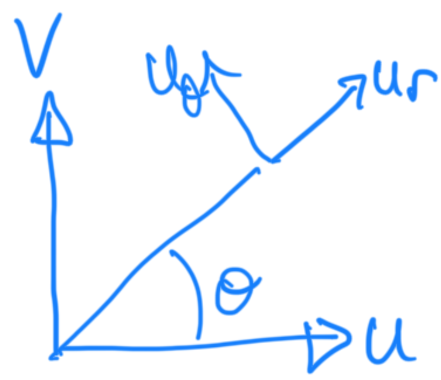
Working out stress tensor components in radial polars

Method ① : Start with Cartesian version & change variables.

In Cartesian coordinates,

$$\sigma_{11} = -p + 2\mu \frac{\partial u}{\partial x}$$
$$\sigma_{12} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\sigma_{22} = -p + 2\mu \frac{\partial v}{\partial y}$$



need to change $(x,y) \to (r,\theta)$ and $(u,v) \to (u_r, u_\theta)$

then work out $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta}$.

(i) For $(x,y) \to (r,\theta)$, using $r = \sqrt{x^2+y^2}$, $\theta = \operatorname{Arctan}\frac{y}{x}$,

we calculate

$$r_x = \cos\theta, \quad r_y = \sin\theta, \quad \theta_y = \frac{\cos\theta}{r}, \quad \theta_x = -\frac{\sin\theta}{r}$$

So that

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}, \quad \frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial v}{\partial x} = \cos\theta \frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial y} = \sin\theta \frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}$$

(ii) Resolving the vectors, we also have

$$u = u_r \cos\theta - u_\theta \sin\theta$$
$$v = u_r \sin\theta + u_\theta \cos\theta$$

(iii) Calculate σ_{11} in terms of u_r, u_θ, r, θ.

$$\sigma_{11} = -p + 2\mu\frac{\partial u}{\partial x} = -p + 2\mu\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= -P + 2\mu\left(\cos\theta \frac{\partial}{\partial r}(u_r\cos\theta - u_\theta\sin\theta) - \sin\theta\frac{\partial}{\partial\theta}(u_r\cos\theta - u_\theta\sin\theta)\right)$$

$$\Rightarrow \boxed{\sigma_{11} = -P + 2\mu\left(\cos^2\theta\frac{\partial u_r}{\partial r} + \sin\theta\cos\theta\left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) + \sin^2\theta\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\right)}$$

(iv) Calculate σ_{12} in terms of u_r, u_θ, r, θ

$$\sigma_{12} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} + \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \mu\left(\sin\theta\frac{\partial}{\partial r}(u_r\cos\theta - u_\theta\sin\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(u_r\cos\theta - u_\theta\sin\theta) + \cos\theta\frac{\partial}{\partial r}(u_r\sin\theta + u_\theta\cos\theta) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(u_r\sin\theta + u_\theta\cos\theta)\right)$$

$$\boxed{\sigma_{12} = \mu\left(2\sin\theta\cos\theta\left(\frac{\partial u_r}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r}\right) + (\cos^2\theta - \sin^2\theta)\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\right)}$$

(v) $$\sigma_{22} = -P + 2\mu\frac{\partial v}{\partial y} = -P + 2\mu\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= -P + 2\mu\left(\sin\theta\frac{\partial}{\partial r}(u_r\sin\theta + u_\theta\cos\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(u_r\sin\theta + u_\theta\cos\theta)\right)$$

$$\boxed{\sigma_{22} = -P + 2\mu\left(\sin^2\theta\frac{\partial u_r}{\partial r} + \cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) + \cos^2\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)\right)}$$

(vi) Calculate

$$\underline{t}(\underline{e_r}) = \begin{pmatrix} \sigma_{11} n_1 + \sigma_{12} n_2 \\ \sigma_{21} n_1 + \sigma_{22} n_2 \end{pmatrix} \qquad \underline{e_r} = \underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\, \sigma_{11} + \sin\theta\, \sigma_{12} \\ \cos\theta\, \sigma_{21} + \sin\theta\, \sigma_{22} \end{pmatrix}$$

