B5.2: Applied Partial Differential Equations

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Contents

1 Introduction

1.1 What is a PDE?

A **partial differential equation (PDE)** is an equation for some quantity u (the dependent variable) which is a function of several independent variables, involving partial derivatives of u with respect to at least some of these independent variables. In this course, we usually limit attention to just two independent variables, so $u = u(x, y)$, and at most two derivatives. Then our PDE can be written in general terms as follows:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{1.1}$$

Here, and henceforth, we follow the convention of using subscripts as shorthand for partial derivatives, *i.e.*,

$$u_x \equiv \frac{\partial u}{\partial x}, \qquad u_{xy} \equiv \frac{\partial^2 u}{\partial y \partial x}, \tag{1.2}$$

and so forth. We will also assume that u is sufficiently smooth for all the required partial derivatives to exist and to be independent of the order in which the derivatives are performed, *i.e.*,

$$\frac{\partial^2 u}{\partial y \partial x} \equiv \frac{\partial^2 u}{\partial x \partial y}, \qquad u_{xy} \equiv u_{yx}. \tag{1.3}$$

In applications, one of the independent variables may represent time, in which case it is conventional to denote it by t and write $u(x, t)$ instead of $u(x, y)$. Finally, we note that the general form (1.1) can describe a **system of PDEs** involving several dependent variables, say $u_1, u_2, \cdots, u_m$, if we interpret u and F as vectors. In this course we denote vectors by bold symbols, so we would write

$$\boldsymbol{F}(x, y, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_{xx}, \boldsymbol{u}_{xy}, \boldsymbol{u}_{yy}) = \boldsymbol{0}, \tag{1.4}$$

where $\boldsymbol{u} : \mathbb{R}^2 \to \mathbb{R}^m$ and $\boldsymbol{F} : \mathbb{R}^{2+6m} \to \mathbb{R}^m$.

The **order** of a PDE is the degree of the highest (partial) differential coefficient in the equation. In general, equation (1.1) therefore defines a second-order PDE, while (1.4) defines a second-order $m \times m$ system. By cross-differentiation, one can transform a PDE system into a scalar PDE of higher order and *vice versa*.

Example 1.1 *The* Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

are a first-order 2×2 *system. By cross-differentiation one can easily find that* u *satisfies Laplace's equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.5}$$

which is a second-order scalar PDE (v satisfies Laplace's equation as well). On the other hand, equation (1.5) can be transformed to a first-order system by (for example) defining $p = u_x$, $q = u_y$, so that p and q satisfy

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = 0, \qquad \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = 0.$$

1.2 Linear PDEs

A **linear** equation is one that does not include any product of the dependent variables or their derivatives. For example:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{first order linear PDE (the simplest wave equation),} \tag{1.6a}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{second order linear PDE (Poisson's equation).} \tag{1.6b}$$

A linear PDE is **homogeneous** if it is satisfied by $u = 0$ (*i.e.*, if the "right-hand side is zero"). So (1.5) and (1.6a) are homogeneous but (1.6b) is not unless $f \equiv 0$.

In general we could write a linear inhomogeneous ("non-homogeneous") PDE in the form

$$\mathcal{L}u = f, \tag{N}$$

where $\mathcal{L}$ is a linear partial differential operator. Restricting to just two independent variables, we could express a general differential operator of order n in the form

$$\begin{aligned}\mathcal{L} &= a_{0,0} + a_{1,0}\frac{\partial}{\partial x} + a_{1,1}\frac{\partial}{\partial y} + a_{2,0}\frac{\partial^2}{\partial x^2} + a_{2,1}\frac{\partial^2}{\partial x \partial y} + a_{2,2}\frac{\partial^2}{\partial y^2} + \cdots \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i} a_{i,j}\frac{\partial^{i-j}}{\partial x^{i-j}}\frac{\partial^j}{\partial y^j},\end{aligned} \tag{1.7}$$

in which the coefficients $a_{i,j}$ are functions of (x, y) only. The homogeneous version of (N) is simply

$$\mathcal{L}u = 0, \tag{H}$$

Linear equations have the following properties.

- The general solution of the inhomogeneous problem (N) can be written as

$$u = p + v, \tag{1.8}$$

  where v is the general solution of the homogeneous problem (H) and p is *any* solution of (N). The decomposition (1.8) is analogous to the standard "particular solution + complementary function" approach to linear ordinary differential equations (ODEs).

- **Principle of superposition.** A linear homogeneous equation has the useful property that, if u_1 and u_2 both satisfy (H), then so does $\alpha u_1 + \beta u_2$ for any $\alpha, \beta \in \mathbb{R}$. This property underpins many standard methods to construct solutions to linear equations (for example Fourier series methods).

These properties do **not** hold for nonlinear equations.

Example 1.2 Laplace's equation (1.5) *is a second-order linear homogeneous PDE that can describe steady heat flow or gravitational or electromagnetic fields in free space. In general it may be written as*

$$\nabla^2 u = 0, \tag{1.9}$$

where Laplacian *is defined as*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cdots = \sum_j \frac{\partial^2 u}{\partial {x_j}^2}, \tag{1.10}$$

which is also sometime denoted by $\triangle u$*.*

Example 1.3 Poisson's equation *is simply Laplace's equation with an inhomogeneous source term, given by* (1.6b) *or in general*

$$\nabla^2 u = f. \tag{1.11}$$

This PDE can describe the electric potential due to a charge density f *or the gravitational potential due to a mass density* f*.*

Example 1.4 The Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{1.12}$$

is a linear homogeneous PDE that may be regarded as a stationary wave equation, with the parameter k *known as the* wave-number*.*

Example 1.5 The wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{1.13}$$

describes waves on a string, for example, with c *representing the wave-speed. The multidimensional wave equation*

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \tag{1.14}$$

describes wave propagation in multiple dimensions, for example sound waves, waves on a stretched membrane, or electromagnetic waves.

Example 1.6 The heat equation (or diffusion equation)

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \tag{1.15}$$

describes the flow of heat or the diffusion of a chemical in one dimension, with κ denoting the diffision coefficient. Multi-dimensional diffusion is described by the generalised heat equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u, \tag{1.16a}$$

or

$$\frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u) \tag{1.16b}$$

if κ is not constant.

Both the wave equation and the heat equation are *linear*, *homogeneous* PDEs that involve time t as an independent variable.

1.3 Nonlinear PDEs

An equation that is not linear is **nonlinear**. Nonlinear PDEs can be further categorised according to where the nonlinearity occurs in the equation.

- A nonlinear PDE is **semilinear** if the coefficients of the highest derivatives are functions of the independent variables only, for example

$$(x+\alpha)\frac{\partial u}{\partial x} + xy\frac{\partial u}{\partial y} = u^3, \tag{1.17a}$$

$$x\frac{\partial^2 u}{\partial x^2} + y(x+y)\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = u^4. \tag{1.17b}$$

- A nonlinear PDE is **quasilinear** if it is linear in the highest derivatives, with coefficients depending only on the independent variables, x, y say, and derivatives of lower order, for example

$$\frac{\partial u}{\partial x} + u\,\frac{\partial u}{\partial y} = u^2, \tag{1.18a}$$

$$\left(1+u_y^2\right) u_{xx} - 2u_x u_y u_{xy} + \left(1+u_x^2\right) u_{yy} = 0. \tag{1.18b}$$

- A PDE is **fully nonlinear** if it is nonlinear in the highest derivatives, for example

$$u_x^2 + u_y^2 = 1, \tag{1.19a}$$

$$u_{xx}u_{yy} - u_{xy}^2 = 1. \tag{1.19b}$$

Example 1.7 Fisher's equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \tag{1.20}$$

is a second-order semilinear *PDE describing the propagation of reaction-diffusion waves.*

Example 1.8 *The first-order quasilinear PDE*

$$\frac{\partial u}{\partial t} + (2u+c)\frac{\partial u}{\partial x} = 0 \tag{1.21}$$

describes nonlinear wave propagation, for example in traffic flow. This PDE is an example of a conservation law

$$u_t + q(u)_x = s, \tag{1.22}$$

where $q(u) = u^2 + cu$ *is known as the* flux function *for the density* $u(x,t)$*, and* s *is a source term (zero in this example).*

1.4 When can a PDE be solved?

Having defined a PDE, the general idea is to solve for the function $u(x,y)$. However, a PDE does not in general determine u uniquely by itself — boundary conditions are also required. A PDE supplemented by appropriate boundary conditions is often referred to as a *system* or a *problem*. For example, a simple problem involving Laplace's equation is

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad & y > 0 \\ u = f(x) \qquad & y = 0 \\ u \to 0 \qquad & y \to \infty. \end{aligned}\right\}. \tag{1.23}$$

We would like to solve this problem for u, but do we have sufficient information to do so? Maybe there are no solutions of Laplace's equation satisfying these particular boundary conditions, or maybe the boundary conditions are insufficient to determine u uniquely or robustly?

Faced with a problem like (1.23), the most important question to ask is: is it *well posed*? To be well posed, a problem must have the following three properties:

1. a solution $u(x,y)$ exists;

2. the solution is unique;

3. the solution depends continuously on the boundary data.

The first of these is obvious: there is no point in trying to find a solution that does not exist. If a problem is physically motivated, and u represents a physical quantity, then we would expect u to have a unique well-defined value at each point. If it does not, it suggests that a boundary condition or other constraint is missing from the problem.

To illustrate the final condition, suppose we vary the function $f(x)$ in (1.23) by a tiny amount and ask whether the corresponding variation in the solution is similarly small. If it is not, then the solution of the problem is impossible in practice, since any numerical errors in $f(x)$, however small, can lead to arbitrarily large errors in the solution.

1.5 Some standard methods for PDEs

The D'Alembert solution

Suppose we wish to solve the one-dimensional wave equation (1.13) subject to the initial conditions

$$u(x,0) = h(x), \qquad\qquad u_t(x,0) = v(x) \tag{1.24}$$

for $x \in \mathbb{R}$. It is easily shown that the change of variables $u(x,t) = U(\xi,\eta)$, where $\xi = x+ct$ and $\eta = x - ct$, transforms (1.13) into

$$\frac{\partial^2 U}{\partial\xi\partial\eta} = 0, \tag{1.25}$$

whose general solution is $U(\xi,\eta) = f(\xi) + g(\eta)$, where f and g are arbitrary functions. Thus the general solution of the one-dimensional wave equation is given by

$$u(x,t) = f(x+ct) + g(x-ct). \tag{1.26}$$

Now apply the initial conditions (1.24) to get

$$f(x) + g(x) = h(x), \qquad\qquad cf'(x) - cg'(x) = v(x). \tag{1.27}$$

The functions f and g are determined by solving these equations simultaneously, and then substituted back into (1.26) to get the *D'Alembert solution*

$$u(x,t) = \frac{1}{2}\big(h(x+ct) + h(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\,\mathrm{d}s, \tag{1.28}$$

which satisfies the wave equation and the initial conditions (1.24) for any given functions h and v.

The D'Alembert solution may be used, for example, to show that the above initial-value problem for the wave equation is well posed. By construction, under very mild assumptions about h and v, the solution for u exists and is unique. Moreover, it is straightforward to show from (1.28) that the solution depends continuously on the data in the sense that a small change in h and/or v results in a comparably small change in the solution u.

Separation of variables

This is a method for solving *linear* PDEs on a fixed finite domain (in at least one variable). The idea is to write the solution in the form

$$u(x, y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y), \tag{1.29}$$

where the functions X_n and Y_n satisfy ordinary differential equations.

Example 1.9 *The general solution of*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.30}$$

subject to $u(x, 0) = u(x, a) = 0$ may be written in the form

$$u(x, y) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{a}\right) + b_n \sin\left(\frac{n\pi x}{a}\right) \right\} \sin\left(\frac{n\pi y}{a}\right). \tag{1.31}$$

The arbitrary constants a_n and b_n may be found by Fourier analysis if u and $\partial u/\partial x$ are given on (say) $x = 0$.

Transforms

Fourier and Laplace transforms (and there are many others) are useful for solving linear PDEs on infinite or semi-infinite domains. They turn differential operators into algebraic operators and thus reduce PDEs to ODEs.

Example 1.10 *The heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1.32}$$

subject to $u \to 0$ as $x \to \pm\infty$ and $u = u_0(x)$ when $t = 0$ may be solved by taking a Fourier transform in x. The Fourier transform of u is defined by

$$\hat{u}(t, k) = \int_{-\infty}^{\infty} u(x, t) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x, \tag{1.33}$$

and u may be recovered from $\hat{u}$ by using the inversion formula

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(t, k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x. \tag{1.34}$$

The heat equation is transformed to

$$\frac{\partial \hat{u}}{\partial t} = -k^2 \hat{u} \qquad \Rightarrow \qquad \hat{u} = \hat{u}_0 \mathrm{e}^{-k^2 t}, \tag{1.35}$$

where $\hat{u}_0$ *is the Fourier transform of* u_0*. Then the* convolution theorem *gives*

$$u(x,t) = u_0(x) \star f(x,t) = \int_{-\infty}^{\infty} u_0(\xi) f(x-\xi,t)\,\mathrm{d}\xi, \tag{1.36}$$

where

$$\hat{f}(t,k) = \mathrm{e}^{-k^2 t}. \tag{1.37}$$

It is straightforward to invert this transform and thus find

$$f(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \tag{1.38}$$

which is the Green's function *for the heat equation.*

Example 1.11 *Consider the heat equation on a half-line:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x > 0,\ t > 0, \tag{1.39}$$

subject to $u = 0$ *when* $t = 0$*,* $u \to 0$ *as* $x \to \infty$ *and* $u(0,t) = f(t)$*. Define the Laplace transform of* u *by*

$$\bar{u}(x,p) = \int_0^{\infty} u(x,t)\mathrm{e}^{-pt}\,\mathrm{d}t, \tag{1.40}$$

so $\bar{u}$ *satisfies*

$$\frac{\partial^2 \bar{u}}{\partial x^2} = p\bar{u} \qquad \Rightarrow \qquad \bar{u} = \bar{f}\mathrm{e}^{-x\sqrt{p}}, \tag{1.41}$$

where $\bar{f}$ *is the Laplace transform of* f *and the branch of* $\sqrt{p}$ *with positive real part is chosen. Then the convolution theorem gives*

$$u(x,t) = f \star g = \int_0^t f(t-\tau) g(x,\tau)\,\mathrm{d}\tau \tag{1.42}$$

where

$$\bar{g} = \mathrm{e}^{-x\sqrt{p}}. \tag{1.43}$$

To find the function g*, we can use the inversion formula*

$$g(x,t) = \frac{1}{2\pi\mathrm{i}} \int_C \bar{g}(x,p)\mathrm{e}^{pt}\,\mathrm{d}p \tag{1.44}$$

and bend the integration contour C *around the branch cut along the negative real* p *axis. However, there's a shortcut. If* $u_0(x,t)$ *is the solution corresponding to* $f(t) \equiv 1$*, then* (1.42) *gives*

$$u_0(x,t) = \int_0^t g(x,\tau)\,\mathrm{d}\tau \qquad \Rightarrow \qquad g(x,t) = \frac{\partial}{\partial t} u_0(x,t). \tag{1.45}$$

Then u_0 *may be found by other methods, for example by looking for a similarity solution (see* Example 1.12*):*

$$u_0(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) \qquad \Rightarrow \qquad g(x,t) = \frac{x}{2\sqrt{\pi} t^{3/2}} \exp\left(-\frac{x^2}{4t}\right), \tag{1.46}$$

where erfc *is the* complementary error function*.*

Similarity solutions

If the PDE and boundary conditions have a certain symmetry, then a *similarity solution* may be sought in the form

$$u(x,y) = x^{\alpha} f(\eta), \qquad \eta = \frac{y}{x^{\beta}}, \tag{1.47}$$

so the partial differential equation for u becomes an ordinary differential equation for f. In general, this form of solution (1.47) will work only for particular choices of the exponents α and β, if at all.

Example 1.12 *Consider the heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1.48}$$

subject to $u(x,0) = u(\infty,t) = 0$, $u(0,t) = t^m$ for some constant m.
Method 1: *We can try to plug in a similarity solution for u and then see what choices for the exponents (if any) will make the ansatz work. Motivated by the boundary condition at $x = 0$, try the substitution*

$$u(x,t) = t^m f\left(x/t^{\beta}\right) = f(\eta), \quad \textit{say.} \tag{1.49}$$

Differentiate using the chain rule to get

$$\frac{\partial u}{\partial t} = mt^{m-1} f(\eta) - \beta\eta t^{m-1} f'(\eta), \qquad \frac{\partial^2 u}{\partial x^2} = t^{m-2\beta} f''(\eta), \tag{1.50}$$

so the heat equation is transformed to

$$f''(\eta) + t^{2\beta-1}\left(\beta\eta f'(\eta) - mf(\eta)\right) = 0. \tag{1.51}$$

It is possible for f to be a function of η alone only if $\beta = 1/2$.
Method 2: *It is insightful instead to seek a* symmetry *of the problem by rescaling the dependent and independent variables as follows:*

$$X = \lambda x, \qquad T = \lambda^b t, \qquad U(X,T) = \lambda^c u(x,t), \tag{1.52}$$

where $\lambda > 0$ and the exponents b and c are so far arbitrary. Then the problem for u becomes

$$\lambda^{b-c} U_T = \lambda^{2-c} U_{XX}, \tag{1.53a}$$

subject to

$$U(X,0) = 0 = U(\infty,T) \qquad \lambda^{-c} U(0,T) = \lambda^{-mb} T^m \tag{1.53b}$$

Now we see that λ can be eliminated from the rescaled problem (1.53) by choosing $b = 2$ and $c = 2m$. In this case, the problem (1.53) for $U(X,T)$ is identical *to the original unscaled problem for $u(x,t)$: this particular scaling represents a* symmetry *that leaves the problem* invariant.

Since the problems for u *and* U *are identical, their solutions (if they exist) must also be identical, that is, we must have*

$$u(x,t) \equiv \lambda^{-2m} u\left(\lambda x, \lambda^2 t\right), \tag{1.54}$$

where $\lambda > 0$ *is arbitrary. For any particular value of* t*, we could set* $\lambda = t^{-1/2}$ *and thus find*

$$u(x,t) = t^m u\left(x/t^{1/2}, 1\right) = t^m f\left(x/t^{1/2}\right), \tag{1.55}$$

say, which reproduces (1.49) *with* $\beta = 1/2$*.*

Whichever approach is used, we find that f *satisfies*

$$f''(\eta) + \frac{\eta}{2} f'(\eta) - m f(\eta) = 0 \qquad f(0) = 1, \qquad f(\infty) = 0. \tag{1.56}$$

The simplest case is $m = 0$*, when the solution is*

$$f(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta/2}^{\infty} \mathrm{e}^{-s^2}\,\mathrm{d}s = \operatorname{erfc}(\eta/2), \tag{1.57}$$

the complementary error function.

2 First order quasilinear equations

2.1 Definitions

In this section we consider partial differential equations (PDEs) of the following form

$$a(x,y,u)\frac{\partial u}{\partial x}+b(x,y,u)\frac{\partial u}{\partial y}=c(x,y,u). \tag{2.1}$$

Here x and y are independent variables, a, b and c are given smooth (*i.e.*, continuously differentiable) functions, and $u(x,y)$ is a scalar function for which we would like to solve. Equation (2.1) is known as a *first-order quasilinear partial differential equation*: first-order since there are no second or higher derivatives and quasilinear because it is linear in its highest derivatives. Equations of this type arise in many areas of mathematical modelling, including fluid mechanics and traffic flow. They also provide a relatively straightforward introduction to some important concepts, such as *Cauchy data*, *characteristics* and *weak solutions*, that will be applied to more complicated equations later in the course.

Two special cases of (2.1) are worth mentioning. First, if a and b are independent of u, then (2.1) becomes the *semilinear* equation

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=c(x,y,u). \tag{2.2}$$

If it also happens that c is a linear function of u, then we have

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=\alpha(x,y)u+\beta(x,y), \tag{2.3}$$

which is a *linear* equation. It is generally the case that linear equations are significantly better-behaved and easier to solve than nonlinear ones.

