Numerical Solution of Partial Differential Equations

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2025

Lecture 2

Elliptic boundary-value problems

A second-order linear PDE for a function $u = u(x, y)$:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} = f(x,y) \qquad \text{is}$$

- ELLIPTIC if $b^2 - ac < 0$;
- PARABOLIC if $b^2 - ac = 0$; (and at least one of a or c is nonzero);
- HYPERBOLIC if $b^2 - ac > 0$.

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Ellipticity amounts to requiring that a and c are of the same sign, say $a > 0$ and $c > 0$ (or $a < 0$ and $c < 0$), and $ac - b^2 > 0$, which is equivalent (by Sylvester's criterion) to demanding that

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is a positive definite matrix, i.e. $\xi^{\mathrm{T}} A \xi > 0$ for all $\xi \in \mathbb{R}^2 \setminus \{0\}$.

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$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{i,j}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x), \quad x \in \Omega,$$

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where the coefficients $a_{i,j}$, b_i, c and f are such that

$$\begin{aligned} a_{i,j} &\in C^1(\overline{\Omega}), && i,j = 1,\ldots,n; \\ b_i &\in C(\overline{\Omega}), && i = 1,\ldots,n; \\ c &\in C(\overline{\Omega}), && f \in C(\overline{\Omega}), \quad \text{and} \end{aligned}$$

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \tilde{c}\sum_{i=1}^{n}\xi_i^2 \qquad \forall\, \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n, \quad \forall\, x \in \overline{\Omega};$$

here $\tilde{c}$ is a positive constant independent of x and ξ.

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(d) A more general version of (b) and (c) is

$$\sum_{i,j=1}^{n} a_{i,j} \frac{\partial u}{\partial x_i} \cos \alpha_j + \sigma(x) u = g \qquad \text{on } \partial\Omega,$$

where α_j is the angle between the unit outward normal vector ν to $\partial\Omega$ and the Ox_j axis (*oblique derivative boundary cond.*).

Classical solutions

Consider the homogeneous Dirichlet boundary-value problem:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{i,j}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \quad \text{for } x \in \Omega, \tag{1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

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The theory of partial differential equations tells us that (1), (2) has a unique classical solution, provided that $a_{i,j}$, b_i, c, f and $\partial\Omega$ are sufficiently smooth.

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$$\left.\begin{array}{rcll} -\Delta u &=& \operatorname{sgn}\left(\frac{1}{2} - |x|\right), & x \in \Omega, \\ u &=& 0, & x \in \partial\Omega. \end{array}\right\} \qquad (*)$$

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This problem has no classical solution, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, for otherwise Δu would be a continuous function on Ω, which is not possible because $\operatorname{sgn}(1/2 - |x|)$ is not a continuous function on Ω.

Definition (Weak solution)

Let $a_{i,j} \in C(\overline{\Omega})$, $i, j = 1, \dots, n$, $b_i \in C(\overline{\Omega})$, $i = 1, \dots, n$, $c \in C(\overline{\Omega})$, and let $f \in L^2(\Omega)$. A function $u \in H_0^1(\Omega)$ satisfying

$$\sum_{i,j=1}^{n} \int_\Omega a_{i,j}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v \, \mathrm{d}x + \int_\Omega c(x) u v \, \mathrm{d}x = \int_\Omega f(x) v(x) \, \mathrm{d}x \qquad \forall\, v \in H_0^1(\Omega)$$

is called a *weak solution* of (1), (2).

Example

Suppose that $\Omega = (a, b) \times (c, d) \subset \mathbb{R}^2$ and let $f \in L^2(\Omega)$. We wish to state the weak formulation of the elliptic boundary-value problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Solution. Note that $-\Delta u = -\operatorname{div}(\nabla u)$ and

$$\int_\Omega (-\Delta u)\, v \,\mathrm{d}x = -\int_\Omega \operatorname{div}(\nabla u)\, v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x$$

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Hence, the weak formulation of the boundary-value problem is: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + u\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x \qquad \forall v \in H_0^1(\Omega).$$

Introduction to the theory of finite difference schemes

Let Ω be a bounded open set in $\mathbb{R}^n$ and suppose that we wish to solve the boundary-value problem

$$\begin{aligned} \mathcal{L}u &= f && \text{in } \Omega, \\ \mathcal{B}u &= g && \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{3}$$

where $\mathcal{L}$ is a linear partial differential operator, and $\mathcal{B}$ is a linear operator which specifies the boundary condition.

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where $\mathcal{L}$ is a linear partial differential operator, and $\mathcal{B}$ is a linear operator which specifies the boundary condition. For example,

$$\mathcal{L}u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{i,j}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu,$$

and

$$\mathcal{B}u \equiv u \quad \text{(Dirichlet boundary condition)},$$

or

$$\mathcal{B}u \equiv \frac{\partial u}{\partial \nu} \quad \text{(Neumann boundary condition)},$$

or some other boundary condition.

The first step

Suppose that we have 'approximated' $\overline{\Omega} = \Omega \cup \Gamma$ by a finite set of points

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The parameter $h = (h_1, \ldots, h_n)$ measures the 'fineness' of the mesh (here h_i denotes the mesh-size in the coordinate direction Ox_i): the smaller $\max_{1 \leq i \leq n} h_i$ is, the finer the mesh.

The second step

Having constructed the mesh, we replace the derivatives in $\mathcal{L}$ by divided differences, and we approximate the boundary condition in a similar fashion. This yields the finite difference scheme

$$\begin{aligned} \mathcal{L}_h U(x) &= f_h(x), & x &\in \Omega_h, \\ \mathcal{B}_h U(x) &= g_h(x), & x &\in \Gamma_h, \end{aligned} \tag{4}$$

where f_h and g_h are suitable approximations of f and g.

Now (4) is a system of linear algebraic equations involving the values of U at the mesh-points, and can be solved by Gaussian elimination or an iterative method, provided that it has a unique solution.

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The sequence

$$\{U(x) : x \in \overline{\Omega}_h\}$$

is an approximation to

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the values of the exact solution at the mesh-points.

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- the first, and most basic, is the problem of approximation, that is, whether (4) approximates the boundary-value problem (3) in some sense, and whether its solution $\{U(x) : x \in \overline{\Omega}_h\}$ approximates $\{u(x) : x \in \overline{\Omega}_h\}$, the values of the exact solution at the mesh-points.

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Here we shall be primarily concerned with the first of these two problems — the question of approximation — although we shall also briefly consider the question of iterative solution of systems of linear algebraic equations by a simple iterative method.