Numerical Solution of Partial Differential Equations

Endre Süli

Mathematical Institute
University of Oxford
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Lecture 5

$-\Delta u + cu = f$, with $f \in L_2(\Omega)$

We use the same finite difference mesh as in the case when $f \in C(\overline{\Omega})$, but we shall modify the right-hand side in the finite difference scheme to cater for the fact that f need not be a continuous function on $\overline{\Omega}$.

The idea is to replace $f(x_i, y_j)$ by a 'cell-average' of f:

$$Tf_{i,j} := \frac{1}{h^2} \int_{K_{i,j}} f(x, y) \,\mathrm{d}x \,\mathrm{d}y,$$

where

$$K_{i,j} = \left[x_i - \frac{h}{2}, x_i + \frac{h}{2}\right] \times \left[y_j - \frac{h}{2}, y_j + \frac{h}{2}\right].$$

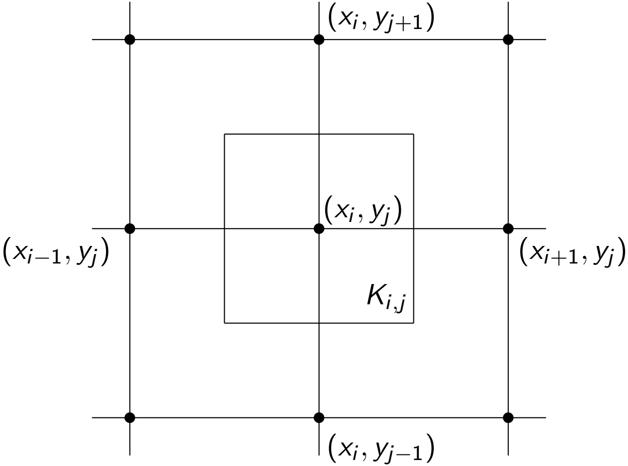

Figure: The cell $K_{i,j}$ surrounding the internal mesh point (x_i, y_j)

Existence and uniqueness of a solution

We define our finite difference approximation of the PDE by

$$\begin{aligned} -(D_x^+ D_x^- U_{i,j} + D_y^+ D_y^- U_{i,j}) + c(x_i, y_j) U_{i,j} &= Tf_{i,j}, && \text{for } (x_i, y_j) \in \Omega_h, \\ U &= 0, && \text{on } \Gamma_h. \end{aligned} \tag{1}$$

As we have not changed the difference operator on the left-hand side, the argument from Lecture 4 concerning the existence and uniqueness of a solution still applies, and therefore (1) has a unique solution, U.

Stability of the finite difference scheme

Theorem

The scheme (1) is stable in the sense that

$$\|U\|_{1,h} \leq \frac{1}{c_0}\|Tf\|_h. \tag{2}$$

PROOF. As in the proof of stability in Lecture 4:

$$\begin{aligned} c_0\|U\|_{1,h}^2 &\leq (AU, U)_h = (Tf, U)_h \\ &\leq \|Tf\|_h \|U\|_h \\ &\leq \|Tf\|_h \|U\|_{1,h}, \end{aligned}$$

where the second inequality follows from the Cauchy–Schwarz inequality, and the third inequality is the consequence of the definition of the discrete Sobolev norm $\|\cdot\|_{1,h}$. Hence (2). □

Convergence

Having established the stability of the scheme (1), we consider the question of its accuracy. Let us define the global error, e, as before,

$$e_{i,j} = u(x_i, y_j) - U_{i,j}, \quad 0 \leq i, j \leq N.$$

Clearly,

$$\begin{aligned} Ae_{i,j} &= Au(x_i, y_j) - AU_{i,j} \\ &= Au(x_i, y_j) - Tf_{i,j} \\ &= -(D_x^+ D_x^- u(x_i, y_j) + D_y^+ D_y^- u(x_i, y_j)) + c(x_i, y_j) u(x_i, y_j) \\ &\quad + \left(T \left(\frac{\partial^2 u}{\partial x^2} \right) (x_i, y_j) + T \left(\frac{\partial^2 u}{\partial y^2} \right) (x_i, y_j) - T(cu)(x_i, y_j) \right). \end{aligned} \tag{3}$$

