Numerical Solution of Partial Differential Equations

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Lecture 4

Finite difference approximation of elliptic BVP's

In Lecture 3 we discussed the finite difference approximation of a two-point boundary-value problem. Here we shall carry out a similar analysis for the elliptic boundary-value problem

$$\begin{aligned} -\Delta u + c(x,y)u &= f(x,y) \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega = (0,1) \times (0,1)$, c is a continuous function on $\overline{\Omega}$ and $c(x,y) \geq 0$. We shall consider two separate cases:

- First we shall assume that $f \in C(\overline{\Omega})$. In this case, the error analysis proceeds similarly as in Lecture 3.
- In Lecture 5 we shall consider the case when f is only in $L_2(\Omega)$. In that case the boundary-value problem (1) does not have a classical solution – only a weak solution exists; a different technique is then needed to prove the convergence of the scheme.

The case when $f \in C(\overline{\Omega})$

Definition of the mesh

Let N be an integer, $N \geq 2$, and let $h = 1/N$; the mesh-points are (x_i, y_j), $i, j = 0, \ldots, N$, where $x_i = ih$, $y_j = jh$. These mesh-points form the mesh

$$\overline{\Omega}_h := \{(x_i, y_j) : i, j = 0, \ldots, N\}.$$

We consider the set of interior mesh-points

$$\Omega_h := \{(x_i, y_j) : i, j = 1, ..., N-1\},$$

and the set of boundary mesh-points $\Gamma_h := \overline{\Omega}_h \setminus \Omega_h$.

Definition of the finite difference scheme

$$
\begin{aligned}
-(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) + c(x_i, y_j)U_{i,j} = f(x_i, y_j) &\quad \text{for } (x_i, y_j) \in \Omega_h, \\
U = 0 &\quad \text{on } \Gamma_h.
\end{aligned} \tag{2}
$$

In an expanded form, this can be written as follows:

$$-\left\{ \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \right\} + c(x_i, y_j)U_{i,j} = f(x_i, y_j), \tag{3}$$

$$\text{for } i, j = 1, \ldots, N-1,$$

$$U_{i,j} = 0 \quad \text{if } i = 0,\ i = N \text{ or if } j = 0,\ j = N. \tag{4}$$

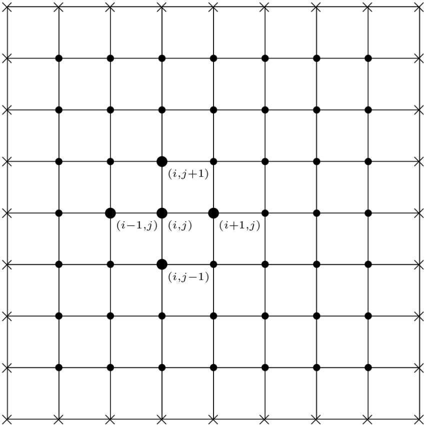

Figure 1: The mesh $\Omega_h(\cdot)$, the boundary mesh $\Gamma_h(\times)$, and a typical five-point difference stencil.

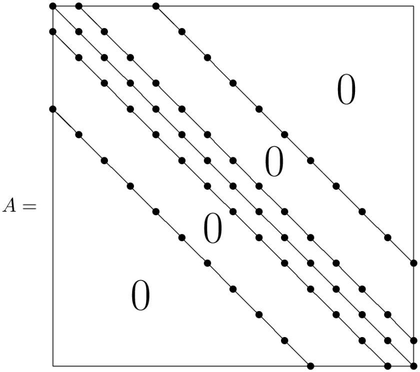

Figure 2: The sparsity structure of the banded matrix A.

A typical row of A has 5 non-zero entries, corresponding to the 5 values of U in the finite difference stencil shown in Figure. 1. The sparsity structure of A is shown in Figure 2.

Existence and uniqueness of solutions

Next we show that the finite difference scheme (2) has a unique solution.