Equations like (2.1) often describe the evolution of a quantity u (representing *e.g.* traffic density or fluid velocity) in space and time. In such cases, to emphasise the fact that one independent variable represents time, we can use x and t as independent variables instead of x and y, writing (2.1) as

$$a(x,t,u)\frac{\partial u}{\partial t}+b(x,t,u)\frac{\partial u}{\partial x}=c(x,t,u). \tag{2.4}$$

2.2 Characteristics

Geometric definition

We can think of the solution $u(x,y)$ we are seeking as defining a surface $z=u(x,y)$ in three-dimensional space. The normal to this surface is in the direction

$$\boldsymbol{n}\propto\boldsymbol{\nabla}\big(u(x,y)-z\big)=\begin{pmatrix}u_x\\u_y\\-1\end{pmatrix} \tag{2.5}$$

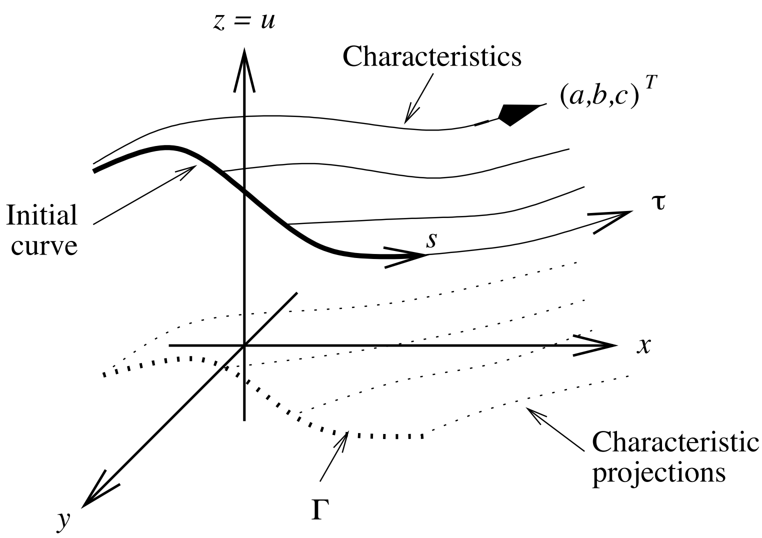

Figure 2.1: Schematic showing the characteristics, parameterised by τ and pointing in the direction $(a, b, c)^T$, emerging from the initial curve, which is parameterised by s. The projection of the initial curve onto the (x, y) plane is Γ and the projection of the characteristics onto the (x, y) plane are the characteristic projections.

and the PDE (2.1) can therefore be written as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \boldsymbol{n} = 0. \tag{2.6}$$

It follows that the vector $(a, b, c)^T$ is everywhere tangent to the solution surface.

We can construct curves that are everywhere tangent to $(a, b, c)^T$ by solving the simultaneous ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = a(x, y, u), \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = b(x, y, u), \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = c(x, y, u). \tag{2.7}$$

Such curves are called *characteristics* of the PDE (2.1). Their projections onto the (x, y) plane, *i.e.*, the plane curves $\big(x(\tau), y(\tau)\big)$ are called *characteristic projections*.

Solution by characteristics

Suppose, as before, that u is specified along some curve Γ in the (x, y) plane, *i.e.*, that we are given $u = u_0(s)$ when $x = x_0(s)$ and $y = y_0(s)$ where s parameterises Γ. As shown in Figure 2.1, this data defines a initial curve in three-dimensional space, through which we require our solution surface to pass. For any fixed value of s, we can find a characteristic that passes through the point $\big(x_0(s), y_0(s), u_0(s)\big)^T$ (and is everywhere tangent to $(a, b, c)^T$) by solving (2.7) with the initial conditions

$$x = x_0(s), \qquad y = y_0(s), \qquad u = u_0(s) \qquad \text{at } \tau = 0. \tag{2.8}$$

As s is varied, these characteristics sweep out the desired solution surface. Put another way, the initial-value problem (2.7), (2.8) determines in principle three scalar functions $x(s,\tau)$, $y(s,\tau)$ and $u(s,\tau)$, and the vector

$$\boldsymbol{r}(s,\tau) = \begin{pmatrix} x(s,\tau) \\ y(s,\tau) \\ u(s,\tau) \end{pmatrix} \tag{2.9}$$

defines the solution surface, parametrised by s and τ.

The theory that exists for ODEs may be applied directly to the system (2.7). For example, Picard's theorem tells us that there is a unique solution to (2.7) satisfying the initial conditions (2.8) provided the right-hand side $(a,b,c)^T$ is bounded and satisfies a Lipschitz condition in u. However, we can easily construct examples for which these conditions fail and the solution either blows up or becomes nonunique at some distance from the initial curve.

Equation (2.9) is called the *parametric* form of the solution. It may be possible to eliminate s and τ from (2.9) to obtain the solution surface in the *implicit* form $G(x,y,u)=0$. Finally, if this implicit equation can be solved for u, then we obtain the solution in the *explicit* form $u=u(x,y)$. Explicit solutions are the most convenient, but are often impossible to obtain in terms of elementary functions.

Example 2.1 *Consider the PDE*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1, \tag{2.10}$$

subject to the boundary data $u=0$ when $x+y=0$. The characteristics satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{\mathrm{d}u}{\mathrm{d}\tau} = 1, \tag{2.11}$$

and the boundary data lead to the initial conditions

$$x = s, \qquad y = -s, \qquad u = 0 \qquad \text{at } \tau = 0. \tag{2.12}$$

Hence we find the parametric solution

$$x = s+\tau, \qquad y = -s+\tau, \qquad u = \tau, \tag{2.13}$$

and it is straightforward in this case to eliminate s and τ to obtain the explicit solution

$$u = \frac{x+y}{2}. \tag{2.14}$$

In Example 2.1, the PDE (2.10) is semilinear. In such cases, the characteristic projections satisfy the ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = a(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = b(x,y), \tag{2.15}$$

which are independent of the solution u. The standard theory of phase planes may be applied to the ODEs (2.15); for example, there is in general a unique characteristic projection through each point in the (x, y) plane except at *critical points* where a and b are both zero. Once (2.15) have been solved to find the characteristic projections in the (x, y) plane, we find that u satisfies the decoupled ODE

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = c\big(x(\tau), y(\tau), u\big) \tag{2.16}$$

along each characteristic projection.

For general quasilinear equations, the characteristic projections depend on the solution; the three ODEs (2.7) are coupled and must be solved simultaneously.

Example 2.2 *Solve the PDE*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 1, \tag{2.17}$$

for $u(x,t)$ in $t > 0$, subject to the initial condition $u = x$ on $t = 0$.

The characteristics are given by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = u, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = 1, \tag{2.18}$$

and the initial data may be parametrised by

$$t = 0, \qquad x = s, \qquad u = s \qquad \textit{at } \tau = 0. \tag{2.19}$$

Solving for t first, we see that $t \equiv \tau$ and thus we may replace τ by t henceforth. The initial-value problem for u has the solution

$$u = s + t, \tag{2.20}$$

so that the problem for x becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = s + t, \qquad x = s \quad \textit{when } t = 0, \tag{2.21}$$

whose solution is

$$x = s + st + \tfrac{1}{2}t^2. \tag{2.22}$$

Now we can solve (2.22) for s and substitute it into (2.20) to obtain the solution in explicit *form:*

$$u = \frac{x + t + \frac{1}{2}t^2}{1 + t}. \tag{2.23}$$

Alternative method of solution

The characteristic equations (2.7) may be rearranged to give

$$\frac{\mathrm{d}x}{a(x, y, u)} = \frac{\mathrm{d}y}{b(x, y, u)} = \frac{\mathrm{d}u}{c(x, y, u)}. \tag{2.24}$$

Suppose we can spot two linearly independent first integrals of these ODEs, of the form $f(x, y, u) = \text{const}$ and $g(x, y, u) = \text{const}$. Then the *general solution* of the PDE (2.1) may be written in the implicit form

$$f(x, y, u) = F\big(g(x, y, u)\big), \tag{2.25}$$

where F is an arbitrary function.

Example 2.3 *Return to the problem considered in* Example 2.2*. The characteristic equations may be written as*

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{u} = \frac{\mathrm{d}u}{1} \tag{2.26}$$

and then rearranged to two ODEs:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = u. \tag{2.27}$$

These may be integrated to give

$$u = t + C_1, \qquad x = \tfrac{1}{2}t^2 + C_1 t + C_2, \tag{2.28}$$

where C_1 and C_2 are constants. Our two first integrals are, therefore, $C_1 = f(x, t, u) = u - t$ and $C_2 = g(x, t, u) = x - \frac{1}{2}t^2 - C_1 t = x - \frac{1}{2}t^2 - (u - t)t$. The general solution is found by setting $f = F(g)$, which leads to

$$u = t + F\left(x + \tfrac{1}{2}t^2 - ut\right), \tag{2.29}$$

where F is an arbitrary function. It may readily be verified that any $u(x, t)$ satisfying the implicit equation (2.29) is a solution of (2.17).

The function F is found by fitting the initial data: $u = x$ when $t = 0$ leads to $F(x) \equiv x$, that is $u = t + x + \frac{1}{2}t^2 - ut$, which reproduces the solution (2.23).

This procedure works because the equation $f(x, y, u) = \text{const}$ defines a one-parameter family of surfaces, as does $g(x, y, u) = \text{const}$, and characteristics are lines of intersection between one member from each of these two families. Now, any surface defined by an equation of the form $f = F(g)$ has the property that f is constant whenever g is constant. It follows that such a surface is composed of a family of characteristics, as indicated in Figure 2.1, and is thus a solution surface for the PDE (2.1).

Example 2.4 *For the PDE*

$$yu\frac{\partial u}{\partial x} - xu\frac{\partial u}{\partial y} = x - y, \tag{2.30}$$

the characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = yu, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -xu, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = x - y, \tag{2.31}$$

may be rearranged to give **[Exercise]**

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(x^2 + y^2\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(u^2 + 2x + 2y\right) = 0. \tag{2.32}$$

It follows that the general solution is

$$u^2 = -2x - 2y + F(x^2 + y^2), \tag{2.33}$$

where F is an arbitrary function.

2.3 Cauchy data

Geometric interpretation

The term *Cauchy data* refers to the boundary data that, when applied to a PDE, in principle determine the solution, at least locally. For the first-order quasilinear PDE (2.1), Cauchy data is the prescription of u on some curve Γ in the (x, y) plane, that is we set $u = u_0(s)$ when $x = x_0(s)$ and $y = y_0(s)$ where s parametrises Γ. The combination of the PDE (2.1) and Cauchy data is called the *Cauchy problem*. For the moment we assume that x_0, y_0 and u_0 are smooth functions of s (although there are interesting cases where this is not true, *e.g.*, where Γ has corners) and that there are no values of s for which $x_0'(s) = y_0'(s) = 0$ (which ensures that s is a sensible parameter for Γ).

We have seen that the method of characteristics, outlined in section 2.2, usually allows a solution surface to be constructed in a neighbourhood of Γ. However, the procedure fails if it happens that the initial curve Γ is at any point *tangent* to $(a, b)^T$. If this occurs, then the characteristic projection, instead of propagating away from the initial curve, points along it. Thus the data $u_0(s)$ given on Γ will not in general agree with the ODE (2.16) satisfied by u in the direction (a, b).

Example 2.5 *We return to the PDE* (2.10) *from* Example 2.1*, namely*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1, \tag{2.34}$$

whose general solution is **[Exercise]**

$$u = \frac{x + y}{2} + F(x - y) \tag{2.35}$$

where F is an arbitrary function.

Now we attempt to fit three different sets of initial data and thus determine the function F.

1. $\underline{u = 0 \text{ on } x + y = 0}$
   This is the case considered in Example 2.1*, in which the initial curve is normal to the characteristic projections. The initial data gives $F \equiv 0$, so the solution* (2.14) *is reproduced.*

2. $\underline{u = 0 \text{ on } x = y}$
   This time the initial curve is *a characteristic projection. When we attempt to fit the initial data we find $F(0) \equiv -x$, which is impossible. In this case there is no solution.*

3. $\underline{u = x \text{ on } x = y}$
   Again, the initial curve is a characteristic projection, but this time we have $x \equiv x + F(0)$ so that F can be virtually anything so long as $F(0) = 0$. This is the nongeneric case in which it just happens that the initial data and the characteristic equation (2.16) agree, and the solution is consequently nonunique.

Example 2.5 illustrates the following three possibilities.

1. If Γ is not tangent to a characteristic projection, then there should be a unique solution, at least locally.

2. If Γ is at any point tangent to a characteristic projection, then there is in general no solution.

3. There is, however, an exceptional case in which the data for u specified on Γ agree with the ODE (2.16) satisfied by u along characteristic projections. If this happens then there is a nonunique solution.

Cauchy–Kovalevskaya theorem

A necessary condition for a unique solution u to exist in a neighbourhood of Γ is for the first derivatives of u to be determined on Γ. Differentiation of u_0 and use of the chain rule leads to

$$\frac{\mathrm{d}u_0}{\mathrm{d}s} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x_0}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y_0}{\mathrm{d}s}. \tag{2.36}$$

The partial differential equation (2.1) and (2.36) form a pair of simultaneous equations for $\partial u/\partial x$ and $\partial u/\partial y$ on the curve Γ. We can therefore solve uniquely for these first derivatives so long as the determinant of the system is nonzero, *i.e.*

$$\begin{vmatrix} a & b \\ \dfrac{\mathrm{d}x_0}{\mathrm{d}s} & \dfrac{\mathrm{d}y_0}{\mathrm{d}s} \end{vmatrix} = a\frac{\mathrm{d}y_0}{\mathrm{d}s} - b\frac{\mathrm{d}x_0}{\mathrm{d}s} \neq 0. \tag{2.37}$$

If this condition is satisfied, then both u and its first derivatives are uniquely determined on the curve Γ, which is clearly the first step in extending the solution away from Γ. Notice that the criterion (2.37) is equivalent to requiring Γ not to be tangent to a characteristic projection, as argued above via geometrical reasoning.

When the determinant in (2.37) *is* zero, there is either no solution for $\partial u/\partial x$ and $\partial u/\partial y$ or an infinite number of solutions (this is an instance of the *Fredholm Alternative*). By eliminating between (2.1) and (2.36) we find that

$$\frac{1}{a}\frac{\mathrm{d}x_0}{\mathrm{d}s} = \frac{1}{b}\frac{\mathrm{d}y_0}{\mathrm{d}s} \neq \frac{1}{c}\frac{\mathrm{d}u_0}{\mathrm{d}s} \qquad \Rightarrow \qquad \text{no solution,} \tag{2.38a}$$

$$\frac{1}{a}\frac{\mathrm{d}x_0}{\mathrm{d}s} = \frac{1}{b}\frac{\mathrm{d}y_0}{\mathrm{d}s} = \frac{1}{c}\frac{\mathrm{d}u_0}{\mathrm{d}s} \qquad \Rightarrow \qquad \text{many solutions.} \tag{2.38b}$$

The latter equality is the exceptional case, seen in Example 2.1, in which the variation of u along Γ just happens to agree with the differential equation (2.16) satisfied along the characteristic projection.

The process outlined above can be continued to obtain higher derivatives of u. If $\partial u/\partial x$ is known, for example, then further differentiation with respect to s gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{\mathrm{d}x_0}{\mathrm{d}s} + \frac{\partial^2 u}{\partial x \partial y}\frac{\mathrm{d}y_0}{\mathrm{d}s}, \tag{2.39}$$

while differentiation of (2.1) with respect to x yields

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial a}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial b}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial a}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial b}{\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=\frac{\partial c}{\partial x}+\frac{\partial c}{\partial u}\frac{\partial u}{\partial x}. \tag{2.40}$$

Now we have a pair of simultaneous equations for $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial x\partial y$. The condition for this system to have a unique solution is identical to (2.37).

So long as a, b and c are analytic, so that this differentiation may be continued indefinitely, we can continue this argument to show that the condition (2.37) allows the derivatives of u to all orders to be defined uniquely at Γ. Thus a Taylor series for $u(x, y)$ may be constructed about the initial data curve Γ, and it can be shown that this series has a nonzero radius of convergence. This is the starting point for the proof of the *Cauchy–Kovalevskaya theorem*, which states that (2.1) has a unique analytic solution in some neighbourhood of Γ, provided a, b and c are analytic and satisfy the condition (2.37).

2.4 Domain of definition

Bounded initial curve

In Example 2.2, we are given $u = x$ along the whole x-axis. In general, however, the initial data may only be given on a finite or semi-infinite initial curve Γ. In such cases, the solution is only defined in the region penetrated by characteristic projections that intersect Γ. This region, which is bounded by the characteristic projections that pass through the end points of Γ, is called the *domain of definition*.

Example 2.6 *We return to the problem considered in* Example 2.2*, and suppose the initial data is only given on a finite line segment:* $u = 0$ *when* $t = 0$, $-1/2 < x < 1/2$. *Then the formulae* (2.20) *and* (2.22) *are only determined from the initial data when* s *is in the range* $(-1/2, 1/2)$. *The characteristic projections for this problem are illustrated in* Figure 2.2. *The* domain of definition *is the region between the characteristic projections that pass through* $(-1/2, 0)$ *and* $(1/2, 0)$, *that is, the region*

$$\tfrac{1}{2}(-1+t+t^2) < x < \tfrac{1}{2}(1-t+t^2).$$

Example 2.7 *Solve the partial differential equation*

$$\frac{\partial u}{\partial t}+xu\frac{\partial u}{\partial x}=u \tag{2.41}$$

for $u(x, t)$ *in* $t > 0$, *subject to the initial condition* $u = x$ *when* $t = 0$, $0 < x < 1$.

The characteristics are given by

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=1, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau}=xu, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau}=u, \tag{2.42}$$

and the initial data may be parameterised by $t = 0$, $x = s$, $u = s$, $0 < s < 1$. *The solution in parametric form is*

$$x = s\exp\bigl(s(\mathrm{e}^t-1)\bigr), \qquad u = s\mathrm{e}^t, \qquad 0 < s < 1, \tag{2.43}$$

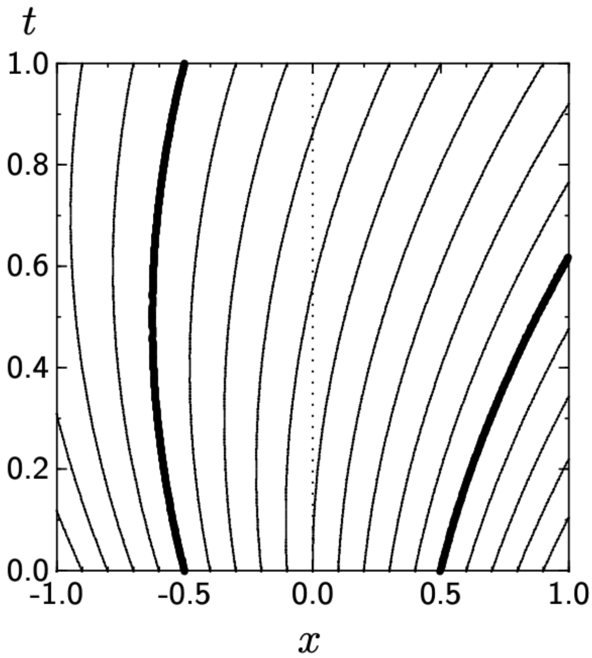

Figure 2.2: The characteristic projections given by equation (2.22). The domain of definition is the region between the two thick black curves.

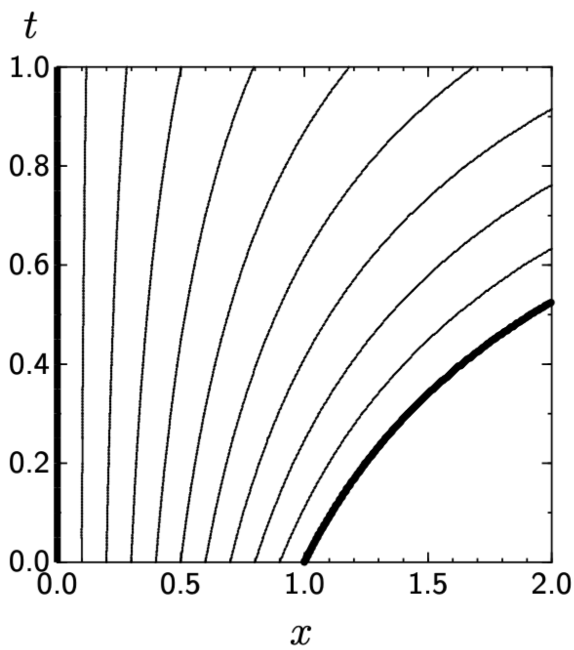

Figure 2.3: The characteristic projections given by equation (2.43). The domain of definition lies between the two thick black curves.

and the characteristic projections are shown in Figure 2.3*. The domain of definition is the region* $0 < x < \exp\left(e^t - 1\right)$.

