2}{r} \Lambda_{ij\,kl}(\vec{n}) \, \ddot{M}^{kl}(t-r)$$

projection depending on angle of incidence

Penrose diagrams

Conformal transformation

$$\tilde{g}_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x)$$

preserves causal structure (though no longer a solution)

null geodesics $\longrightarrow$ null geodesics

Encode the causal structure in a Penrose diagram

3 steps

1) Change coord such that ∞ brought to finite coord distance.
Will introduce some singularities for the points ∞ away.

2) Perform conformal transformation to remove singular points.
on edges

3) Add points @ ∞. Now a boundary.

Example

2d Minkowski

$$ds^2 = -dt^2 + dx^2 \qquad (M, g)$$

Step 1

$$\left.\begin{aligned} u &= t - x \\ v &= t + x \end{aligned}\right\} \longrightarrow ds^2 = du\,dv \qquad u, v \in (-\infty, \infty)$$

$$u = \tan U \qquad v = \tan V \qquad U, V \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$$ds^2 = -\frac{1}{\cos^2 U \cos^2 V} dU\,dV$$

Step 2

Conformal transformation

$$\Omega^2 = \cos^2 U \cos^2 V$$

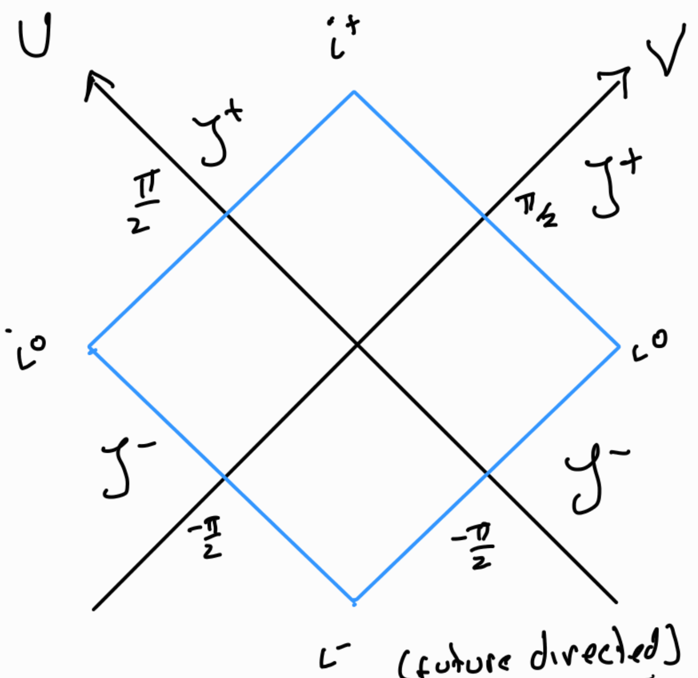
$$\tilde{ds}^2 = -dU dV \qquad (M, \tilde{g})$$

Step 3

Add points @ ∞ $\qquad (\tilde{M}, \tilde{g})$

$$U, V \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \longrightarrow U, V \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

Draw diagram and label



(future directed)

i^+ = future timelike ∞ (timelike curves end here)

i^- = past timelike ∞ (timelike curves start here)

i^0 = spatial ∞ (spacelike curves begin and end.)

$\mathcal{J}^+$ = future null ∞ (null curves end)

$\mathcal{J}^- =$ past null ∞ (null curves begin)

Hypersurfaces

a $d-1$ dim space inside a d-dim spacetime.

Defined by 1 real equation.

Define as $f(x) = 0$

df is normal to hypersurface

dual vector ξ is called hypersurface orthogonal.

Killing horizon

Null hypersurface Σ a killing horizon if ξ is a Killing vector normal to Σ.

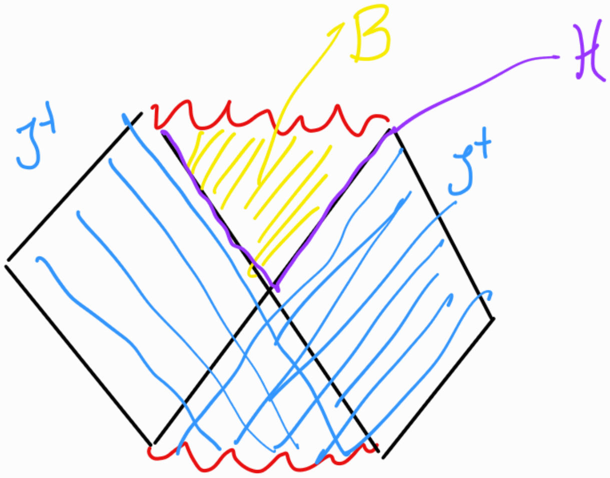
Def of BH

Causal curve. any timelike or null path

Causal future. Given a subset $S \subset M$

$$J^+(S) = \{ p \in M \mid \text{reached from } S \text{ following a future directed causal curve} \}$$

