Numerical Solution of Partial Differential Equations

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Lecture 1

Elements of function spaces

The accuracy of a numerical method for the approximate solution of PDEs depends on its ability to capture the important features of the analytical solution. One such feature is the smoothness of the solution, which depends on the smoothness of the data.

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We present a brief overview of definitions and basic results form the theory of function spaces, focusing in particular on spaces of:

- Continuous functions;
- Integrable functions; and
- Sobolev spaces.

Spaces of continuous functions

$\mathbb{N}$ denotes the set of nonnegative integers.

An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is called a *multi-index*.
The nonnegative integer $|\alpha| := \alpha_1 + \cdots + \alpha_n$ is called the *length* of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. We denote $(0, \ldots, 0)$ by $\mathbf{0}$; clearly $|\mathbf{0}| = 0$.

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Let

$$D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

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EXAMPLE. Suppose that $n = 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_j \in \mathbb{N}$, $j = 1, 2, 3$.
Then, for u, a function of three variables x_1, x_2, x_3:

$$\sum_{|\alpha|=3} D^\alpha u = \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1^2 \partial x_3} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^3}$$
$$+ \frac{\partial^3 u}{\partial x_2^3} + \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} + \frac{\partial^3 u}{\partial x_2^2 \partial x_3} + \frac{\partial^3 u}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u}{\partial x_3^3}.$$

We shall frequently write ∂_{x_j} instead of $\frac{\partial}{\partial x_j}$. ◇

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We denote by $C^k(\Omega)$ the set of all continuous real-valued functions defined on Ω s.t. $D^\alpha u$ is continuous on Ω for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq k$.

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Assuming that Ω is a *bounded* open set, $C^k(\overline{\Omega})$ will denote the set of all u in $C^k(\Omega)$ s.t. $D^\alpha u$ can be extended from Ω to a continuous function on $\overline{\Omega}$, the closure of the set Ω, for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq k$.

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The linear space $C^k(\overline{\Omega})$ can then be equipped with the norm

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} \left| D^\alpha u(x) \right| .$$

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Note: When $k = 0$, we shall write $C(\overline{\Omega})$ instead of $C^0(\overline{\Omega})$.

The *support*, supp u, of a continuous function u on Ω is defined as the closure in Ω of the set

$$\{x \in \Omega : u(x) \neq 0\}.$$

In other words, supp u is the smallest closed subset of Ω such that $u = 0$ in $\Omega \backslash$supp u.

EXAMPLE. Let w be the function defined on $\mathbb{R}^n$ by

$$w(x) = \begin{cases} \mathrm{e}^{-\frac{1}{1-|x|^2}} & , \ |x| < 1, \\ 0, & \text{otherwise}; \end{cases}$$

here $|x| := (x_1^2 + \cdots + x_n^2)^{1/2}$ for $x \in \mathbb{R}^n$.

Clearly, supp w is the closed unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$. ◇

We denote by $C_0^k(\Omega)$ the set of all $u \in C^k(\Omega)$ such that supp $u \subset \Omega$ and supp u is bounded. Let

$$C_0^\infty(\Omega) = \bigcap_{k \geq 0} C_0^k(\Omega).$$

EXAMPLE.
The function w defined in the previous example belongs to $C_0^\infty(\mathbb{R}^n)$. $\diamond$

Spaces of integrable functions

Let p be a real number, $p \geq 1$; we denote by $L_p(\Omega)$ the set of all real-valued functions defined on Ω such that

$$\int_\Omega \left| u(x) \right|^p \, \mathrm{d}x < \infty.$$

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Functions which are equal almost everywhere (i.e., equal, except on a set of measure zero) on Ω are identified with each other.

A subset of $\mathbb{R}^n$ is said to be a *set of measure zero* if it can be contained in the union of countably many open balls of arbitrarily small total volume.

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A subset of $\mathbb{R}^n$ is said to be a *set of measure zero* if it can be contained in the union of countably many open balls of arbitrarily small total volume.

