Numerical Solution of Partial Differential Equations

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Lecture 10

Von Neumann stability

In certain situations, practical stability is too restrictive and we need a less demanding notion of stability.

Definition (von Neumann stability)

We shall say that a finite difference scheme for the unsteady heat equation on the time interval $[0, T]$ is **von Neumann stable** in the ℓ_2 norm, if there exists a positive constant $C = C(T)$ such that

$$\|U^m\|_{\ell_2} \leq C\|U^0\|_{\ell_2}, \qquad m = 1, \ldots, M = \frac{T}{\Delta t},$$

where

$$\|U^m\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j^m|^2\right)^{1/2}.$$

Clearly, practical stability implies von Neumann stability, with stability constant $C = 1$.

As the **stability constant** C in the definition of von Neumann stability may dependent on T, and when it does then, typically, $C(T) \to +\infty$ as $T \to +\infty$, it follows that, unlike practical stability which is meaningful for $m = 1, 2, \dots$, von Neumann stability makes sense on finite time intervals $[0, T]$ (with $T < \infty$) and for the limited range of $0 \leq m \leq T/\Delta t$, only.

Von Neumann stability of a finite difference scheme can be easily verified by using the following result.

Lemma

Suppose that the semidiscrete Fourier transform of the solution $\{U_j^m\}_{j=1}^{\infty}$, $m = 0, 1, \ldots, \frac{T}{\Delta t}$, *of a finite difference scheme for the heat equation satisfies*

$$\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^m(k)$$

and

$$|\lambda(k)| \leq 1 + C_0\Delta t \qquad \forall k \in [-\pi/\Delta x, \pi/\Delta x].$$

Then the scheme is von Neumann stable. In particular, if $C_0 = 0$ *then the scheme is practically stable.*

PROOF: By Parseval's identity for the semidiscrete Fourier transform

$$\begin{aligned}\|U^{m+1}\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}}\|\hat{U}^{m+1}\|_{L_2} = \frac{1}{\sqrt{2\pi}}\|\lambda\hat{U}^m\|_{L_2} \\ &\leq \frac{1}{\sqrt{2\pi}}\max_k|\lambda(k)|\ \|\hat{U}^m\|_{L_2} = \max_k|\lambda(k)|\ \|U^m\|_{\ell_2}.\end{aligned}$$

Hence,

$$\|U^{m+1}\|_{\ell_2} \leq (1 + C_0\Delta t)\|U^m\|_{\ell_2}, \qquad m = 0, 1, \ldots, M-1.$$

Therefore,

$$\|U^m\|_{\ell_2} \leq (1 + C_0\Delta t)^m\|U^0\|_{\ell_2}, \qquad m = 1, \ldots, M.$$

As $(1 + C_0\Delta t)^m \leq \mathrm{e}^{C_0 m\Delta t} \leq \mathrm{e}^{C_0 T}$, it follows that

$$\|U^m\|_{\ell_2} \leq \mathrm{e}^{C_0 T}\|U^0\|_{\ell_2}, \qquad m = 1, 2, \ldots, M,$$

implying von Neumann stability, with $C = \mathrm{e}^{C_0 T}$. $\diamond$

Boundary-value problems for parabolic problems

When a parabolic PDE is considered on a bounded spatial domain, one needs to impose boundary conditions on the boundary of the domain. We shall consider the simplest case, when a Dirichlet boundary is imposed at both endpoints of the spatial domain, which we take to be the nonempty bounded open interval (a, b).

Consider the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad a < x < b, \quad 0 < t \leq T,$$

subject to the initial condition

$$u(x, 0) = u_0(x), \qquad x \in [a, b],$$

and the Dirichlet boundary conditions at $x = a$ and $x = b$:

$$u(a, t) = A(t), \quad u(b, t) = B(t), \qquad t \in (0, T].$$

Remark

The Neumann initial-boundary-value problem for the heat equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad a < x < b, \quad 0 < t \leq T,$$

subject to the initial condition

$$u(x,0) = u_0(x), \qquad x \in [a, b],$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(a,t) = A(t), \quad \frac{\partial u}{\partial x}(b,t) = B(t), \qquad t \in (0, T].$$

θ-scheme for the Dirichlet initial-boundary-value problem

Our aim is to construct a numerical approximation of the Dirichlet initial-boundary-value problem based on the θ-scheme.

Let $\Delta x = (b-a)/J$ and $\Delta t = T/M$, and define

$$x_j := a + j\Delta x, \quad j = 0, \ldots, J, \qquad t_m := m\Delta t, \quad m = 0, \ldots, M.$$

We approximate the Dirichlet initial-boundary-value problem with the θ-scheme:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\,\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} + \theta\,\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},$$

for $j = 1, \ldots, J-1$, $m = 0, 1, \ldots, M-1$,

$$U_j^0 = u_0(x_j), \qquad j = 1, \ldots, J-1,$$

$$U_0^{m+1} = A(t_{m+1}), \quad U_J^{m+1} = B(t_{m+1}), \quad m = 0, \ldots, M-1.$$

To implement this scheme it is helpful to rewrite it as a system of linear algebraic equations to compute the values of the numerical solution on time-level $m+1$ from those on time-level m. We have:

$$\begin{aligned} [1-\theta\mu\delta^2]U_j^{m+1} &= [1+(1-\theta)\mu\delta^2]U_j^m, \\ U_j^0 &= u_0(x_j), \qquad 1 \le j \le J-1, \end{aligned}$$

$$U_0^{m+1} = A(t_{m+1}), \qquad U_J^{m+1} = B(t_{m+1}), \;\; 0 \le m \le M-1,$$

where

$$\delta^2 U_j := U_{j+1} - 2U_j + U_{j-1}.$$

Consider the symmetric tridiagonal $(J-1) \times (J-1)$ matrix:

$$\mathcal{A} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

Let $\mathcal{I} = \mathrm{diag}(1,\, 1,\, 1, \dots,\, 1,\, 1)$ be the $(J-1) \times (J-1)$ identity matrix. Then, the θ-scheme can be written as

$$(\mathcal{I} - \theta\mu\mathcal{A})\mathbf{U}^{m+1} = (\mathcal{I} + (1-\theta)\mu\mathcal{A})\mathbf{U}^m + \theta\mu\mathbf{F}^{m+1} + (1-\theta)\mu\mathbf{F}^m$$

for $m = 0, 1, \dots, M-1$, where

$$\mathbf{U}^m = (U_1^m,\, U_2^m,\, \dots,\, U_{J-2}^m,\, U_{J-1}^m)^{\mathrm{T}}$$

and

$$\mathbf{F}^m = (A(t_m),\, 0,\, \dots,\, 0,\, B(t_m))^{\mathrm{T}}.$$