Problem Sheet 4: Solutions

Question 1. Rotating sphere in Stokes flow.

With $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$, $r = |\boldsymbol{x} - \boldsymbol{x}_0|$ and

$$G_{ij} = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3},$$

the rotational dipole $\boldsymbol{G}^c$ is defined by

$$G^c_{im} := \frac{1}{2}\epsilon_{mlj}\frac{\partial G_{ij}}{\partial x_{0,l}},$$

where

$$\epsilon_{mlj} := \begin{cases} +1 & \text{if} \quad (m,l,j) = (1,2,3) \text{ or } (3,1,2) \text{ or } (2,3,1) \\ -1 & \text{if} \quad (m,l,j) = (1,3,2) \text{ or } (2,1,3) \text{ or } (3,2,1) \\ 0 & \text{if} \quad \text{any of } i,j,k \text{ are equal} \end{cases}$$

Part a. We have

$$\begin{aligned} \frac{1}{2}\epsilon_{mlj}\frac{\partial}{\partial x_{0,l}}G_{ij} &= \frac{1}{2}\epsilon_{mlj}\frac{\partial}{\partial x_{0,l}}\left[\frac{\delta_{ij}}{r} + \frac{\hat{x}_i\hat{x}_j}{r^3}\right], \\ &= \frac{1}{2}\epsilon_{mlj}\left[\delta_{ij}\frac{\partial}{\partial x_{0,l}}\left(\frac{1}{r}\right) - \frac{1}{r^3}\left(\delta_{ij}\hat{x}_j + \delta_{jl}\hat{x}_i\right) + \hat{x}_i\hat{x}_j\frac{\partial}{\partial x_{0,l}}\left(\frac{1}{r^3}\right)\right], \\ &= \epsilon_{mli}\frac{\hat{x}_l}{r^3}. \end{aligned}$$

Part b. Thus $G^c_{im}q_m$ is a solution of Stokes equations for any constant vector $\boldsymbol{q}$ and it decays at spatial infinity. With $\boldsymbol{x}_0$ the centre of the sphere, the sphere is given by $r = a$, where we have

$$a^3 G^c_{im}\Omega_m = a^3\epsilon_{iml}\Omega_m\frac{\hat{x}_l}{r^3} = \epsilon_{iml}\Omega_m\hat{x}_l,$$

which is the velocity on a rotating sphere.

Hence

$$v_i := a^3 G^c_{im}\Omega_m$$

in a solution of the Stokes flow for a rotating sphere with radius a and angular velocity $\boldsymbol{\Omega}$ and, by uniqueness, it is the solution (with constant pressure).

Part c. One can use brute force, though that would be on the long side. The stress field of the Stokes solution

$$u_i = \frac{1}{8\pi\mu}G_{ij}g_j$$

be given by $\sigma_{ij} = T_{ijp} g_p$, with symmetry in the indices i, j. You can show

$$\frac{\partial T_{ijs}}{\partial x_j} = -\delta_{is} \delta(\hat{\boldsymbol{x}}),$$

from the momentum balance for u_i, consistent with the notion that u_i is the flow associated with a point force.

Then the stress field associated with $\mathbf{v} = a^3 \boldsymbol{G}^c \cdot \boldsymbol{\Omega}$, that is

$$v_m = a^3 G^c_{mn} \Omega_n,$$

is given by

$$\Sigma_{ij} = (8\pi\mu) a^3 T^c_{ijp} \Omega_p = (8\pi\mu) \frac{a^3}{2} \epsilon_{plq} \frac{\partial}{\partial x_{0,l}} T_{ijq} \Omega_p = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \epsilon_{plq} \frac{\partial}{\partial x_l} T_{ijq}.$$

Hence the s^{th} component of the moment of the sphere due to the fluid is given by

$$M_s = \int_{Sphere} \epsilon_{sri} x_r \Sigma_{ij} n_j \mathrm{d}S = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{Sphere} \epsilon_{sri} x_r \epsilon_{plq} \left(\frac{\partial}{\partial x_l} T_{ijq} \right) n_j \mathrm{d}S.$$

Using the divergence theorem we have

$$M_s = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{Sphere} \epsilon_{sji} \epsilon_{plq} \left(\frac{\partial}{\partial x_l} T_{ijq} \right) \mathrm{d}V = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{Sphere} \epsilon_{sri} x_r \epsilon_{plq} \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_l} T_{ijq} \right) \mathrm{d}V.$$