By noting that

$$T\left(\frac{\partial^2 u}{\partial x^2}\right)(x_i, y_j) = \frac{1}{h}\int_{y_j-h/2}^{y_j+h/2} \frac{\frac{\partial u}{\partial x}(x_i + h/2, y) - \frac{\partial u}{\partial x}(x_i - h/2, y)}{h}\,\mathrm{d}y$$

$$= \frac{1}{h}\int_{y_j-h/2}^{y_j+h/2} D_x^+ \frac{\partial u}{\partial x}(x_i - h/2, y)\,\mathrm{d}y$$

$$= D_x^+\left[\frac{1}{h}\int_{y_j-h/2}^{y_j+h/2} \frac{\partial u}{\partial x}(x_i - h/2, y)\,\mathrm{d}y\right],$$

and similarly,

$$T\left(\frac{\partial^2 u}{\partial y^2}\right)(x_i, y_j) = D_y^+\left[\frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} \frac{\partial u}{\partial y}(x, y_j - h/2)\,\mathrm{d}x\right],$$

the equality (3) can be rewritten as

$$Ae = D_x^+\varphi_1 + D_y^+\varphi_2 + \psi,$$

where φ_1, φ_2 and ψ are defined on the next slide.

$$
\begin{aligned}
\varphi_1(x_i, y_j) &:= \frac{1}{h}\int_{y_j-h/2}^{y_j+h/2} \frac{\partial u}{\partial x}(x_i - h/2, y)\,\mathrm{d}y - D_x^- u(x_i, y_j),\\
\varphi_2(x_i, y_j) &:= \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} \frac{\partial u}{\partial y}(x, y_j - h/2)\,\mathrm{d}x - D_y^- u(x_i, y_j),\\
\psi(x_i, y_j) &:= (cu)(x_i, y_j) - T(cu)(x_i, y_j).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Ae &= D_x^+\varphi_1 + D_y^+\varphi_2 + \psi \qquad && \text{in } \Omega_h,\\
e &= 0 && \text{on } \Gamma_h.
\end{aligned}
\tag{4}
$$

The stability inequality (1) would only imply the (crude) bound

$$
\|e\|_{1,h} \le \frac{1}{c_0}\|D_x^+\varphi_1 + D_y^+\varphi_2 + \psi\|_h,
$$

which makes no use of the special form of the consistency error

$$
\varphi := D_x^+\varphi_1 + D_y^+\varphi_2 + \psi.
$$

We shall therefore proceed in a different way.

As in the proof of the stability inequality (1), we first note that

$$\begin{aligned} c_0\|e\|_{1,h}^2 &\leq (Ae, e)_h = (\varphi, e)_h \\ &= (D_x^+\varphi_1, e)_h + (D_y^+\varphi_2, e)_h + (\psi, e)_h. \end{aligned} \tag{5}$$

But now, using summation by parts, we shall pass the difference operators D_x^+ and D_y^+ from φ_1 and φ_2, respectively, onto e, using that $e = 0$ on Γ_h.

Indeed, by recalling that $e = 0$ on Γ_h, we have that

$$
\begin{aligned}
(D_x^+\varphi_1, e)_h &= \sum_{j=1}^{N-1} h \left(\sum_{i=1}^{N-1} h \frac{\varphi_1(x_{i+1}, y_j) - \varphi_1(x_i, y_j)}{h} e_{i,j} \right) \\
&= -\sum_{j=1}^{N-1} h \left(\sum_{i=1}^{N} h\varphi_1(x_i, y_j) \frac{e_{i,j} - e_{i-1,j}}{h} \right) \\
&= -\sum_{j=1}^{N-1} h \left(\sum_{i=1}^{N} h\varphi_1(x_i, y_j) D_x^- e_{i,j} \right) \\
&= -\sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 \varphi_1(x_i, y_j) D_x^- e_{i,j} \\
&\leq \left(\sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |\varphi_1(x_i, y_j)|^2 \right)^{1/2} \left(\sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- e_{i,j}|^2 \right)^{1/2} \\
&= \|\varphi_1]|_x \| D_x^- e]|_x.
\end{aligned}
$$