(vii) $\sigma_{rr} = \underline{e_r} \cdot \underline{t}(\underline{e_r})$

$$= \cos^2\theta\, \sigma_{11} + 2\sin\theta\cos\theta\, \sigma_{12} + \sin^2\theta\, \sigma_{22}$$

$$= -P\cos^2\theta + 2\mu\left(\cos^4\theta \frac{\partial u_r}{\partial r} + \sin\theta\cos^3\theta\left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) + \sin^2\theta\cos^2\theta \frac{u_r}{r} + \frac{\cos^2\theta\sin^2\theta}{r}\frac{\partial u_\theta}{\partial\theta} \right)$$

$$+ 2\mu\left(2\sin^2\theta\cos^2\theta\left(\frac{\partial u_r}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r}\right) + \sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)\right)$$

$$-P\sin^2\theta + 2\mu\left(\sin^4\theta \frac{\partial u_r}{\partial r} + \cos\theta\sin^3\theta\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) + \cos^2\theta\sin^2\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right)\right)$$

$$= -P + 2\mu\left[\frac{\partial u_r}{\partial r}\left(\underbrace{\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta}_{=(\cos^2\theta + \sin^2\theta)^2 = 1} \right) \right.$$

$$+ \frac{\partial u_\theta}{\partial r}\left(\cancel{-\sin\theta\cos^3\theta} + \cancel{\cos\theta\sin^3\theta} + \sin\theta\cos\theta(\cancel{\cos^2\theta - \sin^2\theta}) \right)$$

$$\left. + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\left(\cancel{-\sin\theta\cos^3\theta} + \cancel{\cos\theta\sin^3\theta} + \sin\theta\cos\theta(\cancel{\cos^2\theta - \sin^2\theta}) \right) \right.$$

$$+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\left(\cancel{-2\sin^2\theta\cos^2\theta+\cos^2\theta\sin^2\theta} \\ \cancel{+\cos^2\theta\sin^2\theta}\right)$$

$$+\frac{1}{r}u_r\left(\cancel{\sin^2\theta\cos^2\theta-2\sin^2\theta\cos^2\theta} \\ \cancel{+\cos^2\theta\sin^2\theta}\right)$$

$$+\frac{1}{r}u_\theta\left(\cancel{\sin\theta\cos^3\theta}-\cancel{\sin^3\theta\cos\theta} \\ -\sin\theta\cos\theta\left(\cancel{\cos^2\theta-\sin^2\theta}\right)\right)$$

$$\therefore \quad \boxed{\sigma_{rr}=-P+2\mu\frac{\partial u_r}{\partial r}}$$

(viii) $\sigma_{r\theta}(=\sigma_{\theta r})=\underline{e_\theta}\cdot\underline{t}(e_r)$

$$=\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\,\sigma_{11}+\sin\theta\,\sigma_{12}\\ \cos\theta\,\sigma_{21}+\sin\theta\,\sigma_{22}\end{pmatrix}$$

$$=-\sin\theta\cos\theta\,\sigma_{11}+(\cos^2\theta-\sin^2\theta)\sigma_{12} \\ +\sin\theta\cos\theta\,\sigma_{22}$$

$$=-\sin\theta\cos\theta\Bigg(\cancel{-P}+2\mu\Bigg(\cos^2\theta\frac{\partial u_r}{\partial r}+\sin\theta\cos\theta\left(\frac{u_\theta}{r}-\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)+\sin^2\theta\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)\Bigg)\Bigg)$$

$$+\mu(\cos^2\theta-\sin^2\theta)\Bigg(2\sin\theta\cos\theta\left(\frac{\partial u_r}{\partial r}-\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}-\frac{u_r}{r}\right) \\ +(\cos^2\theta-\sin^2\theta)\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)\Bigg)$$

$$+\sin\theta\cos\theta\Bigg(\cancel{-P}+2\mu\Bigg(\sin^2\theta\frac{\partial u_r}{\partial r}+\cos^2\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right) \\ +\cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\Bigg)\Bigg)$$

$$=\mu\Bigg(\left(-2\cancel{\sin\theta\cos^3\theta}+2\left(\cancel{\cos^2\theta-\sin^2\theta}\right)\sin\theta\cos\theta \\ \cancel{+2\sin^3\theta\cos\theta}\right)\frac{\partial u_r}{\partial r}$$

