# C8.7 Optimal Control

## Sheet 4 — HT25

## Section A

1. Consider the control problem with one-dimensional state dynamics given by

$$\mathrm{d}X_t = \frac{1}{3}U_t \mathrm{d}t + \mathrm{d}W_t,$$

where $W$ is a one-dimensional Brownian motion, $U_t$ is a control process taking values in the set $[0,1]$, the horizon is $T = 1$ and the costs are given by

$$g(t,x,u) = \frac{2}{3}ux(1-t) - t + 3x + 1, \qquad \Phi(x) = -x^3.$$

Show that the value function is given by $v(t,x) = -x^2(x+1-t)$, and describe the optimal control.

**Solution:** We can compute the derivatives

$$\partial_t v = x^2; \qquad D_x v = x(2t-3x-2); \qquad D^2_{xx} v = 2(t-3x-1)$$

The Hamiltonian is given by

$$\begin{aligned} H(t,x,D_x v, D^2_{xx} v) &= \min_{u\in[0,1]} \Big\{ g(t,x,u) + u D_x v + \frac{1}{2} D^2_{xx} v \Big\} \\ &= \min_{u\in[0,1]} \Big\{ \frac{2}{3}ux(1-t) - t + 3x + 1 \\ &\qquad + \frac{1}{3}u\Big(x(2t-3x-2)\Big) + \frac{1}{2}\Big(2(t-3x-1)\Big)\Big\} \\ &= \min_{u\in[0,1]} \Big\{ -ux^2 \Big\}. \end{aligned}$$

As $x^2 > 0$, the optimizer is obtained by always taking $u = 1$, and so

$$H(t,x,D_x v, D^2_{xx} v) = -x^2 = -\partial_t v(t,x)$$

so the HJB equation is satisfied. It's easy to check that $v(1,x) = -x^3 = \Phi(x)$, so by the verification theorem, $v$ must be the value function for our control problem, and the optimal strategy is given by taking $u = 1$.


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2. Consider the control problem with state dynamics

$$\mathrm{d}X_t = (a + bU_t)\mathrm{d}t + (c + fU_t)\mathrm{d}W_t$$

where $W$ is a one-dimensional Brownian motion, $a, b \in \mathbb{R}$ and $\mathcal{U} = \mathbb{R}$. Suppose that the costs are given by

$$g(t, x, u) = qx^2 + ru^2 + 2sxu; \qquad \Phi(x) = x^2$$

with $r > 0$, and $q, s \in \mathbb{R}$, and $\Sigma_T > 0$.

Show that, provided $t$ is close enough to $T$, the solution is given by a quadratic $v(t, x) = \Sigma_t x^2 + 2\Psi_t x + \Gamma_t$, and find the ODEs satisfied by $\Sigma_t, \Psi_t, \Gamma_t$, together with the optimal control.

**Solution:** Guessing that the solution is of the stated form, we have the spatial derivatives

$$D_x v = 2\Sigma_t x + 2\Psi; \qquad D^2_{xx} v = 2\Sigma_t.$$

Substituting in the Hamiltonian, we have

$$H(t, x, D_x v, D_{xx} v) = \inf_u \Big\{ qx^2 + ru^2 + 2sxu + (2\Sigma_t x + 2\Psi_t)(a + bu) + \frac{(c + fu)^2}{2}(2\Sigma_t) \Big\}.$$

