Lie Groups

Section C course Hilary 2022

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Example sheet 1

1. Let G be the group of Möbius transformations which map the upper half-plane $\{z = x + iy \in \mathbb{C} : y > 0\}$ to itself. These are of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Show that G is a 3-dimensional non-compact connected Lie group.

Solution The coefficients in a Möbius transformation are only defined up to a scalar multiple, so we cover G with two charts. Since $ad - bc > 0$, a and b are not simultaneously zero, so define U as the subset on which $a \neq 0$ and take coordinates $x = c/a, y = b/a, z = d/a$ in the open subset of $\mathbb{R}^3$ defined by $z - xy > 0$, which is equivalent to $ad - bc > 0$. This is one chart.

For another take V to be the open subset where $b \neq 0$ and set $\tilde{x} = c/b, \tilde{y} = a/b, \tilde{z} = d/b$ so that $\tilde{z} - \tilde{x}\tilde{y} > 0$. Then on $U \cap V$, where $y = b/a \neq 0$,we have

$$\tilde{x} = x/y, \tilde{y} = 1/y, \tilde{z} = z/y$$

which is smooth and invertible.

This makes G into a 3-dimensional manifold with a countable basis of open sets. Composition of Möbius transformations follows multiplication of the 2×2 matrices

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \begin{pmatrix} c & d \\ c' & d' \end{pmatrix},$$

which is polynomial and hence smooth in the coordinates x, y, z etc. Inversion is

$$z \mapsto \frac{dz-b}{-cz+a}$$

which is smooth.

We need to prove that G is Hausdorff; it is sufficient to prove that any $g \in G$ and e, the identity, can be separated by open sets. The identity is given by $a = d$ and $b = c = 0$, or $(x, y, z) = (0, 0, 1)$. Since the topology of an open set in $\mathbb{R}^3$ is Hausdorff it is is separated from anything in U. So if $g \in V$ is not in U then $a = 0$ so $\tilde{y} = 0$. A neighbourhood of this point has $\tilde{y}$ small and hence in $U \cap V$ where $\tilde{y} = 1/y$ we must have $|y|$ large. But then a neighbourhood of $y = 0$ will not intersect this.

The subset U is homeomorphic to the open subset of $\mathbb{R}^3$ defined by $z - xy > 0$, which is connected (think of the half-planes $z > mx$ in the (x, z)-plane as m varies) – and likewise V. Since $U \cap V$ is non-empty, G is connected.

The group G is noncompact, for consider the well-defined function $a^2/(ad-bc)$. Restrict to $b=c=0, a=\lambda\in\mathbb{R}^+, d=1$ and it is the unbounded function λ.

Section B (questions to be handed in for marking)

2. Suppose G_1, G_2 are Lie groups. Show that $G_1\times G_2$ is a Lie group in a natural way. Deduce that the n-dimensional compact torus $T^n = S^1\times\cdots\times S^1$ is a Lie group.

Find a map $\pi:\mathbb{R}^n\to T^n$ that allows you to identify T^n with the quotient group $\mathbb{R}^n/\mathbb{Z}^n$. Which vector fields on $\mathbb{R}^n$ project under $d\pi$ to vector fields on T^n? Do all vector fields on T^n arise in this way?

Which vector fields X on T^n are *left-invariant*, that is, satisfy

$$(dL_g)_h(X|_h) = X_{gh}?$$

3. Use the implicit function/submanifold theorem to prove that the orthogonal group $O(n)$ is a Lie group. Compute the dimension of $O(n)$ and find the tangent space $T_I O(n)$. Show also that $O(n)$ is compact.

4. Show that the *tangent bundle* $TG=\bigsqcup_{g\in G} T_gG$ of a Lie group G is canonically identifiable with $G\times T_IG$. [*Hint: consider the left translation map* $L_g: G\to G$, $L_g(h)=gh$.]

Deduce that any Lie group of dimension n has n non-vanishing vector fields which are linearly independent at each point of G.

Show that the 3-dimensional sphere S^3 is a Lie group by identifying it with

$$SU(2)=\{2\times 2 \text{ complex matrices } A \text{ with } A^*A=I,\ \det A=1\}$$

where A^* denotes the conjugate transpose of A.

Show that the 2-dimensional sphere S^2 *cannot* be a Lie group. [*Hint: you may quote the "hairy ball theorem" from algebraic topology.*]

5. Let $\varphi: M\to N$ be a *diffeomorphism* of manifolds (a smooth map with smooth inverse). For a vector field X on M define the *push-forward* vector field $Z=\varphi_*X$ on N by

$$Z|_y = d\varphi_x(X|_x)$$

where $x=\varphi^{-1}(y)$. Show that for any function $f: N\to\mathbb{R}$,

$$(\varphi_*X)\cdot f = (X\cdot(f\circ\varphi))\circ\varphi^{-1}.$$

Deduce that $[\varphi_*X,\varphi_*Y]\cdot f=\varphi_*[X,Y]\cdot f$, and hence that

$$[\varphi_*X,\varphi_*Y]=\varphi_*[X,Y].$$

Let G be a Lie group. Prove the following characterization of left-invariant vector fields X:

$$(L_g)_*X = X \quad \text{for all } g\in G.$$

Let $\mathrm{Lie}\,G$ denote the set of left-invariant vector fields on G. Deduce that, if $X,Y\in\mathrm{Lie}\,G$, then also $[X,Y]\in\mathrm{Lie}\,G$.

6. Let G be a Lie group, and let G_0 denote the connected path component of G containing the identity (we call G_0 the *identity component* of G).

Show that G_0 is a normal subgroup of G. If $G = O(n)$ what is G_0? Is it true in this example that $G \cong G_0 \times (G/G_0)$ as groups?

Section C (optional extension questions, not to be handed in for marking)

7. (i) By considering the action of a matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} & a_1 \\ A_{21} & A_{22} & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

on the plane $x_3 = 1$, find the condition on A_{ij} for this to define an isometry of $\mathbb{R}^2$, and then show that the set of such matrices is a 3-dimensional Lie group G.

(ii) Is G connected?

(iii) Show that G is diffeomorphic to $\mathbb{R}^2 \times O(2)$ as a manifold.

(iv) Show that G has a subgroup isomorphic as a group to the additive group $\mathbb{R}^2$, and another isomorphic to $O(2)$, but G is not a product of these two groups.