$$+\left(\underline{4\sin^2\theta\cos^2\theta+(\cos^2\theta-\sin^2\theta)^2}\right)\frac{\partial u_\theta}{\partial r}$$

$$=(\cos^2\theta+\sin^2\theta)^2=1$$

$$+\left(4\sin^2\theta\cos^2\theta+\underbrace{(\cos^2\theta-\sin^2\theta)^2}_{=1}\right)\frac{1}{r}\frac{\partial u_r}{\partial\theta}$$

$$+\left(\cancel{-2\sin^3\theta\cos\theta}+\cancel{2\sin\theta\cos^3\theta}-2\sin\theta\cos\theta\left(\cancel{\cos^2\theta}-\cancel{\sin^2\theta}\right)\right)\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$+\left(\cancel{-2\sin^3\theta\cos\theta}+\cancel{2\sin\theta\cos^3\theta}-2\sin\theta\cos\theta\left(\cancel{\cos^2\theta}-\cancel{\sin^2\theta}\right)\right)\frac{u_r}{r}$$

$$+\left(-4\sin^2\theta\cos^2\theta-\underbrace{(\cos^2\theta-\sin^2\theta)^2}_{=-1}\right)\frac{u_\theta}{r}\Bigg)$$

$$\therefore\ \boxed{\sigma_{r\theta}=\mu\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)}$$

(xi)

$$\sigma_{\theta\theta}=\underline{e}_\theta\cdot\underline{t}(\underline{e}_\theta)$$

$$=\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\cdot\begin{pmatrix}-\sigma_{11}\sin\theta+\sigma_{12}\cos\theta\\ -\sigma_{21}\sin\theta+\sigma_{22}\cos\theta\end{pmatrix}$$

$$=\sigma_{11}\sin^2\theta-2\sigma_{12}\cos\theta\sin\theta+\sigma_{22}\cos^2\theta$$

$$=-p\sin^2\theta+2\mu\sin^2\theta\left(\cos^2\theta\frac{\partial u_r}{\partial r}+\sin^2\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\sin\theta\cos\theta\left(\frac{u_\theta}{r}-\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\right)$$

$$-2\mu\cos\theta\sin\theta\left(2\sin\theta\cos\theta\left(\frac{\partial u_r}{\partial r}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{r}\right)+(\cos^2\theta-\sin^2\theta)\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right)\right)$$

$$-p\cos^2\theta+2\mu\cos^2\theta\left(\sin^2\theta\frac{\partial u_r}{\partial r}+\cos^2\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\right)$$

$$
\begin{aligned}
= -p + \mu \Big\{ & \left(\cancel{2\sin^2\theta\cos^2\theta} - \cancel{4\sin^2\theta\cos^2\theta} + \cancel{2\sin^2\theta\cos^2\theta} \right) \frac{\partial u_r}{\partial r} \\
& + \left(\cancel{-2\sin^3\theta\cos\theta} + \cancel{2\cos^3\theta\sin\theta} - 2\cos\theta\sin\theta\,(\cancel{\cos^2\theta - \sin^2\theta}) \right) \frac{\partial u_\theta}{\partial r} \\
& + \left(-2\cancel{\sin^3\theta\cos\theta} + 2\cancel{\cos^2\theta\sin\theta} - 2\cos\theta\sin\theta\,(\cancel{\cos^2\theta - \sin^2\theta}) \right) \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\
& + \left(2\underbrace{\sin^4\theta + 4\cos^2\theta\sin^2\theta + 2\cos^4\theta}_{=2(\cos^2\theta + \sin^2\theta)^2 = 2} \right) \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\
& + \left(\underbrace{2\sin^4\theta + 4\cos^2\theta\sin^2\theta + 2\cos^4\theta}_{=2} \right) \frac{u_r}{r} \\
& + \left(2\cancel{\sin^3\theta\cos\theta} - 2\cancel{\cos^3\theta}\sin\theta + 2\cos\theta\sin\theta\,(\cancel{\cos^2\theta - \sin^2\theta}) \right) \frac{u_\theta}{r} \Big\}
\end{aligned}
$$

$$\therefore \quad \boxed{\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)}$$