Thus,

$$(D_x^+ \varphi_1, e)_h \leq \|\varphi_1]|_x \|D_x^- e]|_x. \tag{6}$$

Similarly,

$$(D_y^+ \varphi_2, e)_h \leq \|\varphi_2]|_y \|D_y^- e]|_y \tag{7}$$

(see Lecture 3 for the definition of the mesh-dependent norms $\|\cdot]|_x$, $\|\cdot]|_y$). By the Cauchy–Schwarz inequality we also have that

$$(\psi, e)_h \leq \|\psi\|_h \|e\|_h. \tag{8}$$

Substitution of the inequalities (6)–(8) into the inequality (5) gives

$$\begin{aligned} c_0 \|e\|_{1,h}^2 &\leq \|\varphi_1]|_x \|D_x^- e]|_x + \|\varphi_2]|_y \|D_y^- e]|_y + \|\psi\|_h \|e\|_h \\ &\leq \left(\|\varphi_1]|_x^2 + \|\varphi_2]|_y^2 + \|\psi\|_h^2\right)^{1/2} \left(\|D_x^- e]|_x^2 + \|D_y^- e]|_y^2 + \|e\|_h^2\right)^{1/2} \\ &= \left(\|\varphi_1]|_x^2 + \|\varphi_2]|_y^2 + \|\psi\|_h^2\right)^{1/2} \|e\|_{1,h}. \end{aligned}$$

Dividing both sides by $\|e\|_{1,h}$ yields the following result.

Lemma

The global error, e, of the finite difference scheme (1) satisfies:

$$\|e\|_{1,h} \le \frac{1}{c_0}(\|\varphi_1]|_x^2 + \|\varphi_2]|_y^2 + \|\psi\|_h^2)^{1/2}, \tag{9}$$

where φ_1, φ_2, and ψ are defined by

$$\varphi_1(x_i, y_j) := \frac{1}{h}\int_{y_j-h/2}^{y_j+h/2} \frac{\partial u}{\partial x}(x_i - h/2, y)\,\mathrm{d}y - D_x^- u(x_i, y_j), \tag{10}$$

for $i = 1, \dots, N$, $j = 1, \dots, N-1$;

$$\varphi_2(x_i, y_j) := \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} \frac{\partial u}{\partial y}(x, y_j - h/2)\,\mathrm{d}x - D_y^- u(x_i, y_j), \tag{11}$$

for $i = 1, \dots, N-1$, $j = 1, \dots, N$; and

$$\psi(x_i, y_j) := (cu)(x_i, y_j) - \frac{1}{h^2}\int_{x_i-h/2}^{x_i+h/2}\int_{y_j-h/2}^{y_j+h/2} (cu)(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{12}$$

for $i, j = 1, \dots, N-1$.

To complete the error analysis, it remains to bound φ_1, φ_2 and ψ. Using Taylor series expansions it is easily seen that

$$|\varphi_1(x_i, y_j)| \leq \frac{h^2}{24}\left(\left\|\frac{\partial^3 u}{\partial x \partial y^2}\right\|_{C(\overline{\Omega})} + \left\|\frac{\partial^3 u}{\partial x^3}\right\|_{C(\overline{\Omega})}\right), \tag{13}$$

$$|\varphi_2(x_i, y_j)| \leq \frac{h^2}{24}\left(\left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_{C(\overline{\Omega})} + \left\|\frac{\partial^3 u}{\partial y^3}\right\|_{C(\overline{\Omega})}\right), \tag{14}$$

$$|\psi(x_i, y_j)| \leq \frac{h^2}{24}\left(\left\|\frac{\partial_2 (cu)}{\partial x^2}\right\|_{C(\overline{\Omega})} + \left\|\frac{\partial^2 (cu)}{\partial y^2}\right\|_{C(\overline{\Omega})}\right), \tag{15}$$

and by using these to bound $\|\varphi_1]|_x$, $\|\varphi_2]|_y$ and $\|\psi]|_h$ on the right-hand side of the ineq. (9) we arrive at the following theorem.