For two functions, V and W, defined on Ω_h, we introduce the inner product

$$(V, W)_h = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} h^2 V_{i,j} W_{i,j},$$

which resembles the L_2-inner product

$$(v, w) = \int_\Omega v(x, y)\, w(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

Lemma

Suppose that V is a function defined on $\overline{\Omega}_h$ and that $V = 0$ on Γ_h; then,

$$(-D_x^+ D_x^- V, V)_h + (-D_y^+ D_y^- V, V)_h = \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{i,j}|^2. \quad (5)$$

PROOF. The identity (5) is a direct consequence of the corresponding univariate summation-by-parts result for $-D_x^+ D_x^-$ shown in Lecture 3, and the analogous identity for $-D_y^+ D_y^-$. $\square$

Returning to the analysis of the finite difference scheme (2), we shall now proceed in much the same way as in the univariate case in Lecture 3. As $c(x, y) \geq 0$ on $\overline{\Omega}$, by the summation-by-parts formula (5) we have that

$$\begin{aligned}(AV, V)_h &= (-D_x^+ D_x^- V - D_y^+ D_y^- V + cV, V)_h \\ &= (-D_x^+ D_x^- V, V)_h + (-D_y^+ D_y^- V, V)_h + (cV, V)_h \\ &\geq \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{i,j}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{i,j}|^2,\end{aligned} \tag{6}$$

for any V defined on $\overline{\Omega}_h$ such that $V = 0$ on Γ_h.

This implies, just as in the one-dimensional analysis presented in Section 3, that A is a non-singular matrix. Indeed if $AV = 0$, then (6) yields:

$$D_x^- V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{h} = 0, \qquad \begin{array}{l} i = 1, \dots, N, \\ j = 1, \dots, N-1; \end{array}$$

$$D_y^- V_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{h} = 0, \qquad \begin{array}{l} i = 1, \dots, N-1, \\ j = 1, \dots, N. \end{array}$$

As $V = 0$ on Γ_h, these imply that $V \equiv 0$. Thus $AV = 0$ if and only if $V = 0$. Hence A is non-singular, and $U = A^{-1}F$ is the unique solution of (2).Thus the unique solution of the finite difference scheme (2) may be found by solving the system of linear algebraic equations $AU = F$.

Stability and convergence of the finite difference scheme

In order to prove the stability of the finite difference scheme (2), we introduce the mesh–dependent norms

$$\|U\|_h := (U, U)_h^{1/2},$$

and

$$\|U\|_{1,h} := (\|U\|_h^2 + \|D_x^- U]|_x^2 + \|D_y^- U]|_y^2)^{1/2},$$

where

$$\|D_x^- U]|_x := \left(\sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- U_{i,j}|^2 \right)^{1/2}$$

and

$$\|D_y^- U]|_y := \left(\sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- U_{i,j}|^2 \right)^{1/2} .$$

$\| \cdot \|_{1,h}$ is the discrete version of the Sobolev norm $\| \cdot \|_{H^1(\Omega)}$.

With this new notation, the inequality (6) can be rewritten in the following compact form:

$$(AV, V)_h \geq \|D_x^- V]|_x^2 + \|D_y^- V]|_y^2. \tag{7}$$

Using the discrete Poincaré–Friedrichs inequality stated in the next lemma, we shall be able to deduce that

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2,$$

where c_0 is a positive constant.

Lemma (Discrete Poincaré–Friedrichs inequality)

Suppose that V is a function defined on $\overline{\Omega}_h$ and such that $V = 0$ on Γ_h; then, there exists a constant c_*, independent of V and h, such that

$$\|V\|_h^2 \leq c_* \left(\|D_x^- V]|_x^2 + \|D_y^- V]|_y^2 \right) \tag{8}$$

for all such V.

Proof.

The inequality (8) is a straightforward consequence of its univariate counterpart proved in Lecture 3; indeed, for each fixed j, $1 \leq j \leq N-1$,

$$\sum_{i=1}^{N-1} h|V_{i,j}|^2 \leq \frac{1}{2} \sum_{i=1}^{N} h|D_x^- V_{i,j}|^2. \tag{9}$$

Analogously, for each fixed i, $1 \leq i \leq N-1$,

$$\sum_{j=1}^{N-1} h|V_{i,j}|^2 \leq \frac{1}{2} \sum_{j=1}^{N} h|D_y^- V_{i,j}|^2. \tag{10}$$

We first multiply (9) by h and sum through j, $1 \leq j \leq N-1$, then multiply (10) by h and sum through i, $1 \leq i \leq N-1$, and finally add these two inequalities to obtain

$$2\,\|V\|_h^2 \leq \frac{1}{2}\left(\|D_x^- V]|_x^2 + \|D_y^- V]|_y^2\right).$$

Hence we arrive at (8) with $c_* = \frac{1}{4}$. $\square$

Now the inequalities (7) and (8) imply that

$$(AV, V)_h \geq \frac{1}{c_*} \|V\|_h^2.$$

Finally, combining this inequality with (7) and recalling the definition of the norm $\| \cdot \|_{1,h}$, we obtain

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2, \tag{11}$$

where $c_0 = (1 + c_*)^{-1} = (1 + (1/4))^{-1} = \frac{4}{5}$.