Method 2: use dyadic product $\otimes$

$$\underline{a} \otimes \underline{b} = \underline{a}\,\underline{b}^T = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

$$\nabla \underline{f} := \nabla_i (f_j \underline{e}_j) \otimes \underline{e}_i$$

$$= \nabla_i (f_j)\, \underline{e}_j \otimes \underline{e}_i + f_j \nabla(\underline{e}_j) \otimes \underline{e}_i$$

[NB ∇_i is ith component of ∇ in the relevant coordinate system]

[example. In Cartesian coordinates, where the $\underline{e}_j$ are constants,

$$\nabla \underline{u} = \nabla_i (u_j)\, \underline{e}_j \otimes \underline{e}_i$$

$$= \frac{\partial u_j}{\partial x_1} \underline{e}_j \otimes \underline{e}_1 + \frac{\partial u_j}{\partial x_2} \underline{e}_j \otimes \underline{e}_2$$

$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} \end{pmatrix} (1, 0) + \begin{pmatrix} \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_2} \end{pmatrix} (0, 1)$$

$$= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

So $\nabla \underline{u} + \nabla \underline{u}^T = \begin{pmatrix} 2\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & 2\frac{\partial u_2}{\partial x_2} \end{pmatrix}$

$\therefore \ \underline{\underline{\sigma}} = -p + \mu(\nabla \underline{u} + \nabla \underline{u}^T) \Rightarrow$

$$\boxed{\begin{aligned} \sigma_{11} &= -p + 2\mu \frac{\partial u_1}{\partial x} \\ \sigma_{12} &= \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ \sigma_{22} &= -p + 2\mu \frac{\partial u_2}{\partial x_2} \end{aligned}}$$

In radial polars, $\underline{u} = u_r \underline{e_r} + u_\theta \underline{e_\theta}$

and $\nabla = \underline{e_r} \frac{\partial}{\partial r} + \frac{\underline{e_\theta}}{r} \frac{\partial}{\partial \theta}$.

Then $\nabla \underline{u} = \frac{\partial}{\partial r}(u_j \underline{e_j}) \otimes \underline{e_r} + \frac{1}{r} \frac{\partial}{\partial \theta}(u_j \underline{e_j}) \otimes \underline{e_\theta}$

$$= \frac{\partial u_j}{\partial r} \underline{e_j} \otimes \underline{e_r} + \frac{1}{r} \frac{\partial u_j}{\partial \theta} \underline{e_j} \otimes \underline{e_\theta}$$

$$+ u_j \frac{\partial \underline{e_j}}{\partial r} \otimes \underline{e_r} + \frac{1}{r} u_j \frac{\partial \underline{e_j}}{\partial \theta} \otimes \underline{e_\theta}$$

[↑ extra bits due to moving basis vectors]

Now $\underline{e_r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $\underline{e_\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

So $\frac{\partial \underline{e_r}}{\partial r} = \underline{0}$, $\frac{\partial \underline{e_\theta}}{\partial r} = \underline{0}$, $\frac{\partial \underline{e_r}}{\partial \theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \underline{e_\theta}$

$$\frac{\partial \underline{e_\theta}}{\partial \theta} = -\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = -\underline{e_r}$$

$$\therefore \nabla \underline{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} \\ \frac{\partial u_\theta}{\partial r} \end{pmatrix} (1,0) + \begin{pmatrix} \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\ \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \end{pmatrix} (0,1)$$

$$+ \frac{1}{r} u_r \, \underline{e_\theta} \otimes \underline{e_\theta} - \frac{1}{r} u_\theta \, \underline{e_r} \otimes \underline{e_\theta}$$

$$= \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \end{pmatrix}$$

$$\text{So } \nabla \underline{u} + \nabla \underline{u}^T = \begin{pmatrix} 2\frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\ \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} & + 2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \end{pmatrix}$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu(\nabla \underline{u} + \nabla \underline{u}^T)$$

$$\therefore \boxed{\begin{aligned} \sigma_{rr} &= -p + 2\mu \frac{\partial u_r}{\partial r} \\ \sigma_{r\theta} &= \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\ \sigma_{\theta\theta} &= -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \end{aligned}}$$

(as in the method answer).

<u>Summary</u>:

Method 1: Conceptually easier but harder algebra

Method 2: Conceptually harder but easier algebra