The first term is zero as there is an contraction between ε with antisymmetry in i, j and T_{ijq}, with symmetry in i, j. Commuting derivatives and noting the above relation for the derivative of T_{ijk} we have

$$M_s = (8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{Sphere} \epsilon_{sri} x_r \epsilon_{pli} \frac{\partial}{\partial x_l} \left(\delta(\hat{\boldsymbol{x}}) \right) \mathrm{d}V = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{Sphere} \epsilon_{sri} \epsilon_{pri} \delta(\hat{\boldsymbol{x}}) \mathrm{d}V,$$

noting the additional surface integral term to deduce the final equality must be zero as the δ-function has no support on the surface of the sphere. Finally, with $\epsilon_{sri} \epsilon_{pri} = 2\delta_{ps}$, we have

$$M_s = -8\pi\mu a^3 \Omega_s,$$

as required.

Question 2. Ciliary Pumping.

Detailed calculation is not necessary to deduce the expression for U. Fourier modes decouple at the first non-trivial order and each derivative acting on mode number n just induces a factor of n. Thus one can determine the contribution to U from the Fourier mode

$$x_e - x = \epsilon(-b_n \cos(n[x+t]), \qquad y_e = \epsilon c_n \sin(n[x+t]) \tag{1}$$

by identifying

$$a = -nb_n, \quad b = nc_n \tag{2}$$

in the example sheet result

$$U_2 = \frac{1}{2}\left(b^2 + 2ab - a^2\right)$$

to obtain the contribution from this mode.

We then can consider the remaining Fourier modes without detailed calculation. The mode

$$x_e - x = \epsilon a_n \sin(n[x+t]), \qquad y_e = \epsilon(-d_n \cos(n[x+t]))$$

is simply a phase shift of the mode in equation (1). By considering a shifted time coordinate

$$t = \bar{t} + \frac{1}{n}\frac{\pi}{2}$$

we can determine the contribution from this mode by the subsitution

$$d_n \to c_n, \quad a_n \to -b_n.$$

followed by the identification (2). Hence we use

$$a = na_n, \quad b = nd_n$$

in the example sheet result

$$U_2 = \frac{1}{2}\left(b^2 + 2ab - a^2\right)$$

to obtain the contribution from this mode.

Summing all contributions, and noting $U = \epsilon^2 U_2$ to leading order, gives

$$U = \frac{1}{2}\epsilon^2 \sum_{n=1}^{\infty} n^2[c_n^2 + d_n^2 - a_n^2 - b_n^2 + 2(a_n d_n - c_n b_n)].$$

We now determine power optimal strokes, defined as those maximising absolute velocity, subject to the constraint of a fixed power consumption W using Lagrange multipliers with the above leading order expressions. Thus we consider

$$L[\{a_n, b_n, c_n, d_n\}] = U[\{a_n, b_n, c_n, d_n\}] - \lambda(P[\{a_n, b_n, c_n, d_n\}] - W) \tag{3}$$

and the extremal conditions

$$\frac{\partial L}{\partial a_n} = \frac{\partial L}{\partial b_n} = \frac{\partial L}{\partial c_n} = \frac{\partial L}{\partial d_n} = 0. \tag{4}$$

Thus

$$a_n^2 + 2a_n d_n - d_n^2 = 0\ ,\ b_n^2 - 2b_n c_n - c_n^2 = 0.$$

and hence $a_n = (-1 \pm \sqrt{2})d_n$ and $b_n = (1 \pm \sqrt{2})c_n$, which yields

$$U = \epsilon^2 \sum_{n=1}^{\infty} n^2 \left((2 \pm 2\sqrt{2})c_n^2 + (2 \mp 2\sqrt{2})d_n^2 \right) \tag{5}$$

$$P = \epsilon^2 \sum_{n=1}^{\infty} n^3 \left((4 \pm 2\sqrt{2})c_n^2 + (4 \mp 2\sqrt{2})d_n^2 \right). \tag{6}$$

Therefore the optimal stroke is achieved when $a_n = b_n = c_n = d_n = 0$ for $n \geq 2$. Without loss of generality, we can set $b_1 = 0$ as it is just a phase difference, and we finally have the optimal strokes $a_1 = (-1 \pm \sqrt{2})d_1$.