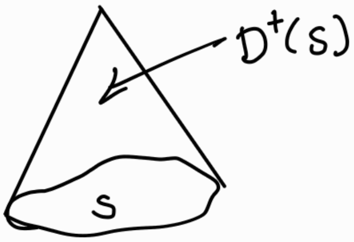
$$B = M \setminus [M \cap J^-(\mathcal{J}^+)] \quad \Big| \quad \mathcal{H} = \partial B$$



Charged and Rotating BHs

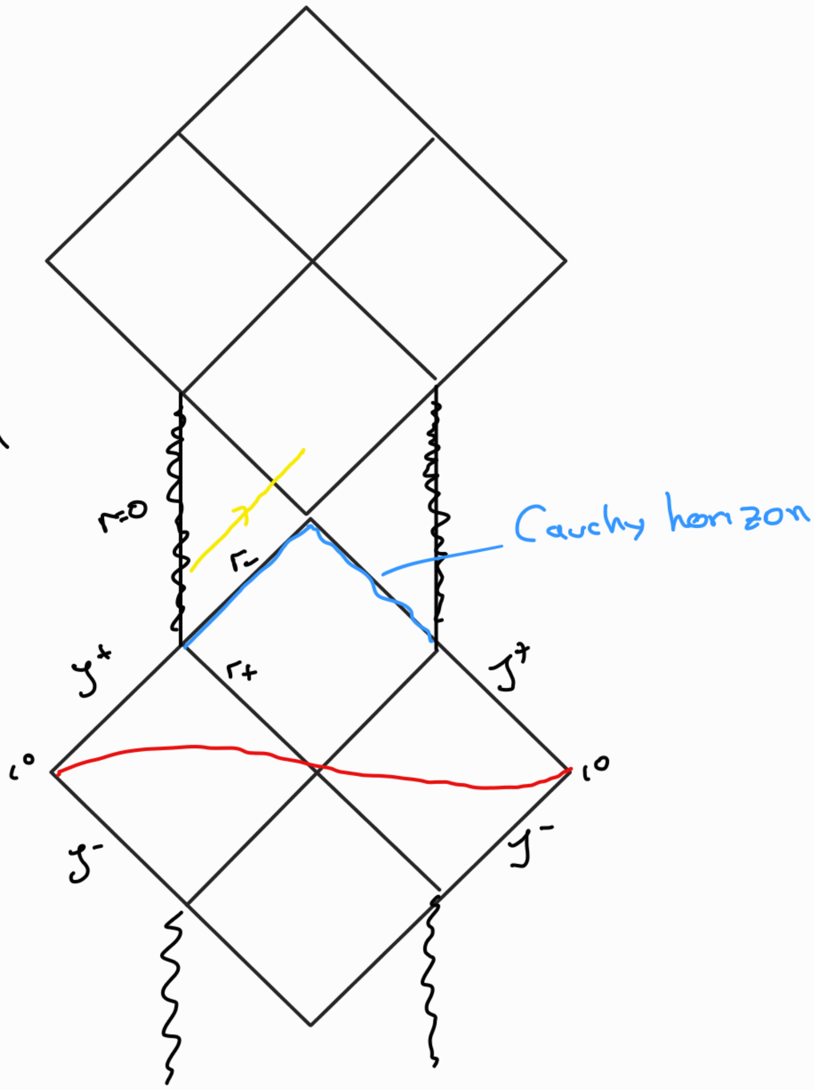
Einstein-Maxwell theory   GR + Electromagnetism

Reissner-Nordström BH — superextremal
— extremal
— sub extremal



<u>Cauchy surface</u>

A closed achronal surface whose domain of dependence is M



Charges

$$Q = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S^2_r} *F \qquad P = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S^2_r} F$$

Kerr

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta} dr^2$$

$$+ \frac{\sin^2\theta}{\rho^2}(a\,dt - (r^2+a^2)\,d\phi)^2 + \rho^2 d\theta^2$$

$$|\partial_t|^2 = -\frac{1}{\rho^2}(\Delta - a^2\sin^2\theta) = 0$$ on stationary limit surface / ergosphere.

Penrose process $\to$ energy extraction.

BH Thermodynamics

0 Surface gravity κ constant on horizon

1st $dM + \frac{1}{8\pi}\kappa\, dA + \Omega_H\, dJ + \Phi_H\, dQ$

2nd $dA \geq 0$

3rd κ cannot be reduced to 0 by a finite # of operations