Taking a first order condition (which gives a minimizer, assuming $r + f^2\Sigma_t > 0$ which holds when $t$ is close enough to $T$) we have

$$0 = 2ru + 2sx + b(2\Sigma_t x + 2\Psi_t) + 2f(c + fu)\Sigma_t$$

which simplifies to give the optimal control,

$$u = -\frac{1}{r + f^2\Sigma_t}\Big((s + b\Sigma_t)x + b\Psi_t + cf\Sigma_t\Big) = K_t x + H_t,$$

where $K_t = -\frac{s+b\Sigma_t}{r+f^2\Sigma_t}$ and $H_t = -\frac{b\Psi_t + cf\Sigma_t}{r+f^2\Sigma_t}$. We substitute this back into the HJB equation to get

$$\begin{aligned}
&- \partial_t(\Sigma_t x^2 + 2\Psi_t x + \Gamma_t) \\
&= H(t, x, D_x v, D_{xx} v) \\
&= qx^2 + r(K_t x + H_t)^2 + 2sx(K_t x + H_t) + (2\Sigma_t x + 2\Psi_t)(a + b(K_t x + H_t)) \\
&\quad + (c + f(K_t x + H_t))^2(\Sigma_t).
\end{aligned}$$

Matching coefficients, we get $\Sigma_T = 1, \Psi_T = \Gamma_T = 0$, and

$$\begin{aligned}
-\partial_t\Sigma_t &= q + rK_t^2 + 2sK_t + 2b\Sigma_t K_t + \Sigma_t f^2 K_t^2, \\
-\partial_t\Psi_t &= rK_t H_t + sH_t + \Sigma_t(a + bH_t) + \Psi_t bK_t + \Sigma_t f K_t(fH_t + c), \\
-\partial_t\Gamma_t &= rH_t^2 + 2\Psi_t(a + bH_t) + (c + fH_t)^2\Sigma_t.
\end{aligned}$$


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3. Consider a controlled diffusion with drift $f$ and volatility $\sigma$. For a given feedback control $U$, we define the linear differential operator

$$\mathcal{L}^U v = g(t,x,U_t) + f(t,x,U_t)^\top (D_x v) + \frac{1}{2}\mathrm{Tr}\Big[(D^2_{xx} v)(\sigma\sigma^\top)(t,x,U_t)\Big]$$

(a) Assuming all relevant equations admit sufficiently smooth solutions, and all stochastic integrals with respect to martingales are martingales, show that the value of the control $J(\cdot,\cdot,U)$ satisfies the PDE

$$-\partial_t J = \mathcal{L}^U J.$$

(b) The comparison principle states that if $w$ and $w'$ satisfy

$$-\partial_t w \geq \mathcal{L}^U w; \qquad -\partial_t w' \leq \mathcal{L}^U w'$$

and $w(T,\cdot) \geq w'(T,\cdot)$, then $w \geq w'$ for all $(t,x)$.

Design a policy iteration scheme to solve the Hamilton–Jacobi–Bellman equation, assuming you are able to solve linear PDEs. For your scheme, assuming the comparison theorem holds, prove the policy improvement lemma.

**Solution:**

(a) Let $w$ be the solution to the PDE. We apply Itô's lemma to calculate the dynamics of $w(t, X_t^{t,x,U})$, which gives (evaluating all terms inside the integral at $(s, X_s^{t,x,U}, U)$)

$$\begin{aligned}
&\Phi(X_T^U) - w(t,x)\\
&= \int_t^T \Big((\partial_t w)\mathrm{d}s + (\partial_x w)^\top \mathrm{d}X_s^{t,x,U} + \frac{1}{2}\mathrm{Tr}\Big[D^2_{xx} w \mathrm{d}\langle X^{t,x,U}\rangle_s\Big]\Big)\\
&= \int_t^T \Big((-\mathcal{L}^U w)\mathrm{d}s + (\partial_x w)^\top f(s, X_s^{t,x,U}, U)\mathrm{d}s + (\partial_x w)^\top \sigma(s, X_s^{t,x,U}, U)\mathrm{d}W_s\Big)\\
&\qquad + \frac{1}{2}\mathrm{Tr}\Big[D^2_{xx} w (\sigma\sigma^\top)(t, X_s^{t,x,U}, U)\Big]\mathrm{d}s\\
&= -\int_t^T g(s, X_s^{t,x,U}, U_s)\mathrm{d}s + \int_t^T (\partial_x w)^\top \sigma(s, X_s^{t,x,U}, U)\mathrm{d}W_s.
\end{aligned}$$

Rearranging and taking an expectation, we have

$$w(t,x) = \mathbb{E}\Big[\int_t^T g(s, X_s^{t,x,U}, U_s)\mathrm{d}s + \Phi(X_T^{t,x,U})\Big|\mathcal{F}_t\Big]$$

so $w(t,x) = J(t,x,U)$.

