Lie Groups

Section C course Hilary 2022

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Example sheet 2

Section A (introductory questions, not for marking, solutions available)

1. The algebra of *quaternions* is defined as

$$\mathbb{H} = \{a + bi + cj + dk \ : \ a, b, c, d \in \mathbb{R}\}$$

where i, j, k satisfy the relations

$$ij = k = -ji \ \text{ and } \ i^2 = j^2 = k^2 = -1.$$

(i) Show that these relations imply $jk = i = -kj$ and $ki = j = -ik$.

(ii) Show that the algebra of quaternions may be identified with the algebra of matrices

$$\left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \ : \ z, w \in \mathbb{C} \right\}.$$

(iii) If $q = a + bi + cj + dk \in \mathbb{H}$, we define the *quaternionic conjugate* to be

$$\bar{q} = a - bi - cj - dk$$

and the *norm* of q to be the nonnegative real number $|q|$ such that $|q|^2 = q\bar{q}$. Show that $q\bar{q}$ is indeed real and nonnegative, so $|q|$ is well-defined. Deduce that $q \neq 0$ has a multiplicative inverse $q^{-1} = \frac{\bar{q}}{|q|^2}$. Show also that

$$|q_1 q_2| = |q_1| \cdot |q_2| \quad \text{and} \quad |q^{-1}| = |q|^{-1}.$$

Viewing $\mathbb{H}$ as a real 4-dimensional vector space, check that $|q|$ is the usual norm on $\mathbb{R}^4$.

Solution

(i) We have $jk = -i^2jk = -ik^2 = i = -k^2i = kji^2 = -kj$ and $ki = -kij^2 = -k^2j = j = -jk^2 = -ik$.

(ii) Let

$$A = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}.$$

An $\mathbb{R}$-algebra isomorphism $\theta : \mathbb{H} \to A$ is given by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}.$$

By inspection θ is compatible with the relations defining $\mathbb{H}$ and is $\mathbb{R}$-linear, so is a genuine homomorphism of $\mathbb{R}$-algebras that is also clearly bijective.

(iii) If $q = a + bi + cj + dk$ then

$$q\overline{q} = (a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}^{\geq 0}.$$

With $q \neq 0$ and $|q| = \sqrt{q\overline{q}}$ we have $q\overline{q}/|q|^2 = 1$ so $q^{-1} = \overline{q}/|q|^2$. Also since we have (by a quick calculation) $\overline{q_1 q_2} = \overline{q_2} \cdot \overline{q_1}$ and $q\overline{q} = \overline{q}q$ then

$$|q_1 q_2|^2 = q_1 \cdot q_2 \cdot \overline{q_2} \cdot \overline{q_1} = q_1 |q_2|^2 \overline{q_1} = |q_1|^2 |q_2|^2.$$

Taking square roots yields $|q_1 q_2| = |q_1||q_2|$. Taking $q_1 = q$ and $q_2 = q^{-1}$ gives $|q||q^{-1}| = |1| = 1$, hence $|q^{-1}| = |q|^{-1}$.

Alternative argument: By direct calculation

$$|q|^2 = \det \theta(q),$$

so the multiplicativity of the quaternionic norm follows from the multiplicativity of the determinant.

2. Calculate the Lie algebras of the following four examples of Lie groups:
(i) the isometric transformations of $\mathbb{R}^2$ of the form $x \mapsto Ax + b$;
(ii) the non-zero quaternions $\mathbb{H}^*$;
(iii) the unit quaternions $\{q \in \mathbb{H} : |q| = 1\}$;
(iv) the group of Möbius transformations of the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ (it may be helpful to consider a homomorphism from a subgroup of $GL(2, \mathbb{R})$ to this group).

Solution (i) The group G of isometric transformations of $\mathbb{R}^2$ of the form $x \mapsto Ax + b$ can be identified with the subgroup of $GL(3, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} & a_1 \\ A_{21} & A_{22} & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where the matrix A is orthogonal. Thus the Lie algebra of G is a subalgebra of the Lie algebra of $GL(3, \mathbb{R})$ with Lie bracket given by commutator of matrices. $O(2)$ has Lie algebra the skew-symmetric 2×2 matrices, so a basis for the Lie algebra of G is

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the Lie brackets are:

$$[X,Y]=0,\ [Y,Z]=-X,\ [Z,X]=-Y.$$

(ii) The nonzero quaternions form an open subset in $\mathbb{R}^4$ so the tangent space at the identity is $\mathbb{R}^4=\mathbb{H}$. By left multiplication the nonzero quaternions form a subgroup of $GL(4,\mathbb{R})$ with the Lie bracket again the commutator. So the Lie algebra is spanned by $1,i,j,k$ and $[1,q]=0$ for all $q\in\mathbb{H}$. The remaining Lie brackets are determined by

$$[i,j]=2k,\ [j,k]=2i,\ [k,i]=2j.$$

(iii) The unit quaternions form the unit sphere in $\mathbb{R}^4$ whose tangent space at 1 is the orthogonal complement of $\mathbb{R}\subseteq\mathbb{H}$, namely the imaginary quaternions. The Lie brackets are as above. (iv) The composition of this group G of Möbius transformations is achieved by multiplying the corresponding 2×2 matrices. This means there is a surjective homomorphism from the subgroup of $GL(2,\mathbb{R})$ consisting of matrices of strictly positive determinant to G and a corresponding surjective map from the Lie algebra of $GL(2,\mathbb{R})$ to the Lie algebra of G. The Lie bracket for the matrix group is again commutator of matrices. The scalar matrices in $GL(2,\mathbb{R})$ give the trivial Möbius transformation, so the Lie algebra homomorphism maps the 3-dimensional Lie algebra of $SL(2,\mathbb{R})$, which consists of the trace zero 2×2 real matrices, surjectively to the 3-dimensional Lie algebra of G. This is therefore an isomorphism of Lie algebras.

