Numerical Solution of Partial Differential Equations

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Lecture 14

The explicit scheme: stability

For $M \geq 2$, we define $\Delta t := T/M$, and for $J \geq 2$ the spatial step is taken to be $\Delta x := (b-a)/J$. We let $x_j := a + j\Delta x$ for $j = 0, 1, \ldots, J$ and $t_m := m\Delta t$ for $m = 0, 1, \ldots, M$.

On the space-time mesh $\{(x_j, t_m) : 0 \leq j \leq J,\ 0 \leq m \leq M\}$ we consider the finite difference scheme

$$
\begin{aligned}
\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} - c^2 \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} &= f(x_j, t_m) \qquad \text{for} \begin{cases} j = 1, \ldots, J-1, \\ m = 1, \ldots, M-1, \end{cases} \\
U_j^0 &= u_0(x_j) \qquad \text{for } j = 0, 1, \ldots, J, \\
U_j^1 &= U_j^0 + \Delta t\, u_1(x_j) \qquad \text{for } j = 1, 2, \ldots, J-1, \\
U_0^m = 0 \quad \text{and} \quad U_J^m &= 0 \qquad \text{for } m = 1, \ldots, M.
\end{aligned} \tag{1}
$$

Once the values of U_j^{m-1} and U_j^m, for $j = 0, \ldots, J$, have been computed (or have been specified by the initial data, in the case of $m = 1$), the subsequent values U_j^{m+1}, $j = 0, \ldots, J$, for $m = 1, \ldots, M - 1$, can be computed explicitly from (1), without having to solve systems of linear algebraic equations; hence the terminology explicit scheme.

Stability of the explicit scheme

It will transpire from the analysis that will follow that the explicit scheme is, unlike the implicit scheme, which was shown to be unconditionally stable, now only conditionally stable: we shall prove its stability in a certain 'energy norm', whose precise definition will emerge during the course of our analysis, — the stability condition for the explicit scheme being that $|c|\Delta t/\Delta x \leq 1$ (Courant–Friedrichs–Lewy (CFL) condition).

We begin by noting that, for any $j \in \{0, \dots, J\}$ and $m \in \{1, \dots, M-1\}$:

$$U_j^m = -\frac{1}{4}(U_j^{m+1} - 2U_j^m + U_j^{m-1}) + \frac{1}{4}(U_j^{m+1} + 2U_j^m + U_j^{m-1})$$

Hence, the left-hand side of equality $(1)_1$ can be rewritten as

$$\begin{aligned} &\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} - c^2 D_x^+ D_x^- U_j^m \\ &= \frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} + \frac{c^2 \Delta t^2}{4} D_x^+ D_x^- \frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} \\ &\qquad - c^2 D_x^+ D_x^- \frac{U_j^{m+1} + 2U_j^m + U_j^{m-1}}{4} \end{aligned}$$

for $j = 1, \dots, J-1$.

Insertion of this into $(1)_1$ then yields

$$\left(I + \frac{1}{4}c^2\Delta t^2 D_x^+ D_x^-\right)\frac{U_j^{m+1} - 2U_j^m + U_j^{m-1}}{(\Delta t)^2} = c^2 D_x^+ D_x^- \frac{U_j^{m+1} + 2U_j^m + U_j^{m-1}}{4} + f(x_j, t_m) \tag{2}$$

for $j = 1, \ldots, J-1$, $m = 1, \ldots, M-1$, where I signifies the identity operator, which maps any mesh function defined on the spatial mesh $\{x_j : j = 1, \ldots, J-1\}$ into itself.

Next, note that

$$\begin{aligned} U_j^{m+1} - 2U_j^m + U_j^{m-1} &= (U_j^{m+1} - U_j^m) - (U_j^m - U_j^{m-1}), \\ U_j^{m+1} + 2U_j^m + U_j^{m-1} &= (U_j^{m+1} + U_j^m) + (U_j^m + U_j^{m-1}), \\ U_j^{m+1} - U_j^{m-1} &= (U_j^{m+1} - U_j^m) + (U_j^m - U_j^{m-1}) \\ &= (U_j^{m+1} + U_j^m) - (U_j^m + U_j^{m-1}), \end{aligned} \tag{3}$$

We shall consider the inner products

$$(U, V) := \sum_{j=1}^{J-1} \Delta x\, U_j\, V_j,$$

$$(U, V\,] := \sum_{j=1}^{J} \Delta x\, U_j\, V_j,$$

and the associated norms, respectively, $\|\cdot\|$ and $\|\cdot\,]|$, defined by $\|U\| := (U, U)^{\frac{1}{2}}$ and $\|U\,]| := (U, U\,]^{\frac{1}{2}}$.

Take the $(\cdot,\cdot)$ inner product of (2) with $U^{m+1} - U^{m-1}$, making use of $(3)_1$ and $(3)_3$ on the left-hand side, and $(3)_2$ and $(3)_4$ on the right-hand side.