As an extra to the question, note the extremal velocity is

$$U = \mp \frac{W}{\sqrt{2}}.$$

Question 3. Ciliate Motility.

Take a reference frame comoving with the swimmer oriented such that the direction of the swimmer velocity is given by $\mathbf{U} = U\mathbf{e}_z$. The non-dimensional Stokes equations are

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0,$$

with

$$\mathbf{u} = -U\mathbf{e}_z, \quad \text{as} \quad |\mathbf{x}| \to \infty, \qquad \mathbf{u} = \epsilon\beta(t)\sin\theta\mathbf{e}_\theta, \quad \text{on} \quad r = 1,$$

where $\mathbf{e}_\theta$ is the unit vector in the direction of increasing spherical polar θ, where $r = |\mathbf{x}|$ and $z = r\cos\theta$, $x = r\sin\theta\cos\varphi$ for instance.

By symmetry we have $\mathbf{U}$ is in the $\mathbf{e}_z$ direction.

Show that

$$\mathbf{u} = \left[-U(t) + \frac{Q(t)}{r^3}\right]\cos\theta\mathbf{e}_r + \left[U(t) + \frac{P(t)}{r^3}\right]\sin\theta\mathbf{e}_\theta, \qquad p = \text{Const}$$

is a solution of the Stokes equation for $Q(t) = 2P(t)$. To do this, you will need to consider the vector Lapalacian of $\mathbf{u}$. This is non-trivial in non-Cartesian coordinates and you may wish to consider using a symbolic algebra package such as Mathematica:

```
f1[r_, theta_, phi_]:=(-U + Q/r^3) * Cos[theta]

f2[r_, theta_, phi_]:=(U + P/r^3) * Sin[theta]

FullSimplify[Laplacian[f1[r, theta, phi], {r, theta, phi}, "Spherical"] - 2 * f1[r, theta, phi]/r^2-
2/(r^2 * Sin[theta]) * (f2[r, theta, phi] * Cos[theta] + Sin[theta] * D[f2[r, theta, phi], theta])]
```

$$\frac{2(-2P + Q)\text{Cos[theta]}}{r^5}$$

```
FullSimplify[Laplacian[f2[r, theta, phi], {r, theta, phi}, "Spherical"]+
2 * D[f1[r, theta, phi], theta]/r^2 - 1/(r^2 * Sin[theta] * Sin[theta]) * f2[r, theta, phi]]
```

$$\frac{2(2P - Q)\text{Sin[theta]}}{r^5}$$

Given $Q(t) = 2P(t)$, the above two expressions are zero and the governing equation is satisfied. One can then read off that $U(t) = 2\epsilon\beta/3$ from the boundary condition, by finding $Q(t)$ in terms of $U(t)$ from setting the coefficient of $\mathbf{e}_r$ to zero and then equating the coefficient of $\mathbf{e}_\theta$ to the boundary conditions at the boundary.

Question 4. Resistive force theory.

For the force balance with a spherical cell body of radius a, we have

$$\mathbf{0} = (\text{Drag force on body}) + (\text{Drag force on flagellum}). \tag{7}$$

and $\boldsymbol{e}_t = (-1, \epsilon h_s)$, $\boldsymbol{e}_n = (\epsilon h_s, 1)$ and the velocity of the flagellum element is given by $\boldsymbol{U} = (U, V + \epsilon h_t)$.

Hence the drag force per unit length on the element ds is given by

$$\begin{aligned}
\boldsymbol{f} &= -\left[C_N \boldsymbol{e}_n \cdot \boldsymbol{U} \boldsymbol{e}_n + C_T \boldsymbol{e}_t \cdot \boldsymbol{U} \boldsymbol{e}_t\right] = -\left[(C_N - C_T)\boldsymbol{e}_n \cdot \boldsymbol{U}\boldsymbol{e}_n + C_T \boldsymbol{U}\right] \\
&= -[(C_N - C_T)\boldsymbol{e}_n \otimes \boldsymbol{e}_n + C_T \boldsymbol{I}]\boldsymbol{U} \\
&= -\left[(C_N - C_T)\begin{pmatrix} \epsilon^2 h_s^2 & \epsilon h_s \\ \epsilon h_s & 1 \end{pmatrix} + C_T \boldsymbol{I}\right]\begin{pmatrix} U \\ \epsilon h_t + V \end{pmatrix} \\
&= -(C_N - C_T)\begin{pmatrix} \epsilon^2 h_s^2 U + \epsilon^2 h_s h_t + \epsilon h_s V \\ \epsilon h_s U + \epsilon h_t + V \end{pmatrix} - C_T \begin{pmatrix} U \\ \epsilon h_t + V \end{pmatrix}
\end{aligned}$$