Note that s *may be eliminated from* (2.43) *to obtain the solution in the* implicit *form*

$$x = u \exp\left(u - t - u e^{-t}\right), \tag{2.44}$$

but there is no explicit *formula for* $u(x,t)$ *in terms of elementary functions.*

Blow-up

The domain in which the solution is defined may be further restricted if the solution develops a singularity, such that the PDE (2.1) ceases to make sense. For example, u may blow up a finite distance from the initial curve Γ. The method of characteristics reduces the partial differential equation (2.1) to the system (2.7) of ordinary differential equations. Since nonlinear ODEs may certainly give rise to solutions that blow up, the same is true of nonlinear PDEs, even those that are semilinear.

Example 2.8 *Consider the equation*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^3, \tag{2.45}$$

subject to $u = y$ *on* $x = 0$, $0 < y < 3$*. The characteristic equations*

$$\frac{dx}{d\tau} = 1, \qquad \frac{dy}{d\tau} = 1, \qquad \frac{du}{d\tau} = u^3, \tag{2.46}$$

and initial data $x = 0$, $y = s$, $u = s$ *on* $\tau = 0$, $0 < s < 3$*, have the solution*

$$x = \tau, \qquad y = s + \tau, \qquad u = \frac{s}{\sqrt{1 - 2s^2\tau}}, \qquad 0 < s < 3. \tag{2.47}$$

We may solve explicitly for u *to obtain*

$$u = \frac{y - x}{\sqrt{1 - 2x(y - x)^2}}. \tag{2.48}$$

This solution blows up on the line $s = 1/\sqrt{2\tau}$*, i.e. on the line*

$$y = x + \frac{1}{\sqrt{2x}}. \tag{2.49}$$

The domain of definition, bounded by this curve and the characteristic projections $y = x$ *and* $y = x + 3$*, is illustrated in* Figure 2.4.

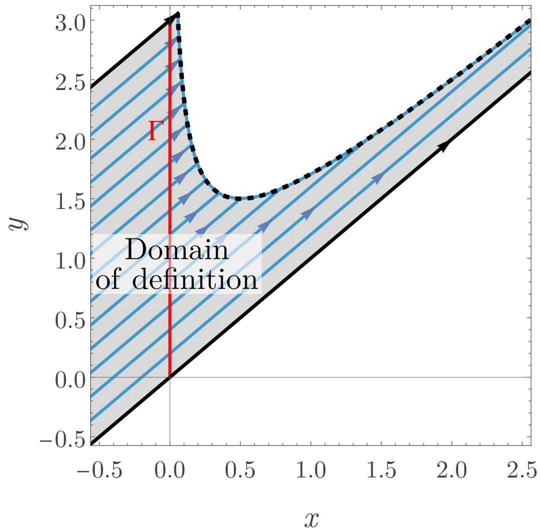

Figure 2.4: Domain of definition for Example 2.8. The initial data curve Γ is shown in red and the bounding characteristic projections ($s = 0$ and $s = 3$) are the thick black lines. The characteristic projections are truncated at $\tau = 1/(2s^2)$ where the solution blows up, indicated by the dashed black curve.

Nonuniqueness

The domain in which the solution is properly defined may also be limited by u ceasing to be a unique function of x and y. Provided the coefficients a, b and c are well-behaved and u does not blow up, the method of characteristics outlined in section 2.2 always allows us in principle to determine the solution in parametric form: $\big(x(s,\tau),\ y(s,\tau),\ u(s,\tau)\big)$. Then u may in principle be found as a function of x and y so long as there is a unique transformation from (s,τ) to (x,y). By the Inverse Function Theorem, a sufficient condition is that the Jacobian of the transformation be finite and nonzero:

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \tau} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial \tau} & \dfrac{\partial y}{\partial s} \end{vmatrix} = a\frac{\partial y}{\partial s} - b\frac{\partial x}{\partial s} \neq 0,\ \infty. \tag{2.50}$$

Note that this reproduces the condition (2.37) for u to be determined in the neighbourhood of Γ.

A unique correspondence between (s,τ) and (x,y) implies that a unique characteristic projection passes through each point in the (x,y) plane. Where J becomes zero, it typically signals that *the characteristic projections start to cross each other*. For semilinear equations, this can only happen at critical points of the phase-plane problem (2.15).

Example 2.9 *Solve the PDE problem*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0, \qquad u = y \text{ on } x = 1,\ 0 < y < 1, \tag{2.51}$$

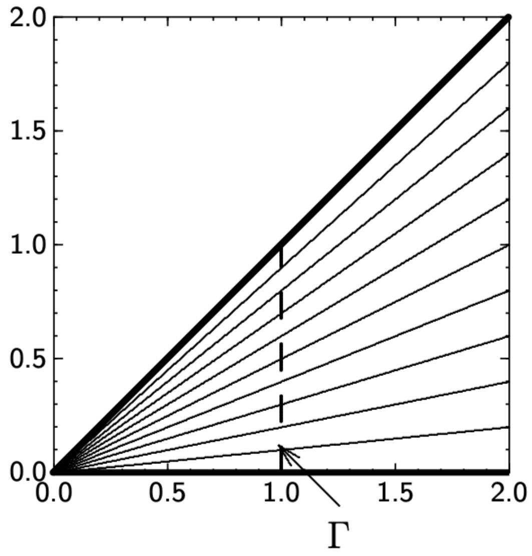

Figure 2.5: The characteristic projections for Example 2.9.

and determine the region in which the solution is defined by the boundary data.

The characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = y, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = 0, \tag{2.52}$$

and initial data $x = 1$, $y = s$, $u = s$, at $\tau = 0$, $0 < s < 1$, lead to

$$x = \mathrm{e}^{\tau}, \qquad y = s\mathrm{e}^{\tau}, \qquad u = s, \qquad 0 < s < 1. \tag{2.53}$$

We can eliminate s and τ to obtain the explicit solution $u = y/x$ in $0 < y/x < 1$. This solution is evidently not uniquely defined at the origin where, as shown in Figure 2.5*, the characteristic projections all cross and where J becomes zero. We cannot continue the solution beyond this point, so the domain of definition is $0 < y/x < 1$, $x > 0$.*

For more general quasilinear equations, the characteristic projections depend on the solution u, so the restriction that they may only cross at critical points no longer holds. The generic situation is that $J = 0$ along curves in the (x, y) plane. On these curves, the solution surface starts to fold over itself such that u ceases to be a single-valued function of x and y. Since u usually represents a physical quantity (such as pressure, temperature or asset price), it cannot be multivalued. Moreover when the solution surface develops a fold, the first derivatives of u become unbounded, so the PDE (2.1) ceases to make sense. For these reasons, we have to cut off the domain of definition along any curves on which J is zero.

Example 2.10 *Solve the PDE problem*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad t > 0, \tag{2.54}$$

$$u = \sin(x), \qquad 0 \le x \le 2\pi, \quad t = 0, \tag{2.55}$$

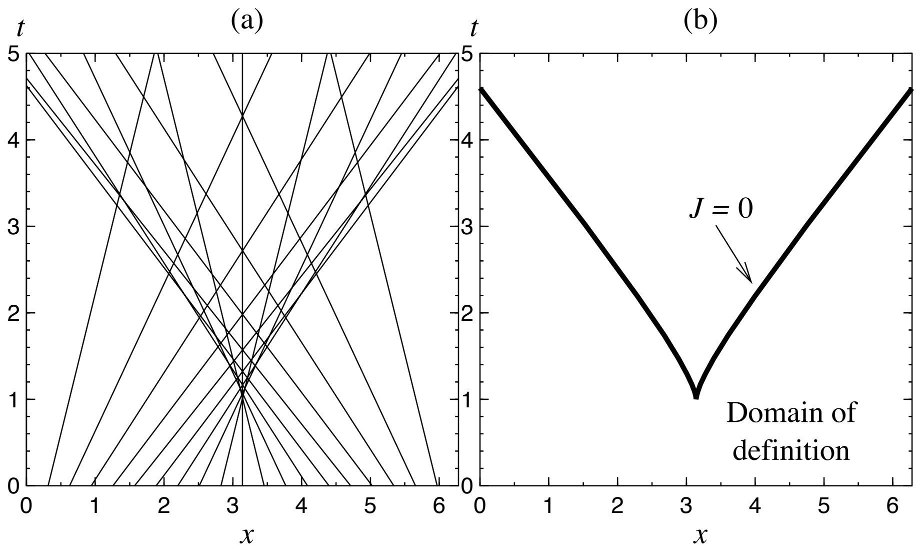

Figure 2.6: (a) The characteristic projections for Example 2.10. (b) The domain of definition, bounded by the curve on which $J = 0$.

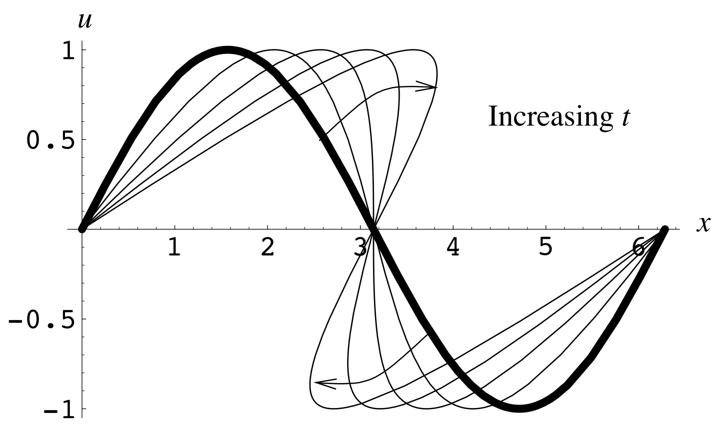

Figure 2.7: The solution $u(x, t)$ from Example 2.10 plotted versus x for $t = 0.5, 1.0, 1.5, 2.0$. The initial solution $u(x, t) = \sin(x)$ is shown as the thicker line.

and find the region in the (x,t) plane where the solution is uniquely defined.

The solution is $u = \sin s$, $t = \tau$, $x = s + \tau \sin s$ in parametric form, or $u = \sin(x - tu)$ in implicit form. The characteristic projections (found by fixing s and varying τ) are straight lines and are illustrated in Figure 2.6(a)*. We can see that they start to cross a finite distance from the initial data $t = 0$. The Jacobian is $\partial(x,y)/\partial(s,\tau) = 1 + t\cos s$. The curve on which $J = 0$ is, therefore, given parametrically by $x = s - \tan(s)$, $t = -\sec(s)$ and is illustrated in* Figure 2.6(b)*. The solution is defined in the region bounded by this curve, the characteristic projections $x = 0$ and $x = 2\pi$, and $t = 0$.*

In Figure 2.7*, we visualise the solution by plotting snapshots of u versus x at different times t. The initial $u = \sin(x)$ steepens as t increases from zero, becoming multi-valued for $t > 1$. When $t = 1$, the Jacobian first reaches zero at $x = \pi$, where $|\partial u/\partial x|$ becomes unbounded.*

A curve on which $J = 0$ may also be viewed as an *envelope* of the characteristic projections. Given a family of curves $F(x, y; \lambda) = 0$, where λ is a scalar parameter, an envelope is a curve that at each point meets one of the family tangentially, and is determined from the simultaneous equations

$$F(x, y; \lambda) = \frac{\partial F}{\partial \lambda}(x, y; \lambda) = 0. \tag{2.56}$$

In Example 2.10, the characteristic projections are given by

$$s + t\sin(s) - x = 0, \tag{2.57}$$

where s is constant along each characteristic projection. Their envelope is found by differentiating with respect to s,

$$1 + t\cos(s) = 0, \tag{2.58}$$

which occurs when the Jacobian $\partial(x,y)/\partial(s,\tau)$ is zero.

3 Weak solutions and shocks for first order PDEs

3.1 Solutions with discontinuous first derivatives

In a so-called *classical* solution, u is smooth so that its first derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuous, and the partial differential equation (2.1) is satisfied at each point in the (x, y) plane where the solution is defined. Now we consider the possibility that the first derivatives of u might be discontinuous across some curve C in the (x, y) plane. The idea is to patch together classical solutions on either side of C although, on C itself, $\partial u/\partial x$ and $\partial u/\partial y$ are not well defined.

Suppose that C is parameterised by $x = x(\xi), y = y(\xi)$. We use the superscript $^{\pm}$ to denote the solution on either side of C. By differentiating both solutions along C we find

$$\frac{\mathrm{d}u^+}{\mathrm{d}\xi} = \frac{\partial u^+}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\xi} + \frac{\partial u^+}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\xi}, \qquad \frac{\mathrm{d}u^-}{\mathrm{d}\xi} = \frac{\partial u^-}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\xi} + \frac{\partial u^-}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\xi}. \tag{3.1}$$

Although the first derivatives of u are discontinuous across C, u itself is assumed to be continuous, so $u^+ = u^-$. It follows that $\mathrm{d}u^+/\mathrm{d}\xi = \mathrm{d}u^-/\mathrm{d}\xi$ and therefore

$$\frac{\mathrm{d}x}{\mathrm{d}\xi}\left[\frac{\partial u}{\partial x}\right]_-^+ + \frac{\mathrm{d}y}{\mathrm{d}\xi}\left[\frac{\partial u}{\partial y}\right]_-^+ = 0, \tag{3.2}$$

where $[f]_-^+ = f^+ - f^-$ represents the jump across C.

Since $u^{\pm}$ are both classical solutions of the partial differential equation (2.1), we have

$$a^{\pm}\frac{\partial u^{\pm}}{\partial x} + b^{\pm}\frac{\partial u^{\pm}}{\partial y} = c^{\pm}. \tag{3.3}$$

Recall that u is continuous across C and, therefore, so are a, b and c: $a^+ = a^- = a$ and so forth. By subtracting the equations on either side of c, we thus find

$$a\left[\frac{\partial u}{\partial x}\right]_-^+ + b\left[\frac{\partial u}{\partial y}\right]_-^+ = 0. \tag{3.4}$$

In (3.2) and (3.4), we have a homogeneous linear system for $[\partial u/\partial x]_-^+$ and $[\partial u/\partial y]_-^+$, which must therefore be zero unless the determinant of the system is zero. In other words, the first derivatives must be continuous unless

$$b\frac{\mathrm{d}x}{\mathrm{d}\xi} - a\frac{\mathrm{d}y}{\mathrm{d}\xi} = 0. \tag{3.5}$$

This is identical to the equation for a characteristic projection. Thus, the first derivatives of u may only be discontinuous across a characteristic projection. Indeed, this may be used as an alternative definition of what a characteristic projection is: it is a curve across which the first derivatives of u may be discontinuous.

Example 3.1 *Consider the partial differential equation*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1, \tag{3.6}$$

subject to the boundary condition

$$u(x,0) = \begin{cases} 0 & x < 0, \\ x & x \geq 0. \end{cases} \tag{3.7}$$

The characteristic equations

$$\mathrm{d}x = \mathrm{d}y = \mathrm{d}u \tag{3.8}$$

give the general solution

$$u = x + f(x - y). \tag{3.9}$$

The boundary condition gives

$$f(s) = \begin{cases} -s & s < 0, \\ 0 & s \geq 0, \end{cases} \tag{3.10}$$

and the solution is therefore

$$u = \begin{cases} y & x < y, \\ x & x \geq y. \end{cases} \tag{3.11}$$

Notice that the first derivatives of u are discontinuous across the characteristic $y = x$ that passes through the origin, but u itself is continuous.

3.2 Weak solutions

In the previous section we showed that classical solutions may be patched together in such a way that the first derivatives of u are discontinuous across a characteristic projection. Now we attempt to do the same for solutions in which u itself has a jump across some curve in the (x, y) plane. Selecting a unique solution is inherently more problematic when u is discontinuous, as the following example illustrates.

Example 3.2 *Consider the partial differential equation*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{3.12}$$

subject to

$$u(0,y) = \begin{cases} 0 & y < 0, \\ 1 & y \geq 0. \end{cases} \tag{3.13}$$

Now we try to find a curve $y = f(x)$ such that the solution is

$$u = \begin{cases} 0 & y < f(x), \\ 1 & y \geq f(x). \end{cases} \tag{3.14}$$

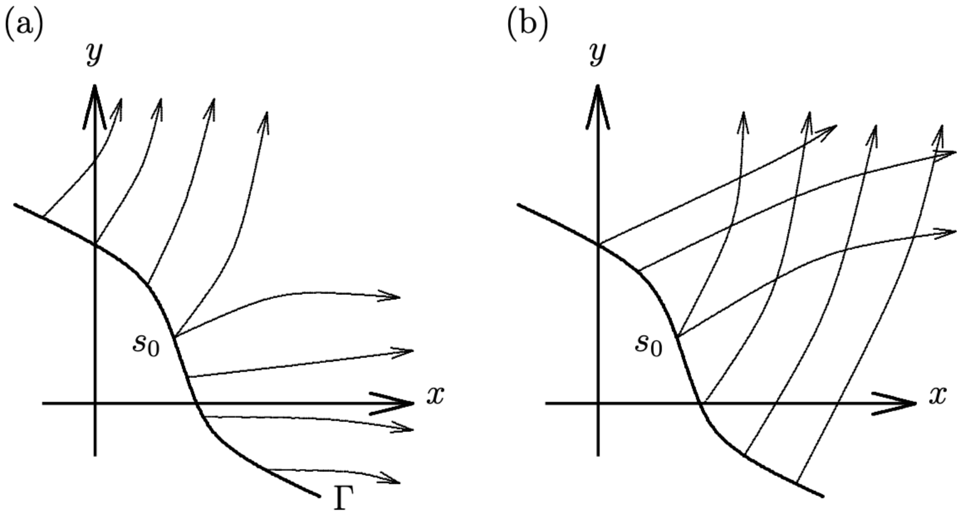

Figure 3.1: Schematics of the possible characteristic projections near a discontinuity in u at the point $s = s_0$ on Γ.

This clearly satisfies the partial differential equation, everywhere except on $y = f(x)$, and the initial condition so long as $f(0) = 0$. Otherwise there does not appear to be any unique way of choosing f.

Note, though, that u is constant along each of the characteristic projections which, for this linear partial differential equation, are given by $y = x + const$. We therefore have $u = 0$ on the characteristic projections leaving $x = 0, y < 0$, and $u = 1$ on those that come from $x = 0$, $y \geq 0$. This implies that $u = 0$ in $y < x$ and $u = 1$ in $y \geq x$, i.e., that the correct choice is $f = x$.

Example 3.2 illustrates a plausible way of selecting a unique solution for semilinear equations, for which the characteristic projections are determined independently of the solution. Suppose the initial data have a discontinuity at some point $s = s_0$ along Γ. On either side of $s = s_0$, a unique solution is determined on each characteristic projection leaving Γ. This suggests that the discontinuity in u simply propagates along the characteristic projection through s_0 (*e.g.*, the line $y = x$ in Example 3.2).

Unfortunately this approach does not work for quasilinear equations. The problem is that the characteristic projections cannot be found in advance: they depend on the solution. If u has a discontinuity at $s = s_0$, then so does the slope of the characteristic projections leaving Γ either side of s_0. Two possible outcomes are illustrated schematically in Figure 3.1. In diagram (a), the slopes of the characteristic projections leaving either side of $s = s_0$ are such that they diverge. There is therefore a region between the limiting characteristic projections in which we do not know the solution. In Figure 3.1(b), the characteristic projections from either side of $s = s_0$ cross, so there is a region in which they overlap and in which there are therefore two possible solutions for u.

To resolve these difficulties, we now reformulate the problem in such a way that it makes sense even if u is discontinuous. The idea is to turn the partial differential equation

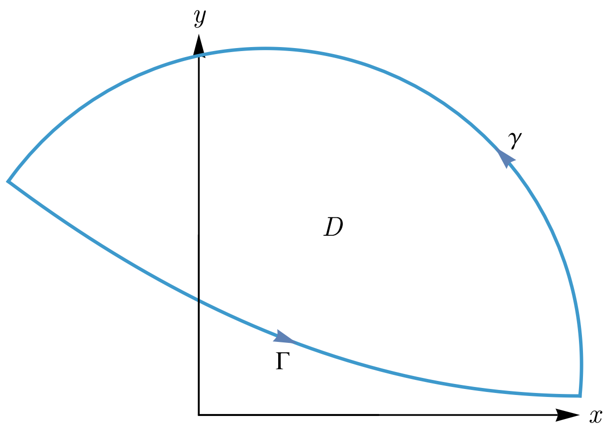

Figure 3.2: Schematic showing the boundary curve Γ, closed by a curve γ to enclose a region D.