$L_p(\Omega)$ is equipped with the norm

$$\|u\|_{L_p(\Omega)} := \left(\int_\Omega \left|u(x)\right|^p \, \mathrm{d}x\right)^{1/p}.$$

A particularly important case is $p = 2$; then,

$$\|u\|_{L_2(\Omega)} = \left(\int_\Omega \left| u(x) \right|^2 \, \mathrm{d}x \right)^{1/2}.$$

The space $L_2(\Omega)$ can be equipped with an inner product

$$(u, v) := \int_\Omega u(x) v(x) \, \mathrm{d}x.$$

Clearly $\|u\|_{L_2(\Omega)} = (u, u)^{1/2}$.

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Clearly $\|u\|_{L_2(\Omega)} = (u, u)^{1/2}$.

Lemma (The Cauchy–Schwarz inequality)

Let $u, v \in L_2(\Omega)$; then

$$|(u, v)| \leq \|u\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}.$$

Remark. The space $L_2(\Omega)$ equipped with the inner product $(\cdot, \cdot)$ (and the associated norm $\|u\|_{L_2(\Omega)} = (u, u)^{1/2}$) is an example of a Hilbert space.

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In general, a linear space X, equipped with an inner product $(\cdot, \cdot)_X$ (and the associated norm $\|u\|_X = (u, u)_X^{1/2}$) is called a Hilbert space if, whenever $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in X, i.e. a sequence of elements of X such that

$$\lim_{n,\, m \to \infty} \|u_n - u_m\|_X = 0,$$

then there exists a $u \in X$ such that $\lim_{m \to \infty} \|u - u_m\|_X = 0$ (i.e., the sequence $\{u_m\}_{m=1}^{\infty}$ converges to u in the norm of X).

Sobolev spaces

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$$\int_\Omega w_\alpha(x)\, v(x)\, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha v(x) \quad \forall\, v \in C_0^\infty(\Omega).$$

Then w_α is called the *weak derivative* of u (of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$) and we write $w_\alpha = D^\alpha u$.

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Clearly, if u is a smooth function then its weak derivatives coincide with those in the classical (pointwise) sense.

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EXAMPLE Let $\Omega = \mathbb{R}^1$, and let $u(x) = (1 - |x|)_+$. Here, for a real number y, $y_+ := \max\{y, 0\}$. Clearly u is not differentiable at $x = 0, \pm 1$. However, because u is locally integrable on Ω it may still have a weak derivative. Indeed, for any $v \in C_0^\infty(\Omega)$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} u(x)\, v'(x)\, \mathrm{d}x &= \int_{-\infty}^{+\infty} (1 - |x|)_+\, v'(x)\, \mathrm{d}x = \int_{-1}^{1} (1 - |x|)\, v'(x)\, \mathrm{d}x \\
&= \int_{-1}^{0} (1 + x)\, v'(x)\, \mathrm{d}x + \int_{0}^{1} (1 - x)\, v'(x)\, \mathrm{d}x \\
&= \int_{-1}^{0} (-1)\, v(x)\, \mathrm{d}x + \int_{0}^{1} (+1)\, v(x)\, \mathrm{d}x \\
&= -\int_{-\infty}^{+\infty} w(x)\, v(x)\, \mathrm{d}x,
\end{aligned}$$

where

$$w(x) = \begin{cases} 0, & x < -1, \\ 1, & x \in (-1, 0), \\ -1, & x \in (0, 1), \\ 0, & x > 1. \end{cases}$$

Thus, $w = u' = Du$ $\diamond$

Let k be a nonnegative integer. We define (with D^α denoting a weak derivative of order $|\alpha|$)

$$H^k(\Omega) := \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), \quad |\alpha| \leq k\}.$$

$H^k(\Omega)$ is called a Sobolev space of order k; it is equipped with the (Sobolev) norm

$$\|u\|_{H^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L_2(\Omega)} \right)^{1/2}$$

and the inner product

$$(u, v)_{H^k(\Omega)} := \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v).$$

Letting

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$|\cdot|_{H^k(\Omega)}$ is called the Sobolev semi-norm (it is only a semi-norm rather than a norm because if $|u|_{H^k(\Omega)} = 0$ for $u \in H^k(\Omega)$ it does not necessarily follow that $u \equiv 0$ on Ω.)