Integrating over the flagellum length, $s \in [0, L]$, gives

$$-U\begin{pmatrix} C_T L + \epsilon^2 (C_N - C_T)\int_0^L \mathrm{d}s h_s^2 \\ \epsilon (C_N - C_T)\int_0^L \mathrm{d}s h_s \end{pmatrix} - V\begin{pmatrix} \epsilon (C_N - C_T)\int_0^L \mathrm{d}s h_s \\ C_N L \end{pmatrix} - \begin{pmatrix} \epsilon^2 (C_N - C_T)\int_0^L \mathrm{d}s h_s h_t \\ \epsilon C_N \int_0^L \mathrm{d}s h_t \end{pmatrix}$$

Clearly the term $\epsilon^2 (C_N - C_T)\int_0^L \mathrm{d}s h_s^2$ is a lower order than $C_T L$ and hence the former is dropped.

The cell body drag follows by setting $h = 0$ and replacing parameters dependent on geometry with those of the cell body. Hence the cell body drag is

$$-U\begin{pmatrix} C_T^b L_b \\ 0 \end{pmatrix} - V\begin{pmatrix} 0 \\ C_N^b L_b \end{pmatrix}$$

Thus, using Eqn.(7),

$$(C_T^b L_b + C_T L)U = -(C_N - C_T)\left[\epsilon^2 \int_0^L \mathrm{d}s h_s h_t + \epsilon V \int_0^L \mathrm{d}s h_s\right],$$

and

$$V = -\frac{1}{C_N^b L_b + C_N L}\left[\epsilon C_N \int_0^L \mathrm{d}s h_t + \epsilon U (C_N - C_T)\int_0^L \mathrm{d}s h_s\right].$$

Substituting the expression for V into the expression for U we have

$$(C_T^b L_b + C_T L + O(\epsilon^2))U = -\epsilon^2 (C_N - C_T)\left[\int_0^L \mathrm{d}s h_s h_t - \frac{C_N}{C_N^b L_b + C_N L}\int_0^L \mathrm{d}s h_s \int_0^L \mathrm{d}s h_t\right].$$

We can drop the $O(\epsilon^2)$ on the left as it is asymptotically small relative to $C_T L$.

Hence we have at leading order

$$U = \epsilon^2 \frac{C_T - C_N}{C_T^b L_b + C_T L} \left[\int_0^L \mathrm{d}s h_s h_t - \frac{C_N}{C_N^b L_b + C_N L} \int_0^L \mathrm{d}s h_s \int_0^L \mathrm{d}s h_t \right]$$

and we recover the expression in the lecture notes provided

$$\frac{C_N}{C_N^b L_b + C_N L} \left| \frac{\int_0^L \mathrm{d}s h_s \int_0^L \mathrm{d}s h_t}{\int_0^L \mathrm{d}s h_s h_t} \right| \ll 1.$$

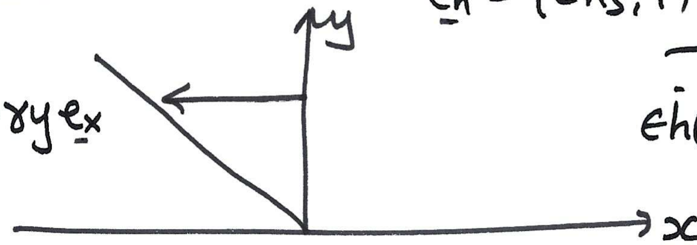
Question 5

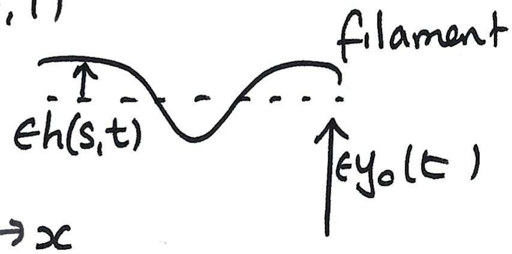
$\underline{e}_t = (-1, \epsilon h_s)$

$\underline{e}_n = (\epsilon h_s, 1)$

displacement y_0 is O(ε) on assumption there is no drift

$\underline{v} = -\gamma y \underline{e}_x$





$\therefore$ On filament, $\quad \underline{v} = -\gamma\epsilon(y_0 + h)\underline{e}_x \, , \quad \underline{u} = (u, \epsilon y_{0t} + \epsilon h_t)$.