Thus we obtain the following equality:

$$
\begin{aligned}
&\left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\
&\quad - \left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^m - U^{m-1}}{\Delta t}, \frac{U^m - U^{m-1}}{\Delta t}\right) \\
&= -c^2\left(-D_x^+ D_x^- \frac{U^{m+1} + U^m}{2}, \frac{U^{m+1} + U^m}{2}\right) \\
&\quad + c^2\left(-D_x^+ D_x^- \frac{U^m + U^{m-1}}{2}, \frac{U^m + U^{m-1}}{2}\right) \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned}
$$

Next, we shall perform summations by parts in the first two terms on the right-hand side, using that, for any mesh-function V defined on $\{x_j : j = 0, \ldots, J\}$ and such that $V_0 = V_J = 0$, one has

$$(-D_x^+ D_x^- V, V) = (D_x^- V, D_x^- V] = \|D_x^- V]|^2.$$

Using this with $V = \frac{1}{2}(U^{m+1} + U^m)$ and $V = \frac{1}{2}(U^m + U^{m-1})$ gives

$$\begin{aligned}
&\left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\
&\quad - \left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^m - U^{m-1}}{\Delta t}, \frac{U^m - U^{m-1}}{\Delta t}\right) \\
&= -c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right]\right\|^2 + c^2 \left\| D_x^- \frac{U^m + U^{m-1}}{2} \right]\right\|^2 \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned}$$

This implies, following a minor rearrangement of terms, that

$$\begin{aligned}
&\left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \\
&\quad + c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right]\Big|^2 \\
&= \left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^m - U^{m-1}}{\Delta t}, \frac{U^m - U^{m-1}}{\Delta t}\right) \\
&\quad + c^2 \left\| D_x^- \frac{U^m + U^{m-1}}{2} \right]\Big|^2 \\
&\quad + (f(\cdot, t_m), U^{m+1} - U^{m-1}).
\end{aligned} \tag{4}$$

The second term on the left-hand side of (4) is nonnegative, as is the second term on the right-hand side.

We would therefore like to ensure that first term on the left-hand side of (4) and the first term on the right-hand side are also nonnegative.

To do so, we shall make a small diversion to investigate this. Letting

$$V_j^m := \frac{U_j^{m+1} - U_j^m}{\Delta t}, \qquad j = 0, \ldots, J,$$

and noting that $V_0^m = V_J^m = 0$, it follows that

$$\begin{aligned}\left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)V^m, V^m\right) &= \|V^m\|^2 + \frac{1}{4}c^2(\Delta t)^2(D_x^+ D_x^- V^m, V^m) \\ &= \|V^m\|^2 - \frac{1}{4}c^2(\Delta t)^2(D_x^- V^m, D_x^- V^m] \\ &= \|V^m\|^2 - \frac{1}{4}c^2(\Delta t)^2\|D_x^- V^m]|^2.\end{aligned}$$

The left-most expression will be nonnegative if and only if

$$\|V^m\|^2 - \frac{1}{4}c^2(\Delta t)^2\|D_x^- V^m]|^2 \geq 0.$$

We will show that this can be guaranteed by requiring that $|c|\Delta t/\Delta x \leq 1$.

For any real numbers α and β, $(\alpha - \beta)^2 \leq 2\alpha^2 + 2\beta^2$. Thus,

$$\begin{aligned}
\| D_x^- V^m]\|^2 &= \sum_{j=1}^{J} \Delta x \, | D_x^- V_j^m |^2 = (\Delta x)^{-1} \sum_{j=1}^{J} (V_j^m - V_{j-1}^m)^2 \\
&\leq 2 \, (\Delta x)^{-1} \sum_{j=1}^{J} (V_j^m)^2 + (V_{j-1}^m)^2 = 4 \, (\Delta x)^{-1} \sum_{j=1}^{J-1} (V_j^m)^2 \\
&= 4 \, (\Delta x)^{-2} \sum_{j=1}^{J-1} \Delta x \, (V_j^m)^2 = 4 \, (\Delta x)^{-2} \| V \|^2.
\end{aligned}$$

Thus we deduce that

$$\left(\left(I + \frac{1}{4} c^2 (\Delta t)^2 D_x^+ D_x^- \right) V^m, V^m \right) \geq \left(1 - \frac{c^2 (\Delta t)^2}{(\Delta x)^2} \right) \| V^m \|^2. \qquad (5)$$

We shall therefore assume that the following CFL condition holds:

$$\frac{|c|\,\Delta t}{\Delta x} \leq 1. \tag{6}$$

We then have from (5) that

$$\left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) \geq 0.$$

Assuming that (6) holds, we define the *nonnegative* expression

$$\mathcal{N}^2(U^m) := \left(\left(I + \frac{1}{4}c^2(\Delta t)^2 D_x^+ D_x^-\right)\frac{U^{m+1} - U^m}{\Delta t}, \frac{U^{m+1} - U^m}{\Delta t}\right) + c^2 \left\| D_x^- \frac{U^{m+1} + U^m}{2} \right]\!\Big\|^2.$$

With this notation (4) becomes

$$\mathcal{N}^2(U^m) = \mathcal{N}^2(U^{m-1}) + (f(\cdot, t_m), U^{m+1} - U^{m-1}). \qquad (7)$$

In the special case when f is identically zero (7) guarantees the stability of the explicit scheme under the CFL condition (6); indeed, (7) implies that

$$\mathcal{N}^2(U^m) = \mathcal{N}^2(U^0), \qquad \text{for all } m = 1, \ldots, M-1.$$

One can check that the mapping

$$U \mapsto \max_{m \in \{0,\ldots,M-1\}} \left[\mathcal{N}^2(U^m) \right]^{1/2}$$

is a norm on the linear space of all mesh functions U defined on the space-time mesh $\{(x_j, t_m) : j = 0, 1, \ldots, J, \;\; m = 0, 1, \ldots, M\}$ such that $U_0^m = U_J^m = 0$ for all $m = 0, 1, \ldots, M$.

Thus we have shown that, if the CFL condition (6) holds and $f \equiv 0$, then the explicit scheme (1) is conditionally stable in this norm.