EXAMPLE

$$H^0(\Omega) = L_2(\Omega).$$

$$H^1(\Omega) := \left\{ u \in L_2(\Omega) : \partial_{x_j} u := \frac{\partial u}{\partial x_j} \in L_2(\Omega), \;\; j = 1, \ldots, n \right\},$$

$$\|u\|_{H^1(\Omega)} := \left\{ \|u\|^2_{L_2(\Omega)} + \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L_2(\Omega)} \right\}^{1/2},$$

$$|u|_{H^1(\Omega)} := \left\{ \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L_2(\Omega)} \right\}^{1/2}.$$

Similarly,

$$H^2(\Omega) := \left\{ u \in L_2(\Omega) : \partial_{x_j} u \in L_2(\Omega),\ \partial^2_{x_i x_j} u \in L_2(\Omega),\ i, j = 1, \ldots, n \right\},$$

$$\|u\|_{H^2(\Omega)} := \left\{ \|u\|^2_{L_2(\Omega)} + \sum_{j=1}^{n} \|\partial_{x_j} u\|^2_{L_2(\Omega)} + \sum_{i,j=1}^{n} \|\partial^2_{x_i x_j} u\|^2_{L_2(\Omega)} \right\}^{1/2},$$

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We define a special Sobolev space,

$$H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\},$$

i.e. $H_0^1(\Omega)$ is the set of all functions u in $H^1(\Omega)$ such that $u = 0$ on $\partial\Omega$, the boundary of the set Ω.

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$H_0^1(\Omega)$ is a Hilbert space, with the same norm and inner product as $H^1(\Omega)$.

We conclude with the following important result.

Lemma (Poincaré–Friedrichs inequality)

Suppose that Ω is a bounded open set in $\mathbb{R}^n$ (with a sufficiently smooth boundary $\partial\Omega$) and let $u \in H_0^1(\Omega)$; then, there exists a positive constant $c_\star(\Omega)$, independent of u, such that

$$\int_\Omega u^2(x)\,\mathrm{d}x \leq c_\star \sum_{i=1}^{n} \int_\Omega \left|\partial_{x_i} u(x)\right|^2 \,\mathrm{d}x. \tag{1}$$

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$$u(x, y) = u(a, y) + \int_a^x \partial_x u(\xi, y)\,\mathrm{d}\xi = \int_a^x \partial_x u(\xi, y)\,\mathrm{d}\xi, \qquad c < y < d.$$

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Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\int_\Omega \left|u(x,y)\right|^2\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \int_c^d \left|\int_a^x \partial_x u(\xi, y)\,\mathrm{d}\xi\right|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\le \int_a^b \int_c^d (x-a)\left(\int_a^x \left|\partial_x u(\xi, y)\right|^2\,\mathrm{d}\xi\right)\,\mathrm{d}x\,\mathrm{d}y \\
&\le \int_a^b (x-a)\,\mathrm{d}x \left(\int_c^d \int_a^b \left|\partial_x u(\xi, y)\right|^2\,\mathrm{d}\xi\,\mathrm{d}y\right) \\
&= \frac{1}{2}(b-a)^2 \int_\Omega \left|\partial_x u(x, y)\right|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Analogously,

$$\int_\Omega \left| u(x,y) \right|^2 \, \mathrm{d}x \, \mathrm{d}y \le \frac{1}{2}(d-c)^2 \int_\Omega \left| \partial_y u(x,y) \right|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

Analogously,

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By adding the two inequalities, we obtain

$$\int_\Omega \big|u(x,y)\big|^2 \,\mathrm{d}x\,\mathrm{d}y \le c_\star \int_\Omega \Big(\big|\partial_x u\big|^2 + \big|\partial_y u\big|^2\Big) \,\mathrm{d}x\,\mathrm{d}y,$$

where $c_\star = \left(\frac{2}{(b-a)^2} + \frac{2}{(d-c)^2}\right)^{-1}$. □