Problem Sheet 3

Section A

No work in this section will be marked. Guided solutions will be published. The material has to be considered as preliminary/bookwork.

QUESTION 1. **Proof of the weak Maximum Principle.**
Let $b \in L^\infty(\Omega, \mathbb{R}^n)$. Give a weak formulation of the condition

$$(\star) \qquad \Delta u + b \cdot \nabla u \leq 0$$

that is well defined for functions $u \in H^1(\Omega)$.

Then show that there exists a number $c_1 > 0$ so that if $u \in H^1(\Omega)$ satisfies the weak form of $(\star)$ for some b with $\|b\|_{L^\infty} \leq c_1$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$.

Hint: You may use that $u^- = -\min(u, 0) \in H_0^1(\Omega)$ with $\nabla u^- = -\nabla u \cdot \chi_{\{u<0\}}$ a.e.

Solution. Seen Δu as an element of $\left(H_0^1(\Omega)\right)^*$ by

$$\langle \Delta u, v \rangle = -\int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall v \in H_0^1(\Omega),$$

a weak formulation of $(\star)$ is given by

$$(1) \qquad \int_\Omega -\nabla u \cdot \nabla v + b \cdot \nabla u \, v \, dx \leq 0 \quad \forall v \in H_0^1(\Omega), \; v \geq 0 \text{ a.e. },$$

which is well defined since $b \in L^\infty$. Note that, since we have an inequality, we restrict to non-negative test functions.

Choose now as test function $v = u^-$. Then

$$\begin{aligned}
0 &\geq \int_\Omega -\nabla u \cdot \nabla(u^-) + b \cdot \nabla u \, u^- \, dx \qquad \text{(Now use the hint)} \\
&= \int_\Omega |\nabla u^-|^2 \cdot \nabla(u^-) - b \cdot \nabla u^- \, u^- \, dx \\
&\geq \|\nabla u^-\|_{L^2}^2 - \|b\|_{L^\infty} \|\nabla u^-|\|_{L^2} \|u^-|\|_{L^2} \\
&\geq \|\nabla u^-\|_{L^2}^2 \left(1 - c_1 \, C_{Poinc}\right),
\end{aligned}$$

where C_{Poinc} is the Poincaré constant of Ω. It follows that, for $c_1 \leq 1/C_{Poinc}$ we must have $\|\nabla u^-\|_{L^2} = 0$, which in turn implies (by Poincaré inequality) $\|u^-\|_{L^2} = 0$, which yields $u^- = 0$ a.e. . □

Section B

Work done in this section will be marked.

QUESTION 2. **A non-linear PDE with a parameter.** Consider the PDE

$$-\Delta u = \exp\left(-\frac{\lambda}{u+1}\right) \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega.$$

Show that this problem can be formulated in an equivalent form so that it makes sense in $H_0^1(\Omega)$, i.e. making a modification to the right-hand-side that would not omit any solution, but would allow the equation to be well-posed. Prove that there exists at least one weak solution, for any λ and that this solution is unique if $\lambda < 0$.

QUESTION 3. **Sub and Super solutions.** Given a smooth, bounded domain $\Omega \subset R^3$, we consider the following reaction-diffusion problem

$$-\Delta u + u(1-u) = -1 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega.$$

- Show that this problem makes sense, and in particular that it can be written (for any $\lambda > 0$) as a fixed point problem for $T := u \to (-\Delta + (\lambda+1))^{-1}(f(u) + \lambda u)$, where T is a continuous map on $H_0^1(\Omega)$.
- Find a sub-solution and a super-solution.
- Show that there exists a $\lambda > 0$ such that $u^2 - 1 + 2\lambda u$ is an increasing function of u when $u \geq \underline{u}$.
- Show that there exists at least one solution in $H_0^1(\Omega)$ by adapting the super/sub solution method given in the lecture notes.

QUESTION 4. **Frechet Derivative**

(a) For a smooth bounded domain Ω, consider the map $F : C^2(\bar{\Omega}) \to C(\bar{\Omega})$ given by

$$F(u) = \Delta u + f(u),$$

where $f \in C^1(R)$. Compute the directional derivatives of F, and show that F is Fréchet differentiable.

(b) Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 4$, be bounded and consider the function $F(u) := (-\Delta)^{-1}(u^2)$. Prove that F is a C^1 function from $H_0^1(\Omega)$ to $H_0^1(\Omega)$.

Section C

No work in this section will be marked. Guided solutions will be published. These problems are not more difficult than those in previous sections. They sit here simply because they are relevant but either slightly off or beyond the main interests of the course.

QUESTION 5. H^2 **regularity** Show that for all $\phi \in C_0^\infty(\Omega)$ with $\Omega \subset R^n$ we have

$$\int_\Omega (\Delta\phi)^2 dx = \sum_{i,j=1}^n \int_\Omega (\partial_{ij}\phi)^2 \, dx.$$

Show that this equality is also true in $H_0^2(\Omega)$, and that the map $v \to \|\Delta v\|_{L^2(\Omega)}$ is a norm on $H_0^2(\Omega)$ equivalent to the usual $H^2(\Omega)$ norm.

Solution. Let $\phi \in C_0^\infty(\Omega)$ with $\Omega \subset R^n$. Integrating by parts twice, we get

$$\int_\Omega (\Delta\phi)^2 dx = \int_\Omega \left(\sum_{i=1}^n (\partial_i^2\phi)\right)\left(\sum_{j=1}^n (\partial_j^2\phi)\right) dx = \sum_{i,j=1}^n \int_\Omega (\partial_{ij}^2\phi)(\partial_{ij}^2\phi) dx = \sum_{i,j=1}^n \int_\Omega (\partial_{ij}\phi)^2 \, dx.$$

Notice that both the maps $u \mapsto \int_\Omega (\Delta u)^2 dx$ and $u \mapsto \sum_{i,j=1}^n \int_\Omega (\partial_{ij}\phi)^2 \, dx$ are continuous from $H_0^2(\Omega)$ to $\mathbb{R}$. Since they coincide on the dense subset $C_0^\infty(\Omega) \subset H_0^2(\Omega)$, they must coincide on the whole $H_0^2(\Omega)$.

By iterating the Poincaré inequality twice, we have that $\left(\sum_{i,j=1}^n \int_\Omega (\partial_{ij}u)^2 \, dx\right)^{1/2}$ defines a norm on $H_0^2(\Omega)$ equivalent to the standard H^2 norm restricted to $H_0^2(\Omega)$. The claim then follows by the identity $\left(\sum_{i,j=1}^n \int_\Omega (\partial_{ij}u)^2 \, dx\right)^{1/2} = \|\Delta u\|_{L^2(\Omega)}$ proved above for every $u \in H_0^2(\Omega)$. □