(2.1) into an integral equation since, although a discontinuous function does not have well-defined derivatives, it may readily be integrated. The first step is to write the partial differential equation in so-called *conservation form*

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = R, \tag{3.15}$$

where P, Q and R are functions of x, y and u.

Example 3.3 *The semilinear equation*

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u) \tag{3.16}$$

may be rewritten as

$$\frac{\partial}{\partial x}(au) - u\frac{\partial a}{\partial x} + \frac{\partial}{\partial y}(bu) - u\frac{\partial b}{\partial y} = c, \tag{3.17}$$

and thus in conservation form (3.15), with $P = au$, $Q = bu$, $R = c + u\partial a/\partial x + u\partial b/\partial y$.

Example 3.4 *The inviscid Burgers equation*

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0 \tag{3.18}$$

may be written in conservation form with $P = u$, $Q = u^2/2$, $R = 0$.

Suppose that u is given as usual on some curve Γ in the (x, y) plane, and we wish to determine the solution for u in some domain D, formed by closing Γ with an additional curve γ, as illustrated in Figure 3.2. Now, we multiply (3.15) through by a continuously differentiable *test function* ψ, assumed to vanish on γ, to obtain

$$\frac{\partial P}{\partial x}\psi + \frac{\partial Q}{\partial y}\psi = R\psi, \tag{3.19}$$

which may be rewritten in the form

$$\frac{\partial}{\partial x}(P\psi) + \frac{\partial}{\partial y}(Q\psi) = P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi. \tag{3.20}$$

Now we integrate both sides of this equation over the region D:

$$\iint_D \frac{\partial}{\partial x}(P\psi) + \frac{\partial}{\partial y}(Q\psi)\,\mathrm{d}x\mathrm{d}y = \iint_D P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi\,\mathrm{d}x\mathrm{d}y. \tag{3.21}$$

We apply Green's theorem to the left-hand side and use the fact that ψ is assumed to vanish on γ:

$$\int_\Gamma \psi\,(P\,\mathrm{d}y - Q\,\mathrm{d}x) = \iint_D P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi\,\mathrm{d}x\mathrm{d}y. \tag{3.22}$$

A so-called *weak solution* of the partial differential equation (3.15) is a function u that satisfies (3.22) for *any* suitably differentiable test function ψ. If u is continuously differentiable, then the steps that led from (3.15) to (3.22) may be reversed. Thus, any continuously differentiable u satisfying (3.22) is a classical solution of (3.15). However, (3.22) makes sense when u is non-differentiable or even discontinuous, while the original partial differential equation (3.15) does not. This is because, by using Green's theorem, we have removed the need to differentiate u: only the test function is differentiated.

3.3 Shocks

Now we show how the weak formulation (3.22) allows us to make sense of solutions in which u is discontinuous across some curve C in the (x, y) plane. Such a curve is called a *shock*; this name arises from the occurrence of such solutions in gas dynamics. If the shock is initiated on Γ, then it will divide our integration domain D into two regions D_1 and D_2, as shown in Figure 3.3. Thus the area integral on the right-hand side of (3.22) can be written as

$$\begin{aligned}\iint_D P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi\,\mathrm{d}x\mathrm{d}y = \iint_{D_1} P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi\,\mathrm{d}x\mathrm{d}y \\ + \iint_{D_2} P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.23}$$

Now, inside each of D_1 and D_2, the solution is supposed to be continuously differentiable, so we can write

$$\begin{aligned}&\iint_{D_i} P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi\,\mathrm{d}x\mathrm{d}y \\ &\qquad = \iint_{D_i} \frac{\partial}{\partial x}(P\psi) + \frac{\partial}{\partial y}(Q\psi) + \psi\left(R - \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.24}$$

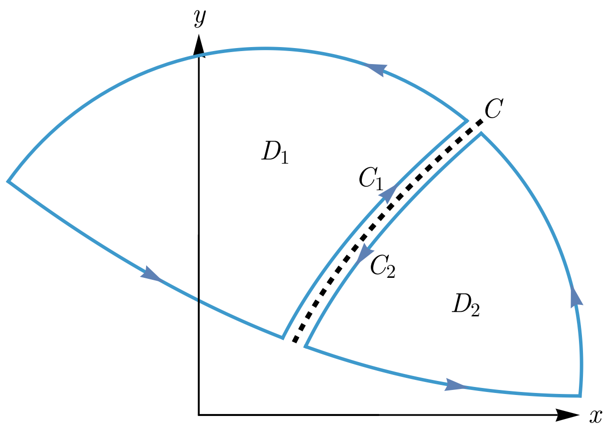

Figure 3.3: Schematic showing the shock C dividing D into two regions D_1 and D_2. The integration paths on either side of C are denoted C_1 and C_2.

where $i = 1$ or 2. The term in brackets is identically zero, because of (3.15), and Green's theorem may be applied to the remainder to give

$$\iint_{D_i} P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi \,\mathrm{d}x\mathrm{d}y = \oint_{\partial D_i} \psi(P\,\mathrm{d}y - Q\,\mathrm{d}x). \tag{3.25}$$

As indicated in Figure 3.3, the integration curves ∂D_1 and ∂D_2 comprise sections of Γ and γ joined to curves C_1 and C_2 adjacent to the shock on either side. When the two integrals are summed, the result is

$$\oint_{\partial D_1+\partial D_2} \psi(P\,\mathrm{d}y - Q\,\mathrm{d}x) = \oint_{\Gamma+C_1-C_2} \psi(P\,\mathrm{d}y - Q\,\mathrm{d}x), \tag{3.26}$$

since ψ is zero on γ. Notice the difference in sign because C_1 and C_2 are traversed in different directions. The right-hand side of (3.22) may therefore be written in the form

$$\iint_{D} P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y} + R\psi \,\mathrm{d}x\mathrm{d}y = \oint_{\Gamma+C_1-C_2} \psi(P\,\mathrm{d}y - Q\,\mathrm{d}x). \tag{3.27}$$

The integral along Γ cancels with the left-hand side of (3.22), and we are left with

$$\int_{C_1} \psi(P\,\mathrm{d}y - Q\,\mathrm{d}x) - \int_{C_2} \psi(P\,\mathrm{d}y - Q\,\mathrm{d}x) = 0, \tag{3.28}$$

or

$$\int_{C} \psi([P]_-^+\,\mathrm{d}y - [Q]_-^+\,\mathrm{d}x) = 0, \tag{3.29}$$

where $[F]_-^+$ denotes the jump in F across the shock C.

Since this relation must hold for any (suitably smooth) test function ψ, the term in brackets must be identically zero and we therefore obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{[Q]_-^+}{[P]_-^+}. \tag{3.30}$$

This so-called *Rankine–Hugoniot* condition determines the slope of the shock in terms of the discontinuities in P and Q across it.

For semilinear equations, as shown in Example 3.3, we have $P = a(x,y)u$, $Q = b(x,y)u$ so (3.30) reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b[u]_-^+}{a[u]_-^+} = \frac{b}{a}. \tag{3.31}$$

This is identical to the slope of characteristic projections so, for semilinear equations, solutions may only be discontinuous across characteristic projections. Thus the solution obtained in Example 3.2 *is* a valid weak solution. For general quasilinear equations, shocks are different from characteristic projections.

Example 3.5 *Find a weak solution of the problem*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad t > 0, \tag{3.32}$$

$$u = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases} \qquad t = 0. \tag{3.33}$$

We look for a solution in which u is piecewise constant: $u = 1$ for $x < X(t)$ and $u = 0$ for $x \geq X(t)$. Our only remaining task is to determine the position $X(t)$ of the shock. In conservation form we have

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) = 0, \tag{3.34}$$

so the Rankine–Hugoniot condition (3.30) gives

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\left[\frac{1}{2}u^2\right]_-^+}{[u]_-^+} = \frac{u_- + u_+}{2} = \frac{1}{2}. \tag{3.35}$$

Hence $X(t) = t/2$ and the solution is

$$u = \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x \geq \frac{1}{2}t. \end{cases} \tag{3.36}$$

The characteristic projections have slope given by $\mathrm{d}x/\mathrm{d}t = u$. *They are illustrated in* Figure 3.4*; note that the characteristic projections travel* into *the shock from either side.*

The great advantage of allowing weak solutions in which u is discontinuous is that we can eliminate multiple-valued solutions like that shown in Figure 2.7. As soon as the Jacobian becomes zero, we insert a shock that prevents the characteristic projections from crossing. In principle, the Rankine–Hugoniot condition (3.30) determines the position of the shock, and we can find the solution on either side by the usual characteristic methods.

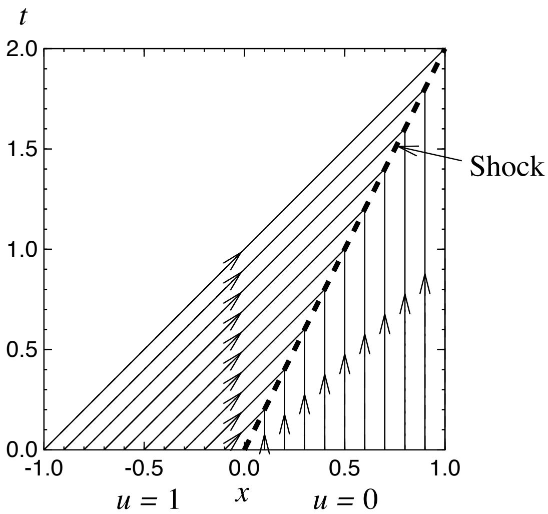

Figure 3.4: The characteristic projections and shock for Example 3.5.

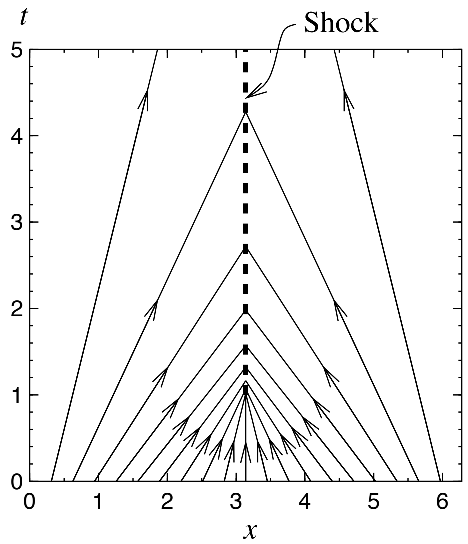

Figure 3.5: Characteristic projections and shock for Example 3.6.

Example 3.6 *We return to the problem from* Example 2.10*, whose solution, in parametric form, is* $x = s + t\sin(s)$*,* $u = \sin(s)$*. Recall that the Jacobian first reaches zero at* $x = \pi$*,* $t = 1$*, and for* $t > 1$ *the solution is multivalued. We can prevent this from occurring by, instead, allowing the solution to be discontinuous, i.e. by initiating a shock at* $x = \pi$*,* $t = 1$*.*

Suppose the shock is at $x = X(t)$*, and that the characteristic projections entering the shock from either side have parameters* $s_\pm(t)$*, so that*

$$X = s_+ + t\sin(s_+) = s_- + t\sin(s_-). \tag{3.37}$$

We also have the Rankine–Hugoniot condition

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{u_+ + u_-}{2} = \frac{\sin(s_+) + \sin(s_-)}{2} \tag{3.38}$$

and $X(1) = \pi$*. The system* (3.37), (3.38) *may be solved to give*

$$t = \frac{\pi - s_+(t)}{\sin[s_+(t)]} = \frac{\pi - s_-(t)}{\sin[s_-(t)]}, \qquad\qquad X(t) \equiv \pi. \tag{3.39}$$

The characteristic projections are shown in Figure 3.5*, along with the shock at* $x = \pi$*,* $t > 1$*. Compare with* Figure 2.6(a) *and notice that, by introducing a shock, we prevent the characteristic projections from crossing and thus keep the solution single-valued.*

Snapshots of the solution $u(x, t)$ *are plotted versus* x *in* Figure 3.6 *for* $t = 1.0,\ 1.1,\ 1.2,\ 1.3$*. Notice the growing discontinuity in* u *when* $t > 1$*.*

3.4 Nonuniqueness of weak solutions

We have shown how one may construct weak solutions by patching together classical solutions and applying the Rankine–Hugoniot condition (3.30) across any shocks where u is discontinuous. This allows us to avoid the unphysical possibility of u becoming a multiple-valued function. Unfortunately (3.30) is in general *not* enough to determine the solution uniquely. There are two further problems to be addressed.

Problem 1

The Rankine–Hugoniot condition (3.30) depends on the functions P and Q that appear in the conservation form (3.15). However, there may be many different ways of expressing the same PDE in conservation form, and a different conservation form leads to a different Rankine–Hugoniot condition.

Example 3.7 *The PDE*

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0 \tag{3.40}$$

may be written in many different conservation forms, including

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left(\tfrac{1}{2}u^2\right) = 0 \qquad \textit{or} \qquad \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) + \frac{\partial}{\partial y}\left(\tfrac{1}{3}u^3\right) = 0. \tag{3.41}$$

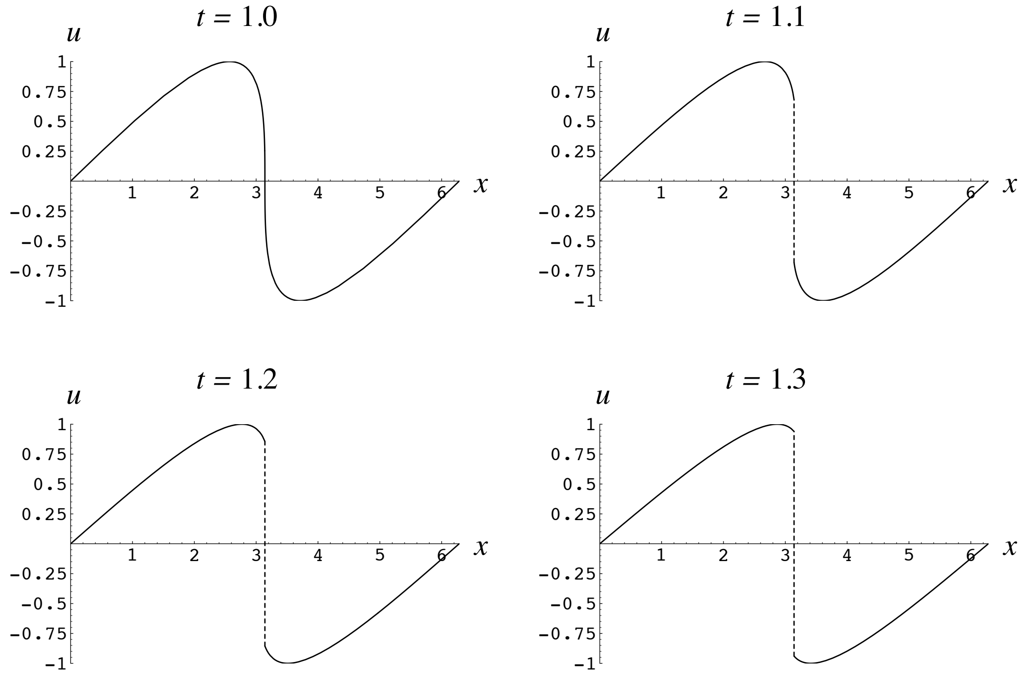

Figure 3.6: The solution $u(x,t)$ from Example 3.6, plotted versus x for $t = 1.0, 1.1, 1.2, 1.3$.

These lead to the two different Rankine–Hugoniot conditions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left[\frac{1}{2}u^2\right]_-^+}{[u]_-^+} = \frac{u_+ + u_-}{2} \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left[\frac{1}{3}u^3\right]_-^+}{\left[\frac{1}{2}u^2\right]_-^+} = \frac{2\left(u_+^2 + u_+u_- + u_-^2\right)}{3\left(u_+ + u_-\right)}, \tag{3.42}$$

respectively.

Solution

The key is to make sure that the functions P and Q in the conservation law correspond to real physical quantities (*e.g.*, mass, momentum, energy, *etc.*). Then the Rankine–Hugoniot condition (3.30) ensures that these are conserved across shocks.

Example 3.8 *Consider the following model for traffic flow. Let $u(x,t)$ be the density of cars on a stretch of road, where x is distance along the road and t is time. Suppose that the speed v of each car is related to the local density by $v = (1-u)$. Then the flux of cars is given by $uv = u(1-u)$ and the equation representing conservation of cars is*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(u(1-u)\big) = 0. \tag{3.43}$$

The Rankine–Hugoniot condition associated with this conservation law is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\left[u(1-u)\right]_-^+}{[u]_-^+}, \tag{3.44}$$

which may be rearranged to give

$$\left[u\left(\frac{\mathrm{d}x}{\mathrm{d}t}-(1-u)\right)\right]_-^+=0. \tag{3.45}$$

This equation ensures that the rate at which cars enter a shock from one side equals the rate at which they exit the other.

If (3.43) is rewritten as

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\right)+\frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2-\tfrac{2}{3}u^3\right)=0, \tag{3.46}$$

then the corresponding Rankine–Hugoniot condition has no physical interpretation, and the net flux of cars would not *be preserved across a shock.*

Problem 2

The strategy described above selects a particular weak formulation (*i.e.*, a particular choice of P and Q) from the many that may be available. The second problem is that this weak formulation may still admit many solutions.

Example 3.9 *The problem*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0, \qquad u(x,0)=\begin{cases}0 & x<0\\ 1 & x\geq 0\end{cases} \tag{3.47}$$

has (at least) two *possible weak solutions, namely*

$$u_1=\begin{cases}0 & x<0\\ x/t & 0\leq x\leq t\\ 1 & x>t\end{cases} \qquad \text{and} \qquad u_2=\begin{cases}0 & x<t/2\\ 1 & x\geq t/2.\end{cases} \tag{3.48}$$

It is readily verified that 0, x/t and 1 all satisfy the PDE, and the discontinuities in the first derivatives of u_1 occur across characteristic projections, so u_1 is a valid solution. Similarly, u_2 clearly satisfies the PDE on either side of the shock at $x=t/2$, and the Rankine–Hugoniot condition $\mathrm{d}x/\mathrm{d}t=\frac{1}{2}$ is satisfied, so u_2 is also a valid weak solution.

There are several ways around the problem illustrated in Example 3.9, including the following.

1. <u>Entropy</u>
   Here we pose a function of u that must increase across a shock.

2. <u>Viscosity</u>
   This involves introducing a higher derivative to *regularise* the equation (*i.e.*, smooth out the shock).

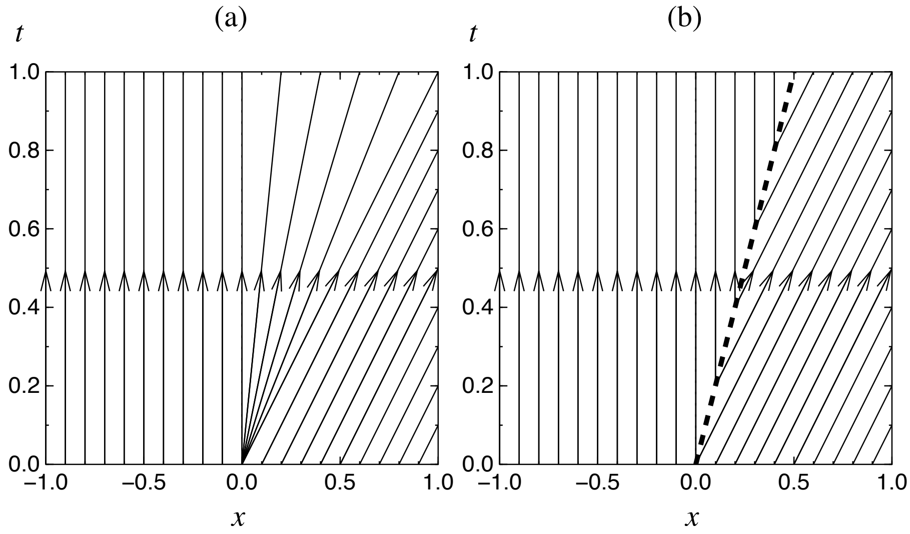

Figure 3.7: Characteristic projections for the two possible solutions, (a) u_1 and (b) u_2, to Example 3.9.

Example 3.10 *As shown in* Example 3.9*, the equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3.49}$$

may have multiple weak solutions. However, the Burgers Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2} \tag{3.50}$$

has a unique solution, given suitable initial and boundary data, for any positive value of ϵ*. So a unique weak solution of* (3.49) *may be selected by solving* (3.50) *and then letting* $\epsilon \to 0$.