(b) A simple policy iteration scheme would take $v_n(t,x) = J(t,x,U_n)$ for the evaluation step, and $U_{n+1} \in \arg\min\{\mathcal{L}^U v_n\}$ for the improvement step. Using this definition, we clearly have $v_n(T,x) = \Phi(x)$ for all $n$, and also

$$-\partial_t v_n = \mathcal{L}^{U_n} v_n \geq \mathcal{L}^{U_{n+1}} v_n.$$


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However, this implies that

$$-\partial_t v_n \geq \mathcal{L}^{U_{n+1}} v_n; \qquad -\partial_t v_{n+1} = \mathcal{L}^{U_{n+1}} v_{n+1}$$

so the comparison theorem proves that $v_n \geq v_{n+1}$, which implies we have the policy improvement lemma.


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## Section B

4. Consider the Merton problem, where an investor has nonnegative wealth $X$, invests a fraction $U^s$ of their wealth into a risky asset following a geometric Brownian motion, and consumes a fraction $U^c$ of their wealth per unit time. This means that their wealth satisfies

$$\mathrm{d}X_t = -U^c X_t \mathrm{d}t + U^s X_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$$

for $\mu, \sigma > 0$, and costs

$$g(t, x, u) = -\frac{(u^c x)^{1-\gamma}}{1-\gamma}; \qquad \Phi(x) = -\frac{x^{1-\gamma}}{1-\gamma}.$$

where $u^c, u^s > 0$ and $\gamma > 0$, $\gamma \neq 1$. Show that the optimal strategy is to invest a constant proportion of wealth $U^s$ in the risky asset, and consume at a deterministic time-dependent rate $U^c$, determined by the solution to an ODE.

*You may wish to use an ansatz of the form $v(t, x) = -w(t)\frac{x^{1-\gamma}}{1-\gamma}$, for some positive function $w$.*

**Solution:** We take the suggested ansatz $v(t, x) = -w(t)\frac{x^{1-\gamma}}{1-\gamma}$. Then our derivatives are

$$\partial_t v = \frac{w'(t)}{w(t)} v; \qquad D_x v = -w(t)x^{-\gamma}; \qquad D^2_{xx} v = \gamma w(t) x^{-1-\gamma}.$$

The Hamiltonian is

$$\begin{aligned} H(...) &= \inf_{u^c, u^s} \Big\{ -\frac{(u^c x)^{1-\gamma}}{1-\gamma} + (-u^c x + \mu u^s x)(-w(t)x^{-\gamma}) + \frac{1}{2}(u^s x \sigma)^2 (\gamma w(t) x^{-1-\gamma}) \Big\} \\ &= \inf_{u^c, u^s} \Big\{ -\frac{(u^c)^{1-\gamma}}{1-\gamma} + (u^c - \mu u^s) w(t) + \frac{\gamma}{2}(u^s \sigma)^2 w(t) \Big\} x^{1-\gamma} \end{aligned}$$

Taking a first order condition gives the pair of equations

$$\begin{aligned} 0 &= -\mu w(t) + \gamma u^s \sigma^2 w(t) \\ 0 &= -(u^c)^{-\gamma} + w(t) \end{aligned}$$

from which we obtain the optimal controls

$$U^s_t = \frac{\mu}{\gamma \sigma^2}; \qquad U^c_t = w(t)^{-1/\gamma}$$

Observe that $u^s$ is constant, and $u^c$ is independent of $X_t$. Substituting into the Hamiltonian, we have

$$H(...) = \Big( -\frac{(w(t))^{-(1-\gamma)/\gamma - 1}}{1-\gamma} + w(t)^{-1/\gamma} - \frac{\mu^2}{2\gamma\sigma^2} \Big) w(t) x^{1-\gamma}$$