Take a basis

$$X=\begin{pmatrix}0&1\\0&0\end{pmatrix},Y=\begin{pmatrix}0&0\\1&0\end{pmatrix},Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and the Lie brackets are [X, Y] = Z,   [Y, Z] = 2Y,   [Z, X] = 2X.

Section B (questions to be handed in for marking)

3. We define the Lie group

$$\mathrm{Sp}(n)=\{A\in GL(n,\mathbb{H}):A^*A=Id_n\}$$

where A^* denotes the quaternionic conjugate transpose of A (ie the ij entry of A^* is the quaternionic conjugate of the ji entry of A). Show that

$$\mathrm{Sp}(1)=\mathrm{SU}(2)$$

and hence that Sp(1) is topologically the 3-sphere.

For $q\in\mathbb{H}\setminus\{0\}$ define

$$\mathcal{A}_q:\mathbb{H}\to\mathbb{H},\quad p\mapsto qpq^{-1}.$$

Show that $\mathcal{A}_q$ is an orthogonal map (viewing $\mathbb{H}$ as $\mathbb{R}^4$).

By considering the orthogonal complement of $\mathbb{R}=\mathbb{R}\cdot 1\subset\mathbb{H}$, deduce that $\mathrm{SU}(2)\cong\mathrm{Sp}(1)\subset\mathbb{H}\setminus\{0\}$ acts on $\mathbb{R}^3$ by rotations. Explain briefly why this gives a homomorphism $\mathrm{Sp}(1)\cong\mathrm{SU}(2)\to\mathrm{SO}(3)$ with kernel $\{\pm1\}$.

4. Check these properties of $\exp : \mathrm{Lie}(G) \to G$ for a Lie group G:
(i) $\mathrm{Image}(\exp) \subseteq G_0$ where $G_0 =$ connected component of $1 \in G$;
(ii) $\exp((t+s)v) = \exp(tv)\exp(sv)$ for all $t, s \in \mathbb{R}$;
(iii) $(\exp v)^{-1} = \exp(-v)$;
(iv) if $g = \exp(v)$ then it has an n-th root;
(v) the exponential map $\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is not surjective.

5. Prove directly that **ad** is a Lie algebra homomorphism by using the fact that $\mathbf{ad}(X) \cdot Z = [X, Z]$. Show that

$$v_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is a basis for $\mathfrak{so}(3) \subset \mathrm{Mat}_{3\times 3}(\mathbb{R})$.
By computing all brackets $[v_i, v_j]$, show that

$$\mathfrak{so}(3) \cong (\mathbb{R}^3, \text{cross product}), \qquad v_i \mapsto \text{standard basis vector } e_i$$

is a Lie algebra isomorphism.

Via this isomorphism we identify $\mathrm{End}(\mathfrak{so}(3))$ with 3×3 matrices. Compute the matrices $\mathbf{ad}(v_i)$. By computing $\langle v_i, v_j \rangle$ show that the **Killing form**

$$\langle v, w \rangle = \mathrm{Trace}(\mathbf{ad}(v)\mathbf{ad}(w)) \in \mathbb{R}$$

is a negative definite scalar product on $\mathfrak{so}(3)$.

Section C (optional extension questions, not to be handed in for marking)

6. Show that for a matrix group G, we have $\exp(gXg^{-1}) = g\exp(X)g^{-1}$ for all $g \in G$ and $X \in \mathfrak{g}$.

Consider the subgroup T of the unitary group $U(n)$ consisting of diagonal matrices. Show that T is a torus $T^n \cong (S^1)^n$ and that T lies in the image of the exponential map $\exp : \mathfrak{u}(n) \to U(n)$.

Deduce that $\exp : \mathfrak{u}(n) \to U(n)$ is surjective.

7. The three-dimensional Heisenberg group consists of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. Show that its Lie algebra consists of matrices

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$$

and calculate the exponential map for this group. Is this exponential map surjective?

8. If $A \in GL(n, \mathbb{C})$ is diagonalizable, show that $A = \exp B$ for a complex matrix B.
Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

with $\lambda \neq 0 \in \mathbb{C}$. Show, by writing this in the form $\lambda(I + N)$, that in this case too there exists B such that $A = \exp B$.

The Jordan normal form states that any complex $n \times n$ matrix is conjugate to a matrix with blocks of the above form down the diagonal. Deduce that the exponential map for the Lie group $GL(n, \mathbb{C})$ is surjective.