Using the inequality (11) we can now prove the stability of the finite difference scheme (2).

Theorem

The finite difference scheme (2) is stable in the sense that

$$\|U\|_{1,h} \leq \frac{1}{c_0}\|f\|_h. \tag{12}$$

PROOF. The proof is identical to that of the analogous stability inequality from Lecture 3 in the univariate case. From (11) and (2) we have that

$$\begin{aligned} c_0\|U\|_{1,h}^2 &\leq (AU, U)_h = (f, U)_h \leq \left|(f, U)_h\right| \\ &\leq \|f\|_h\|U\|_h \leq \|f\|_h\|U\|_{1,h}, \end{aligned}$$

and hence we arrive at the desired inequality (12). $\square$

Convergence in the class of classical solutions

Next, we turn to the study of accuracy of the finite difference scheme (2). We define the global error, e, by

$$e_{i,j} := u(x_i, y_j) - U_{i,j}, \quad 0 \le i, j \le N.$$

Assuming that $u \in C^4(\overline{\Omega})$, Taylor expansions with remainder terms in the x and y directions give:

$$\begin{aligned}
Ae_{i,j} &= Au(x_i, y_j) - AU_{i,j} = Au(x_i.y_j) - f_{i,j} \\
&= \Delta u(x_i, y_j) - (D_x^+ D_x^- u(x_i, y_j) + D_y^+ D_y^- u(x_i, y_j)) \\
&= \left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - D_x^+ D_x^- u(x_i, y_j) \right] + \left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j) \right] \\
&= -\frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) - \frac{h^2}{12} \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j), \qquad 1 \le i, j \le N-1,
\end{aligned}$$

where $\xi_i \in [x_{i-1}, x_{i+1}]$, $\eta_j \in [y_{j-1}, y_{j+1}]$.

We define the consistency error (or truncation error) of the finite difference scheme (2) by

$$\varphi_{i,j} := Au(x_i, y_j) - f_{i,j}.$$

Then, by the calculations above,

$$\varphi_{i,j} = -\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)\right), \quad 1 \le i, j \le N-1,$$

and

$$\begin{aligned} Ae_{i,j} &= \varphi_{i,j}, \quad && 1 \le i, j \le N-1, \\ e &= 0 && \text{on } \Gamma_h. \end{aligned}$$

Thanks to the stability result (12), we therefore have that

$$\|u - U\|_{1,h} = \|e\|_{1,h} \le \frac{1}{c_0}\|\varphi\|_h. \tag{13}$$

To arrive at a bound on the global error $e = u - U$ in the norm $\|\cdot\|_{1,h}$ it therefore remains to bound $\|\varphi\|_h$ and insert the resulting bound in the right-hand side of (13). Indeed, by noting that

$$|\varphi_{i,j}| \le \frac{h^2}{12}\left(\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{C(\overline{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{C(\overline{\Omega})}\right),$$

we deduce that the consistency error, φ, satisfies

$$\|\varphi\|_h \le \frac{h^2}{12}\left(\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{C(\overline{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{C(\overline{\Omega})}\right). \tag{14}$$

Finally (13) and (14) yield the following result.

Theorem

Let $f \in C(\overline{\Omega})$, $c \in C(\overline{\Omega})$, with $c(x,y) \geq 0$, $(x,y) \in \overline{\Omega}$, and suppose that the corresponding weak solution of the boundary-value problem (1) belongs to $C^4(\overline{\Omega})$; then

$$\|u - U\|_{1,h} \leq \frac{5h^2}{48} \left(\left\| \frac{\partial^4 u}{\partial x^4} \right\|_{C(\overline{\Omega})} + \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{C(\overline{\Omega})} \right). \tag{15}$$

PROOF. Recall that $c_0 = (1 + c_*)^{-1}, c_* = \frac{1}{4}$, so that $1/c_0 = \frac{5}{4}$, and combine (13) and (14). □

In other words,

stability + consistency ⇒ convergence.