

Prelims Probability

Sheet 7 — MT25

1. Sketch the cumulative distribution function of the following distributions:
 - (a) the (discrete) uniform distribution on $\{1, 2, \dots, n\}$;
 - (b) the (continuous) uniform distribution on $[a, b]$;
 - (c) the exponential distribution with parameter 1;
 - (d) the normal distribution with mean 0 and variance 1.
2. For each case below, does there exist a constant c such that the given function is a probability density function? If so, find c and find the cumulative distribution function. (In each case, the given function is zero outside the interval $(0, 1)$.)
 - (a) $f_1(x) = cx$ for $0 < x < 1$.
 - (b) $f_2(x) = cx^{-1}$ for $0 < x < 1$.
 - (c) $f_3(x) = cx^{-1/2}$ for $0 < x < 1$.
 - (d) $f_4(x) = c(4x^3 - x)$ for $0 < x < 1$.
3. Let U be a uniformly distributed random variable on $[0, 1]$. Find
 - (a) $\mathbb{E}[U]$ and $\text{var}(U)$
 - (b) $\mathbb{P}(U < a | U < b)$ for $0 < a < b < 1$.
4. Let X be exponentially distributed with parameter λ .
 - (a) Find $\mathbb{P}(X > x)$
 - (b) Find $\mathbb{P}(a \leq X \leq b)$ for $0 < a < b$
 - (c) Show that $\mathbb{P}(X > a + x | X > a) = \mathbb{P}(X > x)$ for $a, x > 0$. [This is the *memoryless property* of the exponential distribution (compare to Question 3 on Sheet 3).]
 - (d) Find $\mathbb{P}(\sin X > \frac{1}{2})$
 - (e) Let $c > 0$. What is the distribution of the random variable cX ? [Try using part (a).]
 - (f) For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the *ceiling* of x ; that is, the smallest integer greater than or equal to x . Show that the discrete random variable $\lceil X \rceil$ has a geometric distribution, and find its parameter. [Hint: write the event $\{\lceil X \rceil = k\}$ as $\{X \in I\}$ for some interval I .]

5. Blood plasma nicotine levels in smokers can be modelled by a normal random variable X with mean 315 and variance 131^2 , the units being nanograms per millilitre.
- (a) What is the probability that a randomly chosen smoker has nicotine levels lower than 300?
 - (b) What is the probability that a randomly chosen smoker has nicotine levels between 300 and 500?
 - (c) If 20 smokers are to be tested what is the probability that at most one has a nicotine level higher than 500?

[If $\Phi(x)$ is the cumulative distribution function of the standard normal distribution then $\Phi(-0.115) = 0.454$, $\Phi(1.412) = 0.921$.]

6. The radius of a circle is uniformly distributed on $[0, b]$. Find the cumulative distribution function, the probability density function, the expectation and the variance of the random variable representing the area of the circle.
7. Let X be a continuous random variable taking values in $[a, b]$ with c.d.f. F_X which is strictly increasing on $[a, b]$.
- (a) Show that the random variable $F_X(X)$ has a uniform distribution on $[0, 1]$.
 - (b) Let U be a uniform random variable on $[0, 1]$. What is the distribution of the random variable $F_X^{-1}(U)$, where F_X^{-1} is the inverse of F_X ?
 - (c) Suppose that $U_1, U_2, \dots, U_n$ are a set of computer-generated pseudo-random numbers (assumed to be drawn from a uniform distribution on $[0, 1]$). How would you use them to simulate a random sample $X_1, X_2, \dots, X_n$ from the distribution with density

$$f(x) = \mu e^{-\mu x}, \quad x \geq 0?$$