Theorem

Let $f \in L_2(\Omega)$, $c \in C^2(\overline{\Omega})$ *with* $c(x,y) \geq 0$, $(x,y) \in \overline{\Omega}$, *and suppose that the corresponding weak solution of the boundary-value problem belongs to* $C^3(\overline{\Omega})$*; then,*

$$\|u - U\|_{1,h} \leq \frac{5}{96} h^2 M_3, \tag{16}$$

where

$$M_3 = \left\{ \left(\left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{C(\overline{\Omega})} + \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{C(\overline{\Omega})} \right)^2 + \left(\left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{C(\overline{\Omega})} + \left\| \frac{\partial^3 u}{\partial y^3} \right\|_{C(\overline{\Omega})} \right)^2 + \left(\left\| \frac{\partial^2 (cu)}{\partial x^2} \right\|_{C(\overline{\Omega})} + \left\| \frac{\partial^2 (cu)}{\partial y^2} \right\|_{C(\overline{\Omega})} \right)^2 \right\}^{1/2}.$$

PROOF. By recalling that $1/c_0 = 5/4$ and substituting the bounds (13)–(15) into the right-hand side of the inequality (9), the inequality (16) immediately follows. $\square$

Comparing (16) with the error bound from Lecture 3, we see that while the smoothness requirement on the solution has been relaxed from $u \in C^4(\overline{\Omega})$ to $u \in C^3(\overline{\Omega})$, second-order convergence has been retained.

Remark

The hypothesis $u \in C^3(\overline{\Omega})$ can be further relaxed by using integral representations of φ_1, φ_2 and ψ instead of Taylor series expansions.

The key idea is to repeatedly use the Newton–Leibniz formula

$$w(b) - w(a) = \int_a^b w'(x)\,\mathrm{d}x$$

in conjunction with repeated partial integration. The details of the calculation are contained in Section 4.1.2 of the Lecture Notes.

Thus,

$$\|\varphi_1]|_x^2 \leq \frac{h^4}{32}\left(\left\|\frac{\partial^3 u}{\partial x^3}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^3 u}{\partial x \partial y^2}\right\|^2_{L_2(\Omega)}\right). \quad (17)$$

Analogously,

$$\|\varphi_2]|_y^2 \leq \frac{h^4}{32}\left(\left\|\frac{\partial^3 u}{\partial y^3}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|^2_{L_2(\Omega)}\right) \quad (18)$$

and

$$\|\psi\|_h^2 \leq \frac{3h^4}{64}\left(\left\|\frac{\partial^2 w}{\partial x^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 w}{\partial y^2}\right\|^2_{L_2(\Omega)} + 4\left\|\frac{\partial^2 w}{\partial x \partial y}\right\|^2_{L_2(\Omega)}\right). \quad (19)$$

By substituting the bounds (17)–(19) into the right-hand side of the inequality (9), noting that $1/c_0 = 4/5$ and recalling the definition of the Sobolev norm $\|\cdot\|_{H^3(\Omega)}$, we obtain the following result.

Theorem

Let $f \in L_2(\Omega)$, $c \in C^2(\overline{\Omega})$, with $c(x, y) \geq 0$, $(x, y) \in \overline{\Omega}$, and suppose that the corresponding weak solution of the boundary-value problem belongs to $H^3(\Omega)$; then,

$$\|u - U\|_{1,h} \leq Ch^2 \|u\|_{H^3(\Omega)}, \tag{20}$$

where C is a positive constant (computable from (17)–(19)).

It can be shown that the error estimate (20) is best possible in the sense that weakening of the assumption that $u \in H^3(\Omega)$ leads to loss of second-order convergence.

An error bound of this type, where the highest possible order of convergence has been attained with the weakest assumption on the smoothness of the solution u is called an *optimal error bound.*

Thus (20) is an optimal error bound for the difference scheme (1).