3. Causality
   This means ensuring that information travels *into* the shock, not out of it.

It may be shown that each of these approaches results in the same unique weak solution being selected. We only use the third option, which does not require us to bring any more physics into the problem, apart from a recognition that one variable (t) represents time. Information travels along characteristic projections, starting from the initial data, in the direction of increasing t. A shock solution is *causal* only if this information travels *into* the shock from either side (as in Figures 3.4 and 3.5). By disallowing shocks that do not satisfy this condition, we narrow down the possible weak solutions to just one.

Example 3.11 *The characteristic projections for the two possible solution* u_1 *and* u_2 *to* Example 3.9 *are shown in* Figure 3.7*. Notice that the shock solution in* Figure 3.7(b) *has characteristic projections travelling* out *of the shock. Thus this solution is not causal and must be discarded. The solution* u_1 *displayed in* Figure 3.7(a) *is the correct weak solution.*

In Example 3.9, the characteristic projections have slope $\mathrm{d}x/\mathrm{d}t = u$, while the slope of the shock is given by the Rankine–Hugoniot condition $\mathrm{d}x/\mathrm{d}t = \frac{1}{2}(u_+ + u_-)$. Thus the characteristic projections travel into the shock from either side so long as

$$u_- > \tfrac{1}{2}(u_+ + u_-) < u_+, \tag{3.51}$$

which may be rearranged to give

$$u_- > u_+. \tag{3.52}$$

This is the condition for a shock to be causal: u must be greater behind the shock than it is ahead. This is why the shock solution in Example 3.5 is acceptable while that in Example 3.9 is not.

3.5 Riemann problems

Examples 3.2, 3.5 and 3.9 are instances of the so-called *Riemann problem*, which is a Cauchy problem in which the Cauchy data has a discontinuity, and we are interested in how the initial discontinuity is (or isn't) propagated by the solution. For a first-order PDE that is quasilinear but not semilinear, the general situation is illustrated in Figure 3.1. Because the characteristic equations depend on the solution u, a discontinuity in u on the boundary curve Γ causes a discontinuity in the characteristic gradients. In the case shown in Figure 3.1(b), this discontinuity causes the characteristic projections to run into each other, and the resulting apparent nonuniqueness in the solution can be removed by inserting a shock, as described above. However, we have not yet fully resolved the situation shown in Figure 3.1(a), where there is a gap between the characteristic projections emanating from the point of discontinuity and thus a region in which the solution seems to be undetermined.

We illustrate how the solution in such a region may be filled in by showing how the continuous solution u_1 from Example 3.9 can be constructed.

Example 3.12 *Solve the PDE*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{3.53a}$$

for $t > 0$, subject to

$$u(x,0) = \mathrm{H}(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \tag{3.53b}$$

The characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = 0, \tag{3.54}$$

so u is constant along characteristic projections, which are therefore straight lines, with slope u. Since $u(x,0) = 0$ for $x < 0$, the characteristic projections leaving $t = 0$ with $x < 0$ are the straight lines $x =$ constant. Similarly, the characteristic projections leaving $t = 0$ with $x > 0$ are the lines $x - t =$ constant, as shown in Figure 3.8(a). *We thus have $u = 0$ for $x < 0$ and $u = 1$ for $x \geq t$, but so far u remains undetermined in $0 \leq x < t$.*

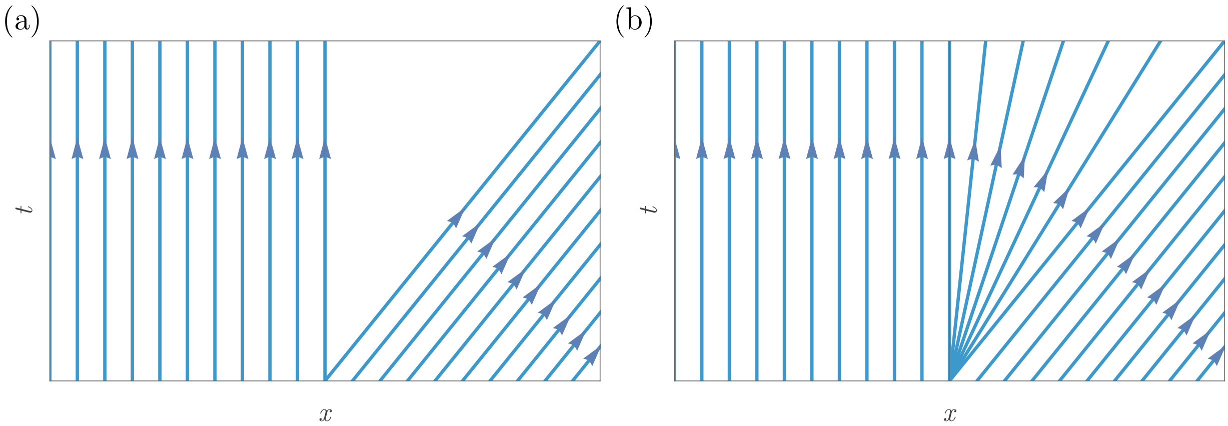

Figure 3.8: The (x, t)-plane for the problem (3.53).

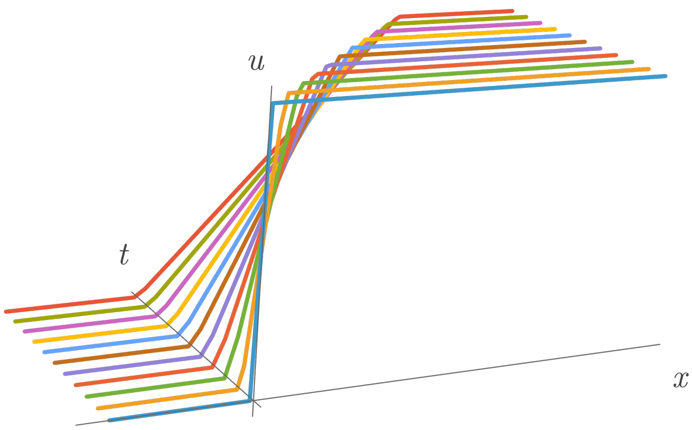

Figure 3.9: The solution $u(x, t)$ given by (3.56).

Since the characteristic projections are diverging from each other, rather than overlapping, it is impossible to insert a shock that satisfies the causality condition discussed in Section 3.4*. We therefore cannot allow the characteristic projections to cross and, since we know they must be straight lines, the only remaining option is for the "missing" characteristic projections to fan out from the origin, as shown in* Figure 3.8(b)*. These straight lines through the origin are of the form*

$$\frac{x}{t} = \textit{constant} = u, \tag{3.55}$$

because we know that $\mathrm{d}x/\mathrm{d}t = u$ *on each characteristic projection. By incorporating this new information into the solution constructed so far, we obtain the full solution*

$$u(x,t) = \begin{cases} 0 & x < 0, \\ \dfrac{x}{t} & 0 \le x < t, \\ 1 & x \ge t, \end{cases} \tag{3.56}$$

as stated in Example 3.9*.*

The structure shown in Figure 3.8(b), with the characteristic projections radiating out from the point of discontinuity in the initial data, is known as an *expansion wave* or an *expansion fan*. The corresponding behaviour of u is shown in Figure 3.9. We see that the initial discontinuity in u instantaneously disappears for $t > 0$, and the remaining slope discontinuities in the solution spread out along characteristics.

The strategy demonstrated here can always be used to "fill in the gap" in a situation like the one shown in Figure 3.1(a). By allowing u to vary between its two values on either side of the discontinuity, we can generate a family of characteristic projections that continuously connect with the characteristic projections on either side of the gap. We illustrate the approach with a more complicated example in which discontinuous initial data can generate either a shock or an expansion wave.

Example 3.13 *Solve the PDE*

$$u_t + (1+x)uu_x = 1 - u \tag{3.57}$$

for $t > 0$*, subject to each of the two initial conditions:*

$$u(x,0) = \mathrm{H}(-x) = \begin{cases} 1 & x \le 0, \\ 0 & x > 0, \end{cases} \tag{3.58a}$$

$$u(x,0) = \mathrm{H}(x) = \begin{cases} 0 & x < 0, \\ 1 & x \ge 0. \end{cases} \tag{3.58b}$$

The characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1+x)u, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = 1 - u. \tag{3.59}$$

The initial data are given parametrically by $t = 0$*,* $x = s$*,* $u = u_0(s)$*, where* u_0 *is either* 0 *or* 1*.*

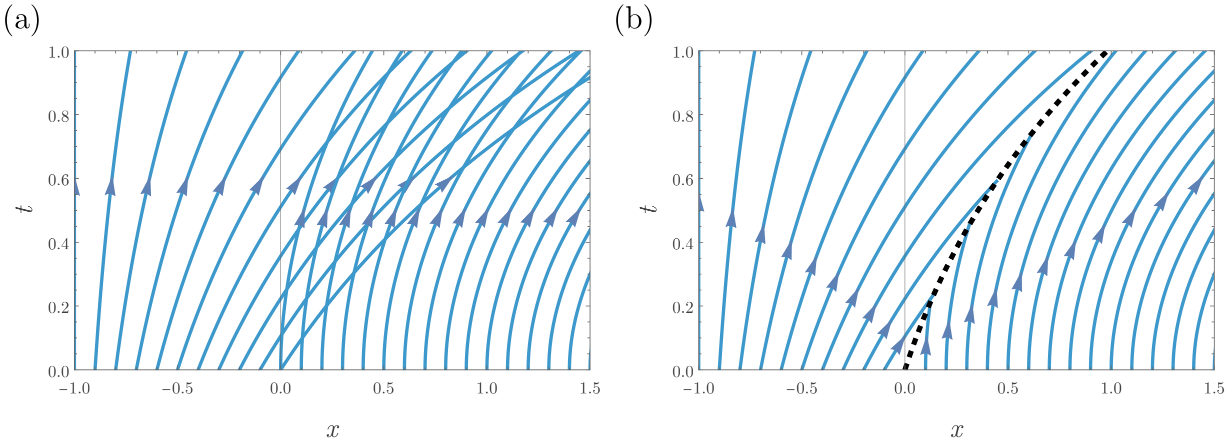

Figure 3.10: The (x, t)-plane for equation (3.57) with initial data (3.58a).

First considering the initial condition (3.58a), we find that the solution along characteristics is given by

$$u = 1, \qquad x = -1 + (s+1)\mathrm{e}^t \qquad \textit{for} \quad s < 0, \tag{3.60a}$$

$$u = 1 - \mathrm{e}^{-t}, \qquad x = -1 + (s+1)\exp\left(t - (1 - \mathrm{e}^{-t})\right) \qquad \textit{for} \quad s > 0. \tag{3.60b}$$

The characteristic projections are plotted in Figure 3.10(a). *Evaluating the characteristic projections starting at $s = 0$, we find that*

$$x = -1 + \mathrm{e}^t \qquad \textit{at} \quad s = 0-, \tag{3.61a}$$

$$x = -1 + \exp\left(t - (1 - \mathrm{e}^{-t})\right) < -1 + \mathrm{e}^t \qquad \textit{at} \quad s = 0+, \tag{3.61b}$$

so the leading characteristic projection from $s < 0$ overtakes the trailing characteristic projection from $s > 0$, causing the solution to be multivalued in the resulting overlap region.

We can construct a single-valued solution by introducing a shock. Assuming that the relevant conservation form for the PDE is $u_t + \left((1+x)u^2/2\right)_x = 1 - u + u^2/2$, we can write down the Rankine–Hugoniot condition

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{(1+X)\left[u^2\right]_-^+}{2\left[u\right]_-^+} = \frac{(1+X)(u_+ + u_-)}{2} = (1+X)\left(1 - \frac{1}{2}\,\mathrm{e}^{-t}\right). \tag{3.62}$$

We integrate (3.62), subject to $X(0) = 0$, to get the shock location as

$$X(t) = -1 + \exp\left(t - \frac{1}{2}\left(1 - \mathrm{e}^{-t}\right)\right), \tag{3.63}$$

which lies between the two limiting characteristic projections (3.61) and thus prevents them from intersecting, as shown in Figure 3.10(b). *The resulting discontinuous but single-valued solution is thus given by*

$$u(x,t) = \begin{cases} 1 & x < X(t), \\ 1 - \mathrm{e}^{-t} & x > X(t). \end{cases} \tag{3.64}$$

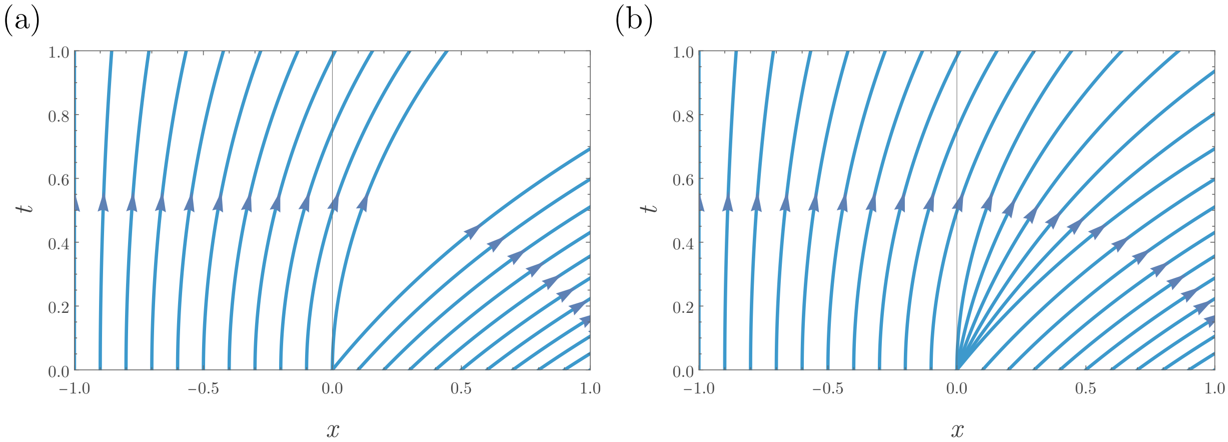

Figure 3.11: The (x,t)-plane for equation (3.57) with initial data (3.58b).

For the second initial condition (3.58b)*, we can reuse the solution* (3.60) *and simply switch the ranges of* s*, i.e.,*

$$u = 1 - \mathrm{e}^{-t}, \qquad x = -1 + (s+1)\exp\left(t - (1 - \mathrm{e}^{-t})\right) \qquad \text{for} \quad s < 0, \tag{3.65a}$$

$$u = 1, \qquad x = -1 + (s+1)\mathrm{e}^{t} \qquad \text{for} \quad s > 0. \tag{3.65b}$$

Now the characteristic projections either side of $s = 0$ *diverge instead of overlapping, so there is a gap in which the solution is thus far undetermined, as shown in* Figure 3.11(a)*. From the characteristic equation* (3.59b)*, we know that*

$$u = 1 - A\mathrm{e}^{-t}, \tag{3.66a}$$

where A *is constant along each characteristic. We then integrate* (3.59a)*, and prevent characteristic projections from overlapping by imposing* $x = 0$ *at* $t = 0$*, to obtain*

$$x = -1 + \exp\left(t - A(1 - \mathrm{e}^{-t})\right) \tag{3.66b}$$

As A *varies between* 0 *and* 1*, the parametric solution* (3.66) *interpolates between the limiting characteristic projections from either side of* $s = 0$ *and thus "fills the gap", as shown in* Figure 3.11(b)

By eliminating A *from* (3.66) *we thus obtain the full explicit solution in this case as*

$$u(x,t) = \begin{cases} 1 - \mathrm{e}^{-t} & x < -1 + \exp\left(t - (1 - \mathrm{e}^{-t})\right), \\ 1 - \dfrac{t - \log(1+x)}{\mathrm{e}^{t} - 1} & -1 + \exp\left(t - (1 - \mathrm{e}^{-t})\right) \leq x \leq -1 + \mathrm{e}^{t}, \\ 1 & x > -1 + \mathrm{e}^{t}. \end{cases} \tag{3.67}$$

One can easily verify that this solution satisfies the PDE (3.57) *in each of the three regions and is continuous for* $t > 0$*.*

4 First order nonlinear equations

We now introduce a method that will enable us to solve fully nonlinear first order PDEs, generalising the method of characteristics discussed in Section 2. We consider a general first-order nonlinear scalar PDE of the form

$$F(p,q,u,x,y)=0, \tag{4.1}$$

where we use

$$\frac{\partial u}{\partial x}=p, \qquad \frac{\partial u}{\partial y}=q \tag{4.2}$$

as shorthand, so that

$$\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}. \tag{4.3}$$

The special case of a quasilinear equation corresponds to F being a linear function of p and q, *i.e.*,

$$F(p,q,u,x,y,)=a(x,y,u)p+b(x,y,u)q-c(x,y,u). \tag{4.4}$$

4.1 Charpit's equations

If we differentiate (4.1) with respect to x and y, we obtain

$$\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=-\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial u}, \tag{4.5a}$$

$$\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=-\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial u}, \tag{4.5b}$$

or, using (4.3),

$$\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial p}{\partial y}=-\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial u}, \tag{4.6a}$$

$$\frac{\partial F}{\partial p}\frac{\partial q}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=-\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial u}. \tag{4.6b}$$

So, if we define *characteristics* or *rays* as curves $\big(x(\tau),y(\tau)\big)$ satisfying

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=\frac{\partial F}{\partial p}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau}=\frac{\partial F}{\partial q} \tag{4.7}$$

then, along these curves,

$$\frac{\mathrm{d}p}{\mathrm{d}\tau}=-\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial u}, \qquad \frac{\mathrm{d}q}{\mathrm{d}\tau}=-\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial u}. \tag{4.8}$$

We therefore have a system of four ODEs for x, y, p and q satisfied along the rays. Recall, though, that in general F depends on u also, so to close the system we also need an ODE for u along the rays, namely

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\tau} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}. \tag{4.9}$$

In summary, we have the following system of ODEs for x, y, p, q and u, known as *Charpit's equations*:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\partial F}{\partial p}, \tag{4.10a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \frac{\partial F}{\partial q}, \tag{4.10b}$$

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial u}, \tag{4.10c}$$

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial u}, \tag{4.10d}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}. \tag{4.10e}$$

It is easily verified that these reduce to the usual characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = a, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = b, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = c, \tag{4.11}$$

for quasi-linear equations where F takes the form (4.4).

4.2 Boundary data

As for quasilinear scalar equations, Cauchy data specifies u along some curve Γ in the (x, y)-plane:

$$x = x_0(s), \qquad y = y_0(s), \qquad u = u_0(s), \tag{4.12}$$

for s in some (possibly infinite) interval. We also require initial conditions for p and q, say $p = p_0(s)$, $q = q_0(s)$, which are obtained by differentiating u_0 with respect to s and using the PDE (4.1):

$$\frac{\mathrm{d}u_0}{\mathrm{d}s} = p_0\frac{\mathrm{d}x_0}{\mathrm{d}s} + q_0\frac{\mathrm{d}y_0}{\mathrm{d}s}, \qquad F\left(p_0, q_0, u_0, x_0, y_0\right) = 0. \tag{4.13}$$

By the implicit function theorem, the two equations (4.13) may be solved uniquely (in principle, if not explicitly) for p_0 and q_0 provided the condition

$$\frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\partial F}{\partial q_0} - \frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial F}{\partial p_0} \neq 0 \tag{4.14}$$

is satisfied. This is the same as insisting that Γ not be parallel to a ray.

Charpit's method consists of solving the ODEs (4.10) for (p, q, u, x, y), with (4.12) and (4.13) as initial data at $\tau = 0$. This gives (p, q, u, x, y) all as functions of s and τ and, in principle, allows us to reconstruct the solution surface $u = u(x, y)$.