$$\underline{u} - \underline{v} = \begin{pmatrix} u - \gamma\epsilon(y_0 + h) \\ \epsilon y_{0t} + \epsilon h_t \end{pmatrix}$$

$$\boxed{\epsilon y_{0t} \equiv V \text{ in Q4}}$$

$\underline{0}$ = Drag force $\qquad \underline{f}$ = drag per unit length

$$\underline{f} = -(C_N - C_T)\, \underline{e}_n \cdot (\underline{u} - \underline{v})\, \underline{e}_n - C_T(\underline{u} - \underline{v})$$

$$= \underbrace{f^{Q4}}_{} + (C_N - C_T)\underbrace{(\underline{e}_n \cdot \underline{v})\, \underline{e}_n}_{-\gamma\epsilon^2 h_s(y_0+h)\begin{pmatrix}\epsilon h_s \\ 1\end{pmatrix}} + \underbrace{C_T \underline{v}}_{-C_T\gamma\epsilon\begin{pmatrix}y_0+h \\ 0\end{pmatrix}}$$

same expression as in Q4 with $L_b \equiv 0$ as no cell body and $V \to \epsilon y_{0t}$

$\int_0^L (y_0 + h)\, ds$

From zero net drag

$$\underline{0} = -u\begin{pmatrix} C_T L + O(\epsilon^2) \\ \epsilon(C_N - C_T)\int_0^L ds\, h_s \end{pmatrix} - \begin{pmatrix} \epsilon(C_N - C_T)\int_0^L ds\, h_s \\ C_N L \end{pmatrix}\epsilon y_{0t}$$

$$- \begin{pmatrix} \epsilon^2 (C_N - C_T)\int_0^L ds\, h_s h_t \\ \epsilon C_N \int_0^L ds\, h_t \end{pmatrix} - \begin{pmatrix} O(\epsilon^3) \quad + \gamma\epsilon C_T \int_0^L (y_0 + h)\, ds \\ \gamma\epsilon^2 (C_N - C_T)\int_0^L ds\, h_s(y_0 + h) \end{pmatrix}$$

$$\therefore\ (C_T L + O(\epsilon^2))\,U = -\epsilon(C_N - C_T)(\epsilon y_{0t})\int_0^L ds\, h_s + O(\epsilon^3)$$
$$-\epsilon^2(C_N - C_T)\int_0^L ds\, h_s h_t + \gamma\epsilon C_T \underbrace{\int_0^L (y_0 + h)\,ds}_{\left(L y_0 + \int_0^L h\,ds\right)}$$

$$(C_N L)(\epsilon y_{0t}) = -\epsilon U (C_N - C_T)\int_0^L ds\, h_s - \epsilon C_N \int_0^L ds\, h_t$$
$$-\gamma\epsilon^2 (C_N - C_T)\int_0^L ds\, h_s (y_0 + h)$$

<u>Leading order</u>

from ϵy_{0t}

$$(C_T L + O(\epsilon^2))\,U = -\epsilon\{O(\epsilon)\} + O(\epsilon^3) + O(\epsilon^2)$$
$$+ \gamma\epsilon C_T L y_0 + \gamma\epsilon C_T \int_0^L h\,ds$$

$$\epsilon y_0 = \int_0^t d\bar{t}\, \epsilon y_{0\bar{t}} = -\frac{\epsilon}{C_N L}\left[(C_N - C_T)\int_0^t d\bar{t}\int_0^L ds\, h_s + C_N \int_0^t d\bar{t}\int_0^L ds\, h_t\right] + O(\epsilon^2)$$

assuming filament starts at $y_0 \equiv 0$

Aside. No drift entails *h(s,t)* is such that these integrals remain O(1) for large time

$$\therefore\ U = -\frac{\gamma\epsilon}{C_N L}\left[(C_N - C_T)\int_0^t d\bar{t}\int_0^L ds\, h_s + C_N\int_0^t d\bar{t}\int_0^L ds\, h_t\right]$$
$$+ \gamma\epsilon / L \int_0^L h\,ds + O(\epsilon^2)$$