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and so the HJB equation simplifies to

$$-\frac{w'(t)}{w(t)}v = \Big(-\frac{w(t)^{-1/\gamma}}{1-\gamma} + w(t)^{-1/\gamma} - \frac{\mu^2}{2\gamma\sigma^2}\Big)(\gamma-1)v$$

which in turn simplifies to the (locally Lipschitz) ODE for $w$

$$-\frac{w'(t)}{w(t)} = \gamma w(t)^{-1/\gamma} + \frac{1-\gamma}{\gamma}\frac{\mu^2}{2\sigma^2}$$

with terminal value $w(T) = 1$. Solving this ODE numerically, we find the optimal consumption rate.


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5. In this question, we consider finite difference approximations of the Hamilton–Jacobi–Bellman equation. We will consider the problem where the state variable is in one dimension, and the Hamiltonian is given by

$$H(t,x,q,a) = \inf_{u\in[-1,1]} \Big\{uq + a\Big\} = \inf_{u\in[-1,1]} \tilde{H}(t,x,q,a,u).$$

For $N \in \mathbb{N}$, consider a discrete grid $\{0, \delta_t, 2\delta_t, ..., N\delta_t\}$ in time, and

$$\{-N\delta_x, (-N+1)\delta_x, ..., 0, ..., (N-1)\delta_x, N\delta_x\}$$

in space, where $\delta_t = T/N$ and $\delta_x = 1/\sqrt{N}$.

We assume the problem stops when we hit the numerical boundary (so $v(t,x) = \Phi(x)$ for all $t$ and all $x \in \{\pm N\delta_x\}$). We assign a terminal value $\Phi(x) = 1 - \frac{x^4}{1+x^4}$.

(a) If $v$ is a $C^{1,2}$ function, show that (as $N \to \infty$), with $k$ chosen such that $t \in [(k-1)\delta t, k\delta_t]$,

$$\frac{v(k\delta_t, x) - v((k-1)\delta_t, x)}{\delta_t} = \partial_t v(t,x) + o(1)$$

and, with $k$ chosen such that $x \in [(k-1)\delta_x, (k+1)\delta_x]$,

$$\frac{v(t,(k+1)\delta_x) - v(t,k\delta_x)}{\delta_x} = D_x v(t,k\delta_x) + o(1)$$

and

$$\frac{v(t,(k+1)\delta_x) - 2v(t,k\delta_x) + v(t,(k-1)\delta_x)}{\delta_x^2} = D^2_{xx} v(t,k\delta_x) + o(1).$$

(b) Using this finite difference scheme, write down an approximation of

$$v_{k-1,j} := v((k-1)\delta_t, j\delta_x)$$

in terms of $v_{k,j}, v_{k,j+1}, v_{k,j-1}$, when $v$ satisfies the HJB equation.

(c) Show that $v_{k-1,j}$ is a componentwise monotone increasing function of $v_{k,\bullet}$ (where $v_{k,\bullet}$ represents the vector $[v_{k,j}]$ for $j = -N, -N+1, ..., N$), provided $3\delta_t/\delta_x^2 < 1$. (This is known as being a *monotone scheme.*) How does this compare with the discrete-time discrete-state control problem?

(d) Now consider the deterministic problem with Hamiltonian

$$H(t,x,q) = \inf_{u\in[-1,1]} \Big\{uq\Big\}.$$

Is the basic finite difference numerical scheme for this problem typically monotone?