Example 4.1 Sugar on a spoon
Consider sugar piled up on a spoon such that its height is given by $u(x, y)$. At criticality, just before the sugar would start to slide off the spoon, the angle between the normal to the surface and the vertical $(0, 0, 1)$ is a prescribed constant, γ, the angle of friction. That is,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 1 = \sec^2\gamma \tag{4.15}$$

After rearranging and normalising, this can be written as the Eikonal equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1, \tag{4.16}$$

which is of the form (4.1) *with*

$$F(p, q) = \tfrac{1}{2}\left(p^2 + q^2 - 1\right). \tag{4.17}$$

Charpit's equations for this particular F are

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = p, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = q, \qquad \frac{\mathrm{d}p}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}q}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = p^2 + q^2 = 1. \tag{4.18}$$

Notice that p and q are constant along rays and, hence, given by their boundary values:

$$p = p_0(s), \qquad\qquad q = q_0(s). \tag{4.19}$$

The remaining ODEs are then readily integrated to give

$$x = x_0(s) + p_0(s)\tau, \qquad y = y_0(s) + q_0(s)\tau, \qquad u = u_0(s) + \tau. \tag{4.20}$$

Notice that the slope of a ray is given by $q_0(s)/p_0(s)$ which is constant along each ray. Thus the rays are straight lines, *along which u increases linearly with τ.*

At the edge of the spoon, the height is zero, so $u_0(s) = 0$. Then we have the system

$$\frac{\mathrm{d}x_0}{\mathrm{d}s}p_0 + \frac{\mathrm{d}y_0}{\mathrm{d}s}q_0 = 0, \qquad\qquad p_0^2 + q_0^2 = 1 \tag{4.21}$$

for p_0 and q_0, whose solution is

$$p_0 = \frac{\mp y_0'}{\sqrt{(x_0')^2 + (y_0')^2}}, \qquad\qquad q_0 = \frac{\pm x_0'}{\sqrt{(x_0')^2 + (y_0')^2}}, \tag{4.22}$$

where $'$ is used as shorthand for $\mathrm{d}/\mathrm{d}s$. The vector (p_0, q_0) is the unit normal *to the boundary Γ. Hence the rays are straight lines perpendicular to Γ and $u(x, y)$ is simply the* distance *of the point (x, y) from Γ.*

Notice that there are two possible solutions corresponding to the $\pm$ in (4.22)*. The correct solution is chosen by ensuring that the rays propagate* into *the region of interest, not out of it. So, in* (4.22)*, we have to choose (p_0, q_0) to be the* inward *pointing normal. Otherwise the solution corresponds to the sandpile outside a spoon-shaped hole in a table.*

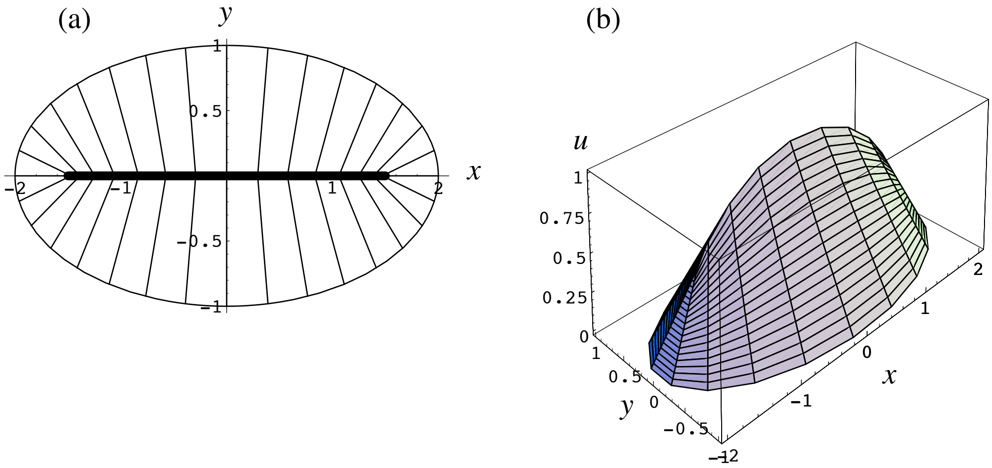

Figure 4.1: (a) Rays for a sugar heap on an elliptical spoon with $a = 2$ and $b = 1$; the bold line marks the ridge. (b) The corresponding pile height $u(x, y)$.

4.3 Proof that Charpit's method works

If we differentiate F along a ray, we find that

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = \frac{\partial F}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}\tau} + \frac{\partial F}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}\tau} + \frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}\tau} + \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\tau} = 0. \tag{4.23}$$

Since the boundary condition (4.13) sets F to zero on the initial curve Γ, it must therefore be zero everywhere along the rays passing through Γ. Hence p, q and u satisfy the equation $F(p, q, u, x, y) = 0$ everywhere in the *domain of definition* where there are rays emanating from Γ.

This is not quite sufficient to prove that u is a solution of the original nonlinear PDE. We still have to show that the functions p and q that result from solving Charpit's equations are equal to $\partial u/\partial x$ and $\partial u/\partial y$ respectively. To do this, we first prove that $\phi \equiv 0$, where

$$\phi = \frac{\partial u}{\partial s} - p\frac{\partial x}{\partial s} - q\frac{\partial y}{\partial s}. \tag{4.24}$$

By differentiating ϕ with respect to τ and rearranging, we obtain

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial F}{\partial s} - \phi\frac{\partial F}{\partial u} + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial \tau} - p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}\right). \tag{4.25}$$

The final term is identically zero by (4.10), and we have already shown that $F \equiv 0$ in the domain of definition, which implies that $\partial F/\partial s \equiv 0$. Hence ϕ satisfies

$$\frac{\partial \phi}{\partial \tau} = -\phi\frac{\partial F}{\partial u} \tag{4.26}$$

with, by (4.13), $\phi = 0$ at $\tau = 0$. Provided $\partial F/\partial u$ is bounded, it follows from Picard's theorem that $\phi \equiv 0$ in the domain of definition.

From this fact and Charpit's equation for u, we obtain two simultaneous equations for $\partial u/\partial x$ and $\partial u/\partial y$:

$$\frac{\partial x}{\partial \tau} p + \frac{\partial y}{\partial \tau} q = \frac{\partial u}{\partial \tau} = \frac{\partial x}{\partial \tau}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial \tau}\frac{\partial u}{\partial y}, \tag{4.27a}$$

$$\frac{\partial x}{\partial s} p + \frac{\partial y}{\partial s} q = \frac{\partial u}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial u}{\partial y}. \tag{4.27b}$$

This system has a unique solution, namely

$$p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial u}{\partial y}, \tag{4.28}$$

provided the determinant of the right-hand side is nonzero, and this determinant is the *Jacobian* of the transformation from (s, τ) to (x, y), that is

$$J = \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial s} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial F}{\partial q} - \frac{\partial y}{\partial s}\frac{\partial F}{\partial p}. \tag{4.29}$$

Hence we have shown that $u(x, y)$ satisfies the nonlinear PDE (4.1) as long as J is nonzero.

This proof applies in particular to the special case where F is given by (4.4), and hence establishes that the method of characteristics works for scalar quasi-linear PDEs.

4.4 Discontinuities

Example 4.1 is an example in which the rays intersect. This happens where the Jacobian J defined by (4.29) first becomes zero. Note that this reproduces the criterion (4.14) for Cauchy data *not* to determine a unique solution on Γ. If rays are allowed to cross, then the solution becomes multi-valued, which is clearly unphysical for a pile of sugar. Instead, we must allow shocks to form across which the solution is discontinuous. Recall that, for nonlinear PDEs, shocks are different from characteristics. The conditions that must be applied across a shock depend on the physical situation being modelled. For the sugar heap problem, it is clear that u must be continuous everywhere, since a discontinuity in u (corresponding to a vertical "cliff") cannot be sustained. This forces the shock, *i.e.* the ridge line, to be along the x-axis as shown in Figure 4.1. In general, the region of validity of the solution obtained by Charpit's method is bounded by curves on which $J = 0$.

4.5 Geometrical optics

The propagation of sound or light waves in two spatial dimensions is governed by the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}, \tag{4.30}$$

where ψ is some state variable such as pressure or electric field and c is the wave speed. We look for time-periodic (or "monochromatic") solutions with constant frequency ω by setting

$$\psi(x, y, t) = \phi(x, y)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{4.31}$$

Then ϕ satisfies the *Helmholtz equation*

$$\nabla^2\phi + k^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0, \tag{4.32}$$

where $k = \omega/c$ is the wavenumber (*i.e.* 2π divided by the wavelength).

The theory of *geometrical optics* arises from the limit $k \to \infty$, which is valid over length-scales much longer than the wavelength. To analyse the behaviour of (4.32) in this limit, we use the so-called *WKBJ method*, which involves writing ϕ in the form

$$\phi(x, y) = A(x, y)\mathrm{e}^{\mathrm{i}ku(x,y)}, \tag{4.33}$$

where A and u represent the *amplitude* and *phase* respectively of the solution. Then (4.32) becomes

$$\nabla^2 A + \mathrm{i}k\left(A\nabla^2 u + 2\boldsymbol{\nabla}A \cdot \boldsymbol{\nabla}u\right) + k^2 A\left(1 - |\boldsymbol{\nabla}u|^2\right) = 0. \tag{4.34}$$

We seek solutions in which A is an asymptotic expansion of the form

$$A \sim A_0 + \frac{A_1}{k} + \frac{A_2}{k^2} + \cdots. \tag{4.35}$$

At leading order, (4.34) implies that u satisfies the Eikonal equation

$$|\boldsymbol{\nabla}u|^2 = 1. \tag{4.36}$$

Then the successive terms in the amplitude expansion satisfy the *transport equations*

$$2\boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}A_0 + A_0\nabla^2 u = 0, \tag{4.37a}$$

$$2\boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}A_n + A_n\nabla^2 u = \mathrm{i}\nabla^2 A_{n-1}, \qquad n \geq 1. \tag{4.37b}$$

The Eikonal equation (4.36) may be solved exactly as in Example 4.1. The rays correspond to light rays and all the familiar properties of geometrical optics, for example that light travels in straight lines, follow from the solution of Charpit's equations.

Example 4.2 Reflecting plane waves
One obvious solution of (4.36) is $u = x$, which corresponds to a plane wave moving in the x-direction. Now we examine what happens if such a wave impinges on a reflecting wall given by a curve Γ in the (x, y)-plane. We decompose the state variable ϕ into an incident *wave ϕ_I, namely a plane wave with constant amplitude a, and a* reflected *wave ϕ_R:*

$$\phi = \phi_I + \phi_R, \qquad \phi_I = a\mathrm{e}^{\mathrm{i}kx}. \tag{4.38}$$

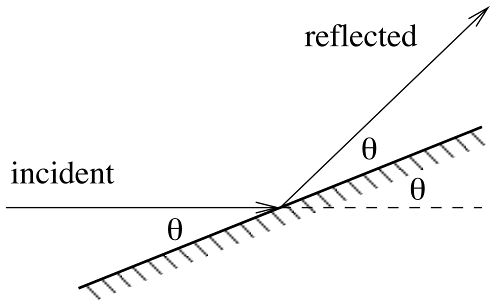

Figure 4.2: Illustration of Snell's law.

Now we apply the WKBJ ansatz to ϕ_R*:*

$$\phi_R = A\mathrm{e}^{\mathrm{i}ku(x,y)}. \tag{4.39}$$

The boundary conditions depend on the physical situation being modelled and exactly what the state variable ϕ *represents. The simplest case is to impose the Dirichlet condition* $\phi = 0$ *on* Γ*, which leads to*

$$u = x, \qquad A_0 = -a \qquad \text{on } \Gamma. \tag{4.40}$$

Other possibilities are $\partial\phi/\partial n = 0$ *or the "Robin condition"* $\partial\phi/\partial n + \lambda\phi = 0$*, but it is readily verified that the leading-order boundary conditions (4.40) are unchanged in either case.*

The solution of Charpit's equations for the Eikonal equation was already obtained in Example 4.1*:*

$$p = p_0(s), \quad q = q_0(s), \quad x = x_0(s) + p_0(s)\tau, \quad y = y_0(s) + q_0(s)\tau, \quad u = u_0(s) + \tau. \tag{4.41}$$

For simplicity we suppose that s *parametrises arc-length along* Γ *so we can write* $x_0' = \cos\theta$, $y_0' = \sin\theta$ *where* θ *is the angle between* Γ *and the* x*-axis. The boundary condition (4.40) implies that* $u_0(s) = x_0(s)$*, and then* p_0 *and* q_0 *are obtained from*

$$x_0'p_0 + y_0'q_0 = x_0', \qquad p_0^2 + q_0^2 = 1. \tag{4.42}$$

This system has two solutions, one of which is $p_0 = 1$, $q_0 = 0$*, corresponding to the incident wave. The reflected wave is given by the other solution*

$$p_0 = 1 - 2(y_0')^2 = \cos(2\theta), \qquad q_0 = 2x_0'y_0' = \sin(2\theta). \tag{4.43}$$

Hence the reflected ray makes an angle of 2θ *with the* x*-axis. This is* Snell's law*: as illustrated in* Figure 4.2*, it implies that the angle of incidence to the wall equals the angle of reflection from it.*

Example 4.3 The caustic in a teacup

As a special case of Example 4.2*, we now consider the case where* Γ *is the unit circle, parametrised (say) by* $x_0(s) = \cos(s)$, $y_0(s) = \sin(s)$.

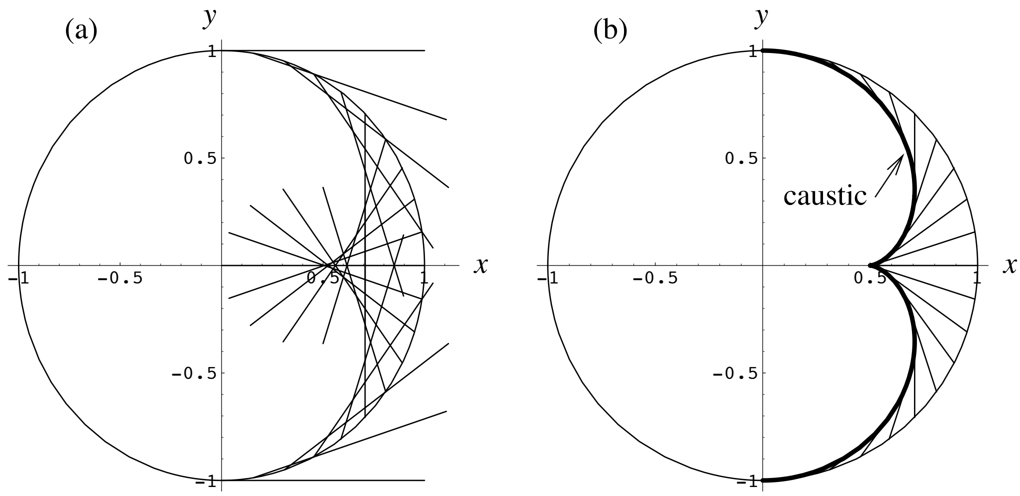

Figure 4.3: (a) Reflected rays for Example 4.3. (b) Rays truncated by a caustic on which $J = 0$.

As shown in Figure 4.3(a)*, the reflected waves start to cross a finite distance from the circle. The envelope of the rays, determined either via the Jacobian condition*

$$J = \frac{\partial x}{\partial s}\frac{\partial F}{\partial q} - \frac{\partial y}{\partial s}\frac{\partial F}{\partial p} = 0, \tag{4.44}$$

or by solving the envelope equations

$$F(x, y; s) = \frac{\partial F}{\partial \lambda} = 0, \tag{4.45}$$

defines a curve of concentrated light, as can be observed by shining a light in a teacup.

A single-valued ray solution may be obtained by truncating rays at any caustic where the Jacobian is zero. It may be shown that the asymptotic ansatz (4.33) breaks down, with $A \to \infty$ as the caustic is approached. The method of *matched asymptotic expansions* yields the appropriate correction in the neighbourhood of a caustic and allows the behaviour in the dark zone beyond the caustic (corresponding to *complex rays*) to be found.

5 First order systems

5.1 Introduction

In this chapter we consider first-order systems of PDEs of the form

$$\boldsymbol{A}(x,y,\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x}+\boldsymbol{B}(x,y,\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial y}=\boldsymbol{c}(x,y,\boldsymbol{u}), \tag{5.1}$$

where now the unknown $\boldsymbol{u}$ is an n-dimensional vector function of x and y, $\boldsymbol{c}$ is also an n-dimensional vector, $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ matrices. Again, we only consider *quasilinear* equations, so that (5.1) is linear in the first derivatives of $\boldsymbol{u}$. Equations of the form (5.1) arise frequently in physical problems (*e.g.*, gas dynamics), in which it is often the case that one of the independent variables represents time, so we will also write (5.1) in the form

$$\boldsymbol{A}(x,t,\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{B}(x,t,\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x}=\boldsymbol{c}(x,t,\boldsymbol{u}). \tag{5.2}$$

The other motivation for studying (5.1) is that many higher-order scalar equations may be written in this form. For example, Laplace's equation may be transformed to the first-order Cauchy–Riemann equations, as shown in Example 1.1.

Example 5.1 *The* shallow-water equations *describe the flow of a thin layer of liquid over a flat surface under gravity. If x measures horizontal distance and t is time, then the dimensionless height $h(x,t)$ and velocity $u(x,t)$ satisfy*

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0, \qquad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial h}{\partial x}=0, \tag{5.3}$$

which may be written in the form (5.2), with

$$\boldsymbol{u}=\begin{pmatrix}h\\u\end{pmatrix}, \qquad \boldsymbol{A}=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad \boldsymbol{B}=\begin{pmatrix}u&h\\1&u\end{pmatrix}, \qquad \boldsymbol{c}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{5.4}$$

5.2 Cauchy data

Definition

For the PDE system (5.1), Cauchy data is to specify $\boldsymbol{u}$ on a curve Γ in the (x,y) plane, *i.e.*,

$$x=x_0(s), \qquad y=y_0(s), \qquad \boldsymbol{u}=\boldsymbol{u}_0(s), \qquad s_1\le s\le s_2, \tag{5.5}$$

and the PDE (5.1) together with the data (5.5) is known as the *Cauchy problem*. As for the scalar case, we now ask whether the Cauchy problem determines the first derivatives of $\boldsymbol{u}$.

Differentiating (5.5) with respect to s, we find

$$\frac{\mathrm{d}\boldsymbol{u}_0}{\mathrm{d}s}=\frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\partial \boldsymbol{u}}{\partial x}+\frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial \boldsymbol{u}}{\partial y}. \tag{5.6}$$

Now we consider (5.1) and (5.6) as a $(2n) \times (2n)$ matrix system for the $(2n)$-dimensional vector $(\partial \boldsymbol{u}/\partial x, \partial \boldsymbol{u}/\partial y)^T$:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ x_0' \boldsymbol{I} & y_0' \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \partial \boldsymbol{u}/\partial x \\ \partial \boldsymbol{u}/\partial y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{u}_0' \end{pmatrix}, \tag{5.7}$$

where $'$ is shorthand for $\mathrm{d}/\mathrm{d}s$ and $\boldsymbol{I}$ is the $n \times n$ identity matrix. The system (5.7) may be solved uniquely for the first derivatives of $\boldsymbol{u}$ provided the determinant of the matrix on the left-hand side is nonzero, and this determinant may be rearranged to give

$$\det\left(x_0' \boldsymbol{B} - y_0' \boldsymbol{A}\right) \neq 0. \tag{5.8}$$

This is the condition on the initial data for the the first derivatives of $\boldsymbol{u}$ to be locally determined. It clearly reduces to the condition found for scalar equations when $n = 1$.

Cauchy–Kovalevskaya theorem

Now we state a generalisation of the theorem previously introduced for scalar PDEs. For simplicity we suppose that a coordinate transformation is used to shift the boundary Γ onto the y-axis, where we specify $\boldsymbol{u}$:

$$\boldsymbol{u} = \boldsymbol{u}_0(y) \qquad \text{on} \quad x = 0, \quad y_1 \leq y \leq y_2. \tag{5.9}$$

Clearly, so long as $\boldsymbol{u}_0$ is differentiable, we can calculate $\partial \boldsymbol{u}/\partial y$ directly:

$$\frac{\partial \boldsymbol{u}}{\partial y} = \frac{\mathrm{d}\boldsymbol{u}_0}{\mathrm{d}y} \qquad \text{on} \quad x = 0, \quad y_1 \leq y \leq y_2. \tag{5.10}$$

We can then use the PDE (5.1) to solve for $\partial \boldsymbol{u}/\partial x$,

$$\frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{A}^{-1}\boldsymbol{c} - \boldsymbol{A}^{-1}\boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial y} = \boldsymbol{f}\left(x, y, \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial y}\right), \quad \text{say}, \tag{5.11}$$

so long as $\boldsymbol{A}$ is *invertible*. Thus the PDE may be written in the form (5.11) provided $|\boldsymbol{A}| \neq 0$, which is the same as condition (5.8), with $x_0 = 0$, $y_0 = s$.