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(e) For the deterministic problem in the previous part, consider the modified scheme

$$\hat{H}(t,x,v_{k,\cdot}) = \begin{cases} \frac{v_{k,j+1}-v_{k,j}}{\delta_x} & \text{if } v_{k,j+1} < v_{k,j}, \\ -\frac{v_{k,j}-v_{k,j-1}}{\delta_x} & \text{if } v_{k,j+1} \geq v_{k,j} \end{cases}$$

Show that using $\hat{H}(v)$ in the place of $H(t,x,q)$ in our numerical approximation gives a monotone scheme, provided $\delta_t/\delta_x < 1$. (Approximations of this type are commonly known as upwind schemes, and are important in order to prove numerical stability).

(f) Implement the three numerical schemes considered above, for various choices of $T$ and $N$, and observe the behaviour of the numerical approximations.

**Solution:**

(a) This is essentially just the definition of the derivative and second derivative.

(b) Expanding the HJB equation with our approximation, we have

$$\begin{aligned} -\frac{v_{k,j}-v_{k-1,j}}{\delta_t} &= H\Big(t,x,\frac{v_{k,j+1}-v_{k,j}}{\delta_x},\frac{v_{k,j+1}-2v_{k,j}+v_{k,j-1}}{\delta_x^2}\Big) \\ &= -\Big|\frac{v_{k,j+1}-v_{k,j}}{\delta_x}\Big| + \frac{v_{k,j+1}-2v_{k,j}+v_{k,j-1}}{\delta_x^2} \end{aligned}$$

which rearranges to

$$v_{k-1,j} = v_{k,j} + \Big(-\Big|\frac{v_{k,j+1}-v_{k,j}}{\delta_x}\Big| + \frac{v_{k,j+1}-2v_{k,j}+v_{k,j-1}}{\delta_x^2}\Big)\delta_t$$

(c) The right hand side can be written as the minimum of

$$v_{k,j} + \Big(\frac{v_{k,j+1}-v_{k,j}}{\delta_x} + \frac{v_{k,j+1}-2v_{k,j}+v_{k,j-1}}{\delta_x^2}\Big)\delta_t$$

and

$$v_{k,j} + \Big(-\frac{v_{k,j+1}-v_{k,j}}{\delta_x} + \frac{v_{k,j+1}-2v_{k,j}+v_{k,j-1}}{\delta_x^2}\Big)\delta_t.$$

The first of these is clearly monotone increasing in $v_{k,j+1}$ and $v_{k,j-1}$, and is monotone increasing in $v_{k,j}$ provided $1-\frac{\delta_t}{\delta_x}-2\frac{\delta_t}{\delta_x^2} > 0$. The second equation is always monotone increasing in $v_{k,j-1}$, is monotone increasing in $v_{k,j+1}$ provided $\delta_x < 1$ (which we know), and is monotone increasing in $v_{k,j}$ provided $1+\frac{\delta_t}{\delta_x}-2\frac{\delta_t}{\delta_x^2} > 0$. Both of these are guaranteed provided $3\delta_t/\delta_x^2 < 1$.

(d) We now have

$$v_{k-1,j} = v_{k,j} + \Big(-\Big|\frac{v_{k,j+1}-v_{k,j}}{\delta_x}\Big|\Big)\delta_t$$

Taking $v_{k,j+1} \to \pm\infty$ we have $v_{k-1,j} \to -\infty$, so this scheme is not monotone.


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(e) Observe that $\hat{H}$ is indeed an approximation of $H(t, x, q)$ (with a different choice of derivative approximator), and $v_{k-1,j}$ is one of

$$v_{k,j} + \Big(\frac{v_{k,j+1} - v_{k,j}}{\delta_x}\Big)\delta_t$$

and

$$v_{k,j} + \Big(-\frac{v_{k,j} - v_{k,j-1}}{\delta_x}\Big)\delta_t$$

These are both monotone functions, provided $\delta_t/\delta_x < 1$

(f) See `PS4_Upwind.ipynb`


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