Now suppose that $\boldsymbol{u}_0(y)$ is analytic at a point $y_0 \in (y_1, y_2)$ and that $\boldsymbol{f}$ is analytic in all its arguments at the point

$$\left(0, y_0, \boldsymbol{u}_0(y_0), \frac{\mathrm{d}\boldsymbol{u}_0}{\mathrm{d}y}(y_0)\right).$$

Then the Cauchy-Kovalevskaya theorem says that the Cauchy problem (5.9, 5.11) has a unique analytic solution $\boldsymbol{u}(x, y)$ in a neighbourhood of $(0, y_0)$.

The proof of this theorem works by constructing a Taylor expansion for $\boldsymbol{u}$ and showing it has a finite radius of convergence. It may readily be extended to the case of a general analytic initial curve Γ if the condition (5.8) is satisfied. However, the result is entirely local: the radius of convergence may be extremely small, meaning that $\boldsymbol{u}$ may become singular or nonunique a short distance from the initial data. Moreover, the hypotheses of the theorem are rather restrictive: it says nothing about existence or uniqueness of solutions when the initial data (say) are nonanalytic.

5.3 Characteristics

Definition

There is not a straightforward geometric interpretation of the PDE system (5.1) as there is in the scalar case. At the very least we have to imagine n solution surfaces — one for each component of $\boldsymbol{u}$. There is also no obvious generalisation of the characteristic ODEs. Recall, though, that for scalar PDEs, characteristics are curves on which Cauchy data does not allow the the first derivatives of u to be determined uniquely, and this definition *does* generalise to n dimensions.

Consider a curve in the (x, y) plane, parameterised by τ, and suppose that $\boldsymbol{u}$ is known on this curve. Then, according to (5.8), the first derivatives of $\boldsymbol{u}$ may be found locally unless

$$\det\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\boldsymbol{B} - \frac{\mathrm{d}y}{\mathrm{d}\tau}\boldsymbol{A}\right) = 0. \tag{5.12}$$

A curve $(x(\tau), y(\tau))$ satisfying (5.12) is called a *characteristic projection.* Since there is no generalisation of the concept of characteristic introduced previously for scalar PDEs, it is usual to drop the word "projection" and refer to these plane curves simply as *characteristics.*

As for scalar PDEs, an alternative definition is that characteristics are curves across which the first derivatives of $\boldsymbol{u}$ may be discontinuous. Suppose $\boldsymbol{u}$ is continuous across a curve C in the (x, y) plane, but its first derivatives take different limiting values (denoted by $^+$ and $^-$) on either side of C. If C is parametrised by $x = x(\xi)$, $y = y(\xi)$. Then, by differentiating $\boldsymbol{u}$ along either side of C, we obtain

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\xi} = \frac{\partial \boldsymbol{u}^+}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\xi} + \frac{\partial \boldsymbol{u}^+}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\partial \boldsymbol{u}^-}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\xi} + \frac{\partial \boldsymbol{u}^-}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\xi} \tag{5.13}$$

and by subtracting these two equations we find

$$\frac{\mathrm{d}x}{\mathrm{d}\xi}\left[\frac{\partial \boldsymbol{u}}{\partial x}\right]_-^+ + \frac{\mathrm{d}y}{\mathrm{d}\xi}\left[\frac{\partial \boldsymbol{u}}{\partial y}\right]_-^+ = \boldsymbol{0}. \tag{5.14}$$

Similarly, because $\boldsymbol{u}$ satisfies the PDE (5.1) on either side of C, we have

$$\boldsymbol{A}\left[\frac{\partial \boldsymbol{u}}{\partial x}\right]_-^+ + \boldsymbol{B}\left[\frac{\partial \boldsymbol{u}}{\partial y}\right]_-^+ = \boldsymbol{0}. \tag{5.15}$$

The homogeneous system (5.14, 5.15) implies that the jumps in $\partial\boldsymbol{u}/\partial x$ and $\partial\boldsymbol{u}/\partial y$ must be zero unless the determinant of the system is zero, which implies that

$$\det\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\boldsymbol{B} - \frac{\mathrm{d}y}{\mathrm{d}\xi}\boldsymbol{A}\right) = 0, \tag{5.16}$$

i.e., that C is a characteristic.

5.4 Classification

The slopes of the characteristics satisfy the eigenvalue problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda \qquad \text{where} \quad \det(\boldsymbol{B} - \lambda \boldsymbol{A}) = 0. \tag{5.17}$$

Thus λ satisfies an nth-order polynomial equation, whose roots may be complex in general. A 2×2 system may be classified as follows, depending on the eigenvalues λ.

- If there are **two distinct real eigenvalues**, then the system is said to be **hyperbolic**.
- If there is **one repeated real eigenvalue**, then the system is **parabolic**.
- If the **eigenvalues are complex**, then the system is called **elliptic**.

Example 5.2 *Consider the* quasilinear second-order PDE

$$a\frac{\partial^2\phi}{\partial x^2} + 2b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} = f, \tag{5.18}$$

where a, b and f are in general functions of x, y, ϕ, $\partial\phi/\partial x$ and $\partial\phi/\partial y$. We can write (5.18) as the first-order system

$$\begin{pmatrix} a & b \\ 0 & -1 \end{pmatrix} \frac{\partial \boldsymbol{u}}{\partial x} + \begin{pmatrix} b & c \\ 1 & 0 \end{pmatrix} \frac{\partial \boldsymbol{u}}{\partial y} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad \textit{where} \quad \boldsymbol{u} = \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{pmatrix}. \tag{5.19}$$

If a, b, c and f are independent of ϕ, then we can ignore the uncoupled equation $\partial\phi/\partial x = u$, and the characteristic slopes satisfy

$$\begin{vmatrix} b - \lambda a & c - \lambda b \\ 1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad a\lambda^2 - 2b\lambda + c = 0. \tag{5.20}$$

The system is thus hyperbolic *if $b^2 > ac$,* parabolic *if $b^2 = ac$,* elliptic *if $b^2 < ac$.*

In dimensions higher than two, there are clearly many possible combinations of real, complex, distinct and repeated roots of the polynomial equation (5.17), and there is no such simple classification. However, we still define an equation as **hyperbolic** if (5.17) has n distinct real roots λ. Since the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ depend on x, y and, in general, also on the solution $\boldsymbol{u}$, the *type* of the equation (*i.e.*, hyperbolic, elliptic, parabolic or some hybrid) may also vary with position.

Now, according to the Cauchy–Kovalevskaya theorem, provided all our coefficients and initial data are analytic and the condition (5.8) is satisfied, there is a unique solution for $\boldsymbol{u}$ in a neighbourhood of Γ. Nevertheless, unless the PDE is hyperbolic, the Cauchy problem is in general *ill posed*. This may manifest itself in several ways. For example, the unique local solution may blow up arbitrarily close to Γ or may be pathalogically sensitive to the initial data.

Example 5.3 *The Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \tag{5.21}$$

are in the form of (5.1), *with*

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.22}$$

The characteristic slopes satisfy

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow \quad \lambda = \pm \mathrm{i}, \tag{5.23}$$

so the system is elliptic.

Suppose we try to solve (5.21) *in* $y > 0$ *subject to the Cauchy data*

$$u = u_0(x) = 0, \qquad v = v_0(x) = \frac{\epsilon\delta^2}{x^2 + \delta^2} \qquad \text{on} \quad y = 0, \tag{5.24}$$

where ϵ *and* δ *are positive constants. Notice that the initial data are analytic and bounded:* $v_0(x)$ *is in the range* $(0, \epsilon]$ *for all* x*. It may be shown*[1] *that the solution of this problem is*

$$u = \frac{2\epsilon\delta^2 xy}{[x^2 + (y-\delta)^2]\,[x^2 + (y+\delta)^2]}, \qquad v = \frac{\epsilon\delta^2\left[x^2 - y^2 + \delta^2\right]}{[x^2 + (y-\delta)^2]\,[x^2 + (y+\delta)^2]}. \tag{5.25}$$

Thus, however small ϵ *is, both* u *and* v *blow up at the point* $(0, \delta)$*: by choosing* δ *small, we can make the solution break down arbitrarily close to the initial curve* $y = 0$*.*

For the remainder of this chapter we restrict our attention to hyperbolic systems, for which the Cauchy problem is generally well posed, and for which characteristic methods analogous to those used for scalar equations can be applied. So, at each point in the (x, y) plane, we assume that (5.12) defines n distinct real eigenvalues λ. Thus, by solving $\mathrm{d}y/\mathrm{d}x = \lambda$ for each of these n characteristic slopes, we can obtain in principle n families of characteristics for an n-dimensional hyperbolic system.

5.5 Integration along characteristics

Reduction to an ODE

Suppose λ is a real eigenvalue of (5.17); recall that λ is in general a function of x, y and $\boldsymbol{u}$, since $\boldsymbol{A}$ and $\boldsymbol{B}$ are. Now the matrix $(\boldsymbol{B} - \lambda\boldsymbol{A})$ is singular, so there exists a *left eigenvector* $\boldsymbol{l}^T$, such that

$$\boldsymbol{l}^T(\boldsymbol{B} - \lambda\boldsymbol{A}) = \boldsymbol{0}^T, \qquad \text{that is} \quad \boldsymbol{l}^T\boldsymbol{B} = \lambda\boldsymbol{l}^T\boldsymbol{A}. \tag{5.26}$$

[1] the Cauchy–Riemann equations imply that $u + \mathrm{i}v$ is a function of $z = x + \mathrm{i}y$; here, u and v are the real and imaginary parts of the complex function

$$f(z) = \frac{\epsilon\delta^2\mathrm{i}}{z^2 + \delta^2}.$$

Multiplying the PDE (5.1) on the left by $\boldsymbol{l}^T$, we obtain

$$\boldsymbol{l}^T\boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{l}^T\boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial y} = \boldsymbol{l}^T\boldsymbol{c}$$

$$\Rightarrow \quad \boldsymbol{l}^T\boldsymbol{A}\left(\frac{\partial \boldsymbol{u}}{\partial x} + \lambda\frac{\partial \boldsymbol{u}}{\partial y}\right) = \boldsymbol{l}^T\boldsymbol{c}. \tag{5.27}$$

Along characteristics, whose slope is $\mathrm{d}y/\mathrm{d}x = \lambda$, we have

$$\boldsymbol{l}^T\boldsymbol{A}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}x} = \boldsymbol{l}^T\boldsymbol{c}. \tag{5.28}$$

This is the equivalent of the ODE satisfied by u along characteristics in the scalar case. There is one ODE of the form (5.28) satisfied along each of the n families of characteristics.

Riemann invariants

Since (5.28) is a single differential equation for the n components of $\boldsymbol{u}$, it is usually not integrable. However, there are special cases in which the ODE (5.28) may be rearranged to take the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[R(x, y, \boldsymbol{u})\big] = 0. \tag{5.29}$$

If so, the function R is conserved along the characteristics satisfying $\mathrm{d}y/\mathrm{d}x = \lambda$, and is called a *Riemann invariant*.

Example 5.4 *Consider the system*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \tag{5.30}$$

which may be written in the form (5.1) *with*

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.31}$$

The characteristic slopes satisfy

$$\begin{vmatrix} -\lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0 \qquad \Rightarrow \quad \lambda = \pm 1, \tag{5.32}$$

so the system is hyperbolic*. Now we have to solve*

$$\boldsymbol{l}^T(\boldsymbol{B} - \lambda\boldsymbol{A}) = \boldsymbol{l}^T\begin{pmatrix} \mp 1 & -1 \\ 1 & \pm 1 \end{pmatrix} = \boldsymbol{0}^T, \tag{5.33}$$

and a suitable left eigenvector is $\boldsymbol{l}^T = (1,\ \pm 1)$*. When we multiply the system*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.34}$$

on the left by $\boldsymbol{l}^T$*, we obtain*

$$\begin{pmatrix}1 & \mp 1\end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix}u \\ v\end{pmatrix} + \begin{pmatrix}\pm 1 & -1\end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix}u \\ v\end{pmatrix} = 0. \tag{5.35}$$

Using the fact that $\mathrm{d}y/\mathrm{d}x = \lambda = \pm 1$*, this may be rearranged to give*

$$\frac{\mathrm{d}}{\mathrm{d}x}(u \mp v) = 0, \tag{5.36}$$

so the Riemann invariants $u \mp v$ *are conserved along the characteristics* $\mathrm{d}y/\mathrm{d}x = \pm 1$.

In this simple case, the Riemann invariants may be used to write down the general solution. Since $u + v$ *is constant when* $y + x$ *is constant, we must have* $u + v = f(y + x)$ *for some function* f*. Similarly,* $u - v$ *must take the form* $g(y - x)$*, so the general solution is*

$$u = \tfrac{1}{2}f(y + x) + \tfrac{1}{2}g(y - x), \qquad v = \tfrac{1}{2}f(y + x) - \tfrac{1}{2}g(y - x), \tag{5.37}$$

where f *and* g *are two arbitrary functions. Note that, if we cross-differentiate to eliminate* v*, we find that* u *satisfies the* wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y^2} = 0. \tag{5.38}$$

For linear hyperbolic PDEs with $\boldsymbol{c} = \boldsymbol{0}$, as in Example 5.4, we can always find a complete set of n Riemann invariants. Furthermore, for linear PDEs, the characteristics may be found independently of the solution. We thus obtain a system of n algebraic equations for the components of $\boldsymbol{u}$ in terms of arbitrary functions that are constant along each family of characteristics. This suggests a plausible method for solving hyperbolic systems numerically. If $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{c}$ are approximated as being locally constant near Γ, then the resulting autonomous linear system has a complete set of Riemann invariants. Thus the solution $\boldsymbol{u}$ a small distance from Γ may be found by solving the resulting system of algebraic equations. By repeating this process, the solution may be continued further still from the initial data. This proposed procedure *suggests* that the Cauchy problem should usually be well posed for hyperbolic systems.

The following two examples show that, when $\boldsymbol{c}$ is nonzero, even linear PDEs have no Riemann invariants in general.

Example 5.5 *The system*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = u + v, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{5.39}$$

may be written as

$$\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\frac{\partial \boldsymbol{u}}{\partial x} + \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\frac{\partial \boldsymbol{u}}{\partial y} = \begin{pmatrix}u + v \\ 0\end{pmatrix},$$

where $\boldsymbol{u} = (u, v)^T$*. As in* Example 5.4*, the characteristic directions are* $\mathrm{d}y/\mathrm{d}x = \lambda = \pm 1$ *and the corresponding left eigenvectors are* $\boldsymbol{l}^T = (1, \pm 1)$*. Now, the ODEs satisfied along the characteristics are*

$$\frac{\mathrm{d}}{\mathrm{d}x}(u \mp v) = u + v \qquad \text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \pm 1, \tag{5.40}$$

only one of which is integrable:

$$\frac{\mathrm{d}}{\mathrm{d}x}(u+v) = u+v \qquad \Rightarrow \qquad \mathrm{e}^{-x}(u+v) = const \qquad \textit{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -1. \tag{5.41}$$

Nevertheless, we can use this single Riemann invariant to find the general solution in this case.

Since $\mathrm{e}^{-x}(u+v)$ *is constant when* $(y+x)$ *is constant, we may write*

$$u + v = \mathrm{e}^{x} f(y+x), \tag{5.42}$$

where f *is an arbitrary function. Along the other family of characteristics we have* $(y-x) = const = k$*, say, so*

$$\frac{\mathrm{d}}{\mathrm{d}x}(u-v) = u+v = \mathrm{e}^{x} f(y+x) = \mathrm{e}^{x} f(k+2x) \qquad \textit{on} \quad y-x=k. \tag{5.43}$$

This may be integrated with respect to x *to give*

$$u - v = \tfrac{1}{2}\mathrm{e}^{(x-y)/2}\int_0^{y+x} \mathrm{e}^{s/2} f(s)\,\mathrm{d}s + g(y-x), \tag{5.44}$$

where g *is a second arbitrary function. The general solution is, therefore,*

$$u = \frac{1}{2}\mathrm{e}^{x} f(y+x) + \frac{1}{2}g(y-x) + \frac{1}{4}\mathrm{e}^{(x-y)/2}\int_0^{y+x} \mathrm{e}^{s/2} f(s)\,\mathrm{d}s, \tag{5.45a}$$

$$v = \frac{1}{2}\mathrm{e}^{x} f(y+x) - \frac{1}{2}g(y-x) - \frac{1}{4}\mathrm{e}^{(x-y)/2}\int_0^{y+x} \mathrm{e}^{s/2} f(s)\,\mathrm{d}s. \tag{5.45b}$$

Example 5.6 *The system*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = u, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{5.46}$$

has characteristic equations

$$\frac{\mathrm{d}}{\mathrm{d}x}(u \mp v) = u \qquad \textit{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \pm 1, \tag{5.47}$$

which cannot be integrated. There are no Riemann invariants and no way to find the general solution explicitly.

The situation is even worse for fully nonlinear systems, where the characteristics depend on the solution $\boldsymbol{u}$. Even when such systems have a complete set of n Riemann invariants, since we do not know in advance the curves along which each is conserved, we cannot in general find an explicit solution.

Example 5.7 *The* shallow-water equations (5.3) *have characteristic slopes given by*

$$\det(\boldsymbol{B} - \lambda\boldsymbol{A}) = \begin{vmatrix} u-\lambda & h \\ 1 & u-\lambda \end{vmatrix} = (u-\lambda)^2 - h = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda = u \pm \sqrt{h}, \tag{5.48}$$

and the corresponding left eigenvectors are $(1,\ \pm\sqrt{h})$. *The characteristic ODEs are*

$$\frac{\mathrm{d}h}{\mathrm{d}t} \pm \sqrt{h}\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \qquad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u \pm \sqrt{h}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(u \pm 2\sqrt{h}\right) = 0 \qquad \text{on} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u \pm \sqrt{h}, \tag{5.49}$$

so the Riemann invariants $u \pm 2\sqrt{h}$ *are preserved along the characteristics* $\mathrm{d}x/\mathrm{d}t = u \pm \sqrt{h}$. *Although the system has two Riemann invariants, we cannot infer the general solution.*

Simple waves

For nonlinear systems, even if a complete set of Riemann invariants exists, it is usually not possible to construct an explicit solution. However, some progress can be made if the boundary conditions are such that one Riemann invariant is constant everywhere (*i.e.*, not just along characteristics). If this occurs, then we have a functional relation $R(x, y, \boldsymbol{u}) = \text{const}$, which allows us to eliminate one of the components of $\boldsymbol{u}$ and thus reduce the dimension of the system by one. For example, two-dimensional systems become scalar PDEs, which may then be solved using the methods from chapter 1. These special solutions are known as *simple waves*.

Example 5.8 *We return to the* shallow-water equations (5.3). *As shown in* Example 5.7, *these have Riemann invariants* $u \pm 2\sqrt{h}$. *Suppose we are given* $u = u_0(x)$, $h = u_0(x)^2/4$ *on* $t = 0$. *Then* $u - 2\sqrt{h}$ *is zero on* $t = 0$ *and, since it is preserved along the characteristics* $\mathrm{d}x/\mathrm{d}t = u - \sqrt{h}$, *must therefore be zero everywhere (in the domain of definition). So we can substitute* $h = u^2/4$ *into* (5.3) *to obtain*

$$\frac{\partial u}{\partial t} + \frac{3u}{2}\frac{\partial u}{\partial x} = 0, \tag{5.50}$$

which is readily solved to give the implicit solution

$$u = u_0(x - 3ut/2). \tag{5.51}$$

Domain of definition

Recall that, for scalar PDEs, where Cauchy data are only given on a finite initial curve Γ, the solution is only determined in the so-called *domain of definition*, penetrated by characteristics emanating from Γ. In n dimensions, the domain of definition is the region penetrated by all n families of characteristics originating at Γ.

Example 5.9 *We return to the system considered in* Example 5.4, *which has the general solution*

$$u = \tfrac{1}{2}f(y+x) + \tfrac{1}{2}g(y-x), \qquad v = \tfrac{1}{2}f(y+x) - \tfrac{1}{2}g(y-x). \tag{5.52}$$

If we are given the Cauchy data $u = u_0(x)$, $v = v_0(x)$ *on* $y = 0$, $0 < x < 1$, *then we can determine the two arbitrary functions* f *and* g:

$$f(s) = u_0(s) + v_0(s), \quad 0 < s < 1, \tag{5.53a}$$

$$g(s) = u_0(-s) - v_0(-s), \quad -1 < s < 0. \tag{5.53b}$$

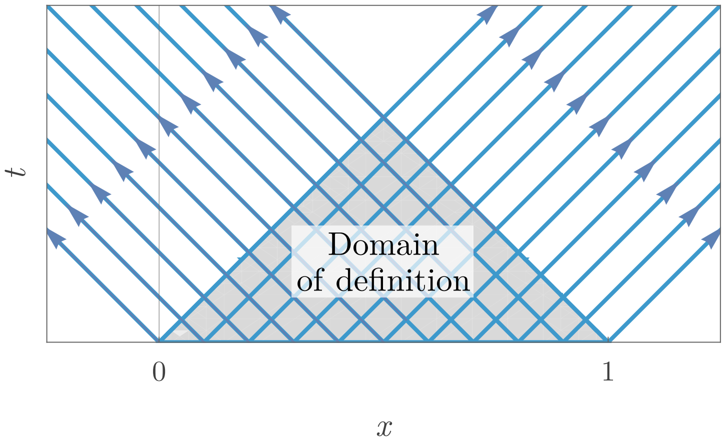

Figure 5.1: Domain of definition for Example 5.9.

The solution for u is, therefore,

$$u = \tfrac{1}{2}\big\{u_0(y+x) + v_0(y+x) + u_0(x-y) - v_0(x-y)\big\}, \tag{5.54}$$

and a similar expression for v may readily be found.

In (5.54), the first two terms correspond to the function f and are defined in $0 < y + x < 1$, while the final two terms, corresponding to g, are defined in $0 < x - y < 1$. The domain of definition *is the intersection of these two regions, where both f and g are defined, that is, $0 < y < \max(x, 1 - x)$, as illustrated in* Figure 5.1.

Figure 5.1 shows that the solution is defined only in the region penetrated by *both* families of characteristics that start from the initial data curve. The region where the solution is defined may be further restricted if the characteristics *from one family* start to intersect with each other, in which case we must allow for the possibility of discontinuous solutions.

5.6 Weak solutions

Formulation

As for scalar PDEs, nonlinear hyperbolic systems may have solutions that lose uniqueness a finite distance from the initial data. This is illustrated by the solution of Example 5.8, for which $\partial u/\partial x$ becomes unbounded in finite time if u_0' is ever negative. To continue such solutions, it is necessary to allow $\boldsymbol{u}$ to be discontinuous across curves in the (x, y) plane, again referred to as *shocks*. Since the PDE (5.1) does not make sense on such a curve, we have to use a weak formulation of the problem. The theory is very similar to the scalar case, so we omit most of the details.

The first step is to write the system in *conservation form*

$$\frac{\partial \boldsymbol{P}}{\partial x} + \frac{\partial \boldsymbol{Q}}{\partial y} = \boldsymbol{R}, \tag{5.55}$$

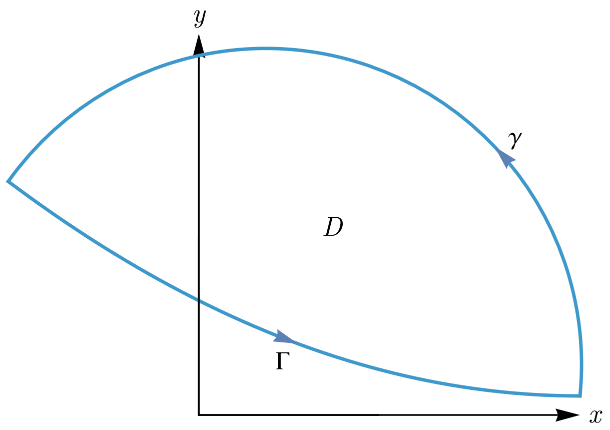

Figure 5.2: Schematic showing the boundary curve Γ, closed by a curve γ to enclose a region D.

where $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are vector-valued functions of x, y and $\boldsymbol{u}$.[2] Now, as illustrated in Figure 5.2, we form a closed region D by closing Γ with a second curve γ, then multiply (5.55) through by a test function ψ, assumed to be suitably differentiable and to vanish on γ. Then we integrate over D and, just as for the scalar case, Green's theorem leads to the following weak formulation of (5.1):

$$\int_{\Gamma} \psi \left(\boldsymbol{P} \, \mathrm{d}y - \boldsymbol{Q} \, \mathrm{d}x \right) = \iint_{D} \boldsymbol{P} \frac{\partial \psi}{\partial x} + \boldsymbol{Q} \frac{\partial \psi}{\partial y} + \boldsymbol{R} \psi \, \mathrm{d}x \mathrm{d}y. \tag{5.56}$$

A function $\boldsymbol{u}(x, y)$ that satisfies (5.56) for all suitable test functions ψ is called a *weak solution* of (5.1). If $\boldsymbol{u}$ is continuously differentiable and satisfies (5.56), then it is also a classical solution of (5.1). However, (5.56) also makes sense if $\boldsymbol{u}$ is discontinuous.

Shocks

Now we look for a weak solution in which $\boldsymbol{u}$ is smooth everywhere except a curve C, across which it is discontinuous. As shown in Figure 5.3, C divides the region D into two subregions D_1 and D_2. The integral on the right-hand side of (5.56) may be split up into two integrals over D_1 and D_2 respectively. Since $\boldsymbol{u}$ is smooth within D_1 and within D_2, Green's

[2] In fact, an arbitrary PDE system cannot always be written in this form, but it is usually possible for physically-motivated problems that are based on conservation laws.

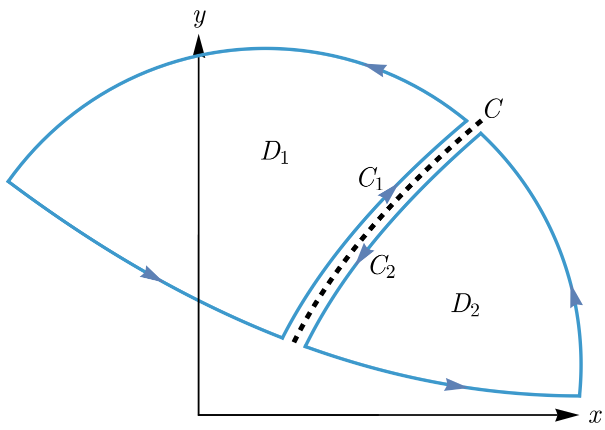

Figure 5.3: Schematic showing the shock C dividing D into two regions D_1 and D_2. The integration paths on either side of C are denoted C_1 and C_2.

theorem may then be used, along with the fact that $\boldsymbol{u}$ satisfies the PDE (5.1), giving

$$\begin{aligned}\iint_D \boldsymbol{P}\frac{\partial\psi}{\partial x} + \boldsymbol{Q}\frac{\partial\psi}{\partial y} + \boldsymbol{R}\psi \,\mathrm{d}x\mathrm{d}y &= \iint_{D_1} \boldsymbol{P}\frac{\partial\psi}{\partial x} + \boldsymbol{Q}\frac{\partial\psi}{\partial y} + \boldsymbol{R}\psi \,\mathrm{d}x\mathrm{d}y \\ &\quad + \iint_{D_2} \boldsymbol{P}\frac{\partial\psi}{\partial x} + \boldsymbol{Q}\frac{\partial\psi}{\partial y} + \boldsymbol{R}\psi \,\mathrm{d}x\mathrm{d}y. \\ &= \oint_{\partial D_1} \psi\left(\boldsymbol{P}\,\mathrm{d}y - \boldsymbol{Q}\,\mathrm{d}x\right) + \oint_{\partial D_2} \psi\left(\boldsymbol{P}\,\mathrm{d}y - \boldsymbol{Q}\,\mathrm{d}x\right). \qquad (5.57)\end{aligned}$$

Then, since ψ is assumed to be zero on γ, (5.56) reduces to

$$\int_C \psi([\boldsymbol{P}]_-^+\,\mathrm{d}y - [\boldsymbol{Q}]_-^+\,\mathrm{d}x) = 0, \qquad (5.58)$$

where $[\;]_-^+$ denotes the jump across the shock. This holds for all test functions ψ, and the slope of the shock must therefore satisfy the *Rankine–Hugoniot condition*

$$[\boldsymbol{P}]_-^+\frac{\mathrm{d}y}{\mathrm{d}x} = [\boldsymbol{Q}]_-^+. \qquad (5.59)$$

The scalar Rankine–Hugoniot condition is clearly reproduced if $n = 1$ but, in higher dimensions, (5.59) gives us n relations between $dy/\mathrm{d}x$ and the jumps in the n components of $\boldsymbol{u}$. For semilinear equations, we have

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{u}, \qquad \boldsymbol{Q} = \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{R} = \boldsymbol{c} + \frac{\partial \boldsymbol{A}}{\partial x}\boldsymbol{u} + \frac{\partial \boldsymbol{B}}{\partial y}\boldsymbol{u}, \qquad (5.60)$$

so the Rankine–Hugoniot condition is

$$[\boldsymbol{A}\boldsymbol{u}]_-^+\frac{\mathrm{d}y}{\mathrm{d}x} = [\boldsymbol{B}\boldsymbol{u}]_-^+ \qquad \Rightarrow \qquad \left(\boldsymbol{B} - \frac{\mathrm{d}y}{\mathrm{d}x}\boldsymbol{A}\right)[\boldsymbol{u}]_-^+ = \boldsymbol{0}. \qquad (5.61)$$

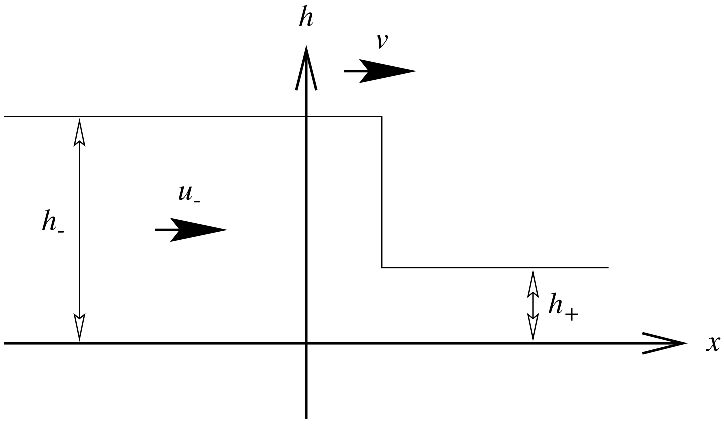

Figure 5.4: Schematic of a bore (with $u_+ = 0$).

Thus $\boldsymbol{u}$ can only be discontinuous if the determinant of the matrix on the left-hand side is zero, which implies that $\mathrm{d}y/\mathrm{d}x$ is equal to a characteristic slope. In other words, shocks occur on characteristics for semilinear equations. This is not true for general quasilinear systems, though.

Example 5.10 *The* shallow-water equations (5.3) *may be written in the conservation form*

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \qquad \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(hu^2 + \tfrac{1}{2}h^2\right) = 0, \tag{5.62a}$$

$$\text{i.e.} \quad \frac{\partial \boldsymbol{P}}{\partial t} + \frac{\partial \boldsymbol{Q}}{\partial x} = \boldsymbol{0}, \qquad \text{where} \quad \boldsymbol{P} = \begin{pmatrix} h \\ u \end{pmatrix}, \qquad \boldsymbol{Q} = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}h^2 \end{pmatrix}, \tag{5.62b}$$

so the Rankine–Hugoniot condition is

$$\begin{bmatrix} h \\ hu \end{bmatrix}_-^+ \frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}h^2 \end{bmatrix}_-^+. \tag{5.63}$$

If the speed of the shock is denoted by $v = \mathrm{d}x/\mathrm{d}t$, then this may be rearranged to give

$$\left[(u-v)h\right]_-^+ = 0, \qquad \left[h(u-v)^2 + \tfrac{1}{2}h^2\right]_-^+ = 0, \tag{5.64}$$

which represent conservation of mass and momentum respectively across the shock.

A shock solution of the shallow-water equations may be used to model a bore in a river, as depicted in Figure 5.4. *Suppose the flux q coming down the bore and the water height h_+ ahead of the bore are given, and that the water is stationary ahead of the bore so that $u_+ = 0$. Then we have*

$$u_- h_- = q, \qquad (u_- - v)h_- = -vh_+, \qquad h_-(u_- - v)^2 + \tfrac{1}{2}h_-^2 = h_+v^2 + \tfrac{1}{2}h_+^2, \tag{5.65}$$

which constitute three equations to determine h_-, u_- (the height and velocity behind the bore) and v (the speed at which the bore propagates).

As for scalar PDEs, there may be many possible ways of writing a quasilinear system in conservation form. To obtain a sensible weak formulation of the problem, one should choose a form such that $\boldsymbol{P}$ and $\boldsymbol{Q}$ represent physical quantities that one expects to be conserved across the shock. Even following this principle, there may be several alternative formulations corresponding to different physical scenarios.

Example 5.11 *The* shallow-water equations *may also be written in the alternative conservation form*

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \qquad \frac{\partial}{\partial t}\left(\tfrac{1}{2}hu^2 + \tfrac{1}{2}h^2\right) + \frac{\partial}{\partial x}\left(\tfrac{1}{2}hu^3 + uh^2\right) = 0, \tag{5.66}$$

in which the two equations represent conservation of mass and energy respectively. The corresponding Rankine–Hugoniot relations,

$$v = \frac{[uh]_-^+}{[h]_-^+} = \frac{\left[\frac{1}{2}hu^3 + uh^2\right]_-^+}{\left[\frac{1}{2}hu^2 + \frac{1}{2}h^2\right]_-^+}, \tag{5.67}$$

give shock conditions that are quite different from (5.64)*. We have a choice of conserving either mass and momentum or mass and energy. In fact, two different kinds of bores are observed in practice:* turbulent bores *that conserve momentum but lose energy, and* undular bores *that conserve energy but not momentum.*

Causality

Once we have chosen a particular conservation form and, thus, a particular weak formulation, there is still the possibility of more than one solution existing if we allow shocks. As for scalar PDEs, there are some shock solutions that, although they satisfy the Rankine–Hugoniot conditions, are unphysical and should be eliminated. There are several methods for doing this, of which we concentrate on *causality*: making sure that information propagates *into* a shock, rather than out of it.

An n-dimensional hyperbolic system has n families of characteristics, so a shock intersects $2n$ of them: n from either side. If there are k *outgoing* characteristics, then there are $(2n-k)$ characteristics going in. We also have the n Rankine–Hugoniot relations, giving a total of $(3n-k)$ pieces of information on the shock. The unknowns are the n components of $\boldsymbol{u}$, on either side of the shock, and the shock slope $\mathrm{d}y/\mathrm{d}x$, giving a total of $(2n+1)$. For the number of equations to equal the number of unknowns, we require $(3n-k) = (2n+1)$, that is

$$k = (n-1). \tag{5.68}$$

This is the condition for a shock to be causal: there must be $(n-1)$ characteristics leaving the shock (and, therefore, $(n+1)$ going in).

For scalar equations, $n = 1$ so there should be no characteristics leaving the shock, 2 going in, as imposed previously. For two-dimensional systems, $n = 2$ so we need one family of characteristics going out of a shock and three going in. Schematics of the characteristics for two alternative shock solutions are shown in Figure 5.5. In diagram (a), three families

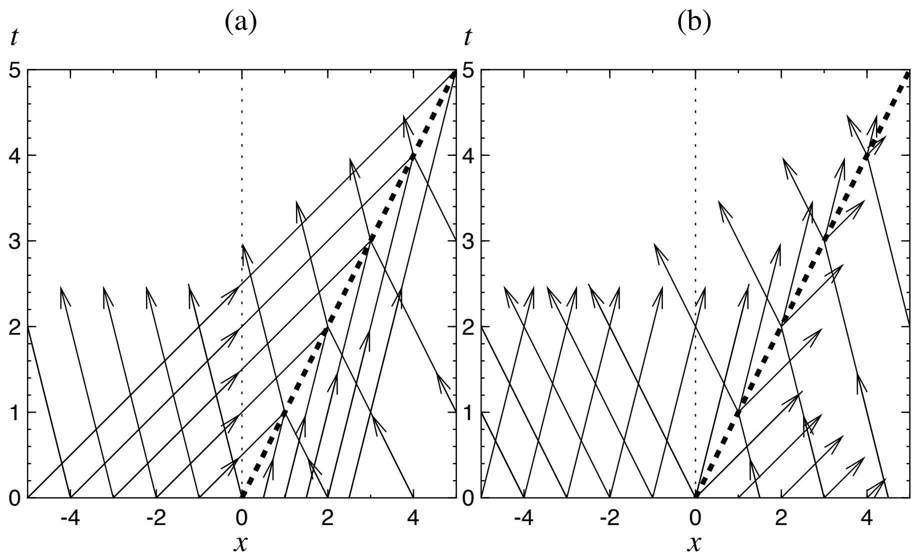

Figure 5.5: Schematics of the characteristics for two alternative shock solutions; the shock is shown as a heavy dashed line.

of characteristics enter the shock and one leaves: this solution is causal. In diagram (b), only one family of characteristics propagates into the shock, so this solution is non-causal and should be discarded.

Example 5.12 *Consider a shock solution of the* shallow-water equations *satisfying the momentum-conserving Rankine–Hugoniot conditions* (5.64)*. Recall that the characteristic velocities are given by* $\mathrm{d}x/\mathrm{d}t = u \pm \sqrt{h}$*. Assume the shock is moving in the positive x-direction. Then, for the solution to be causal, as shown in* Figure 5.5*, there should be one set of characteristics entering the shock from behind and two from the front. In other words, the shock should be moving faster than one of the characteristic speeds behind the shock and faster than both the characteristic speeds ahead of the shock. Thus we obtain the following four inequalities*

$$u_- + \sqrt{h_-} > v, \quad u_- - \sqrt{h_-} < v, \quad u_+ + \sqrt{h_+} < v, \quad u_+ - \sqrt{h_+} < v, \tag{5.69}$$

where, as before, v is the speed of the shock. From these it follows that

$$(u_- - v)^2 < h_- \qquad \text{and} \quad (u_+ - v)^2 > h_+, \tag{5.70}$$

and the Rankine–Hugoniot condition (5.64) *then leads to the condition*

$$h_+ < h_-. \tag{5.71}$$

Thus the height behind a bore must be greater than that ahead; otherwise the bore is not causal and the discontinuity cannot be maintained. That all the inequalities in (5.69) *follow from* (5.71)*, may be verified from the identities*

$$(u_+ - v)^2 = \frac{h_-(h_+ + h_-)}{2h_+}, \qquad (u_- - v)^2 = \frac{h_+(h_+ + h_-)}{2h_-}, \tag{5.72}$$

which follow from (5.64)*.*

6 Second order semi-linear equations

6.1 Introduction

Now we consider second-order scalar equations of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{6.1}$$

This is called a *semilinear* equation because the coefficients a, b and c are independent of u and its derivatives. Note that a second-order scalar equation like (6.1) may always be transformed into a first-order system by setting $p = \partial u/\partial x$, $q = \partial u/\partial y$. Where one of the independent variables is clearly supposed to represent time, we can instead write (6.1) in the form

$$a(x,t)\frac{\partial^2 u}{\partial t^2} + 2b(x,t)\frac{\partial^2 u}{\partial x \partial t} + c(x,t)\frac{\partial^2 u}{\partial x^2} = f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right). \tag{6.2}$$

We are particularly interested in three important canonical examples, namely *the wave equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{6.3a}$$

the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{6.3b}$$

and *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{6.3c}$$

6.2 Cauchy data

Cauchy data for the general second-order quiasi-linear PDE (6.1) is to specify u and its normal derivative on some curve Γ in the (x,y) plane, *i.e.*

$$x = x_0(s), \qquad y = y_0(s), \qquad u = u_0(s), \qquad \frac{\partial u}{\partial n} = v_0(s), \tag{6.4}$$

where s parametrises Γ. This is equivalent to specifying u and both its first derivatives on Γ, since

$$\frac{\mathrm{d}u_0}{\mathrm{d}s} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x_0}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y_0}{\mathrm{d}s} \tag{6.5}$$

and

$$v_0(s) = \frac{\partial u}{\partial n} = \left(\frac{\partial u}{\partial x}\frac{\mathrm{d}y_0}{\mathrm{d}s} - \frac{\partial u}{\partial y}\frac{\mathrm{d}x_0}{\mathrm{d}s}\right)\left\{\left(\frac{\mathrm{d}x_0}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y_0}{\mathrm{d}s}\right)^2\right\}^{-1/2} \tag{6.6}$$

may be solved simultaneously for $\partial u/\partial x$ and $\partial u/\partial y$. We may therefore replace (6.4) with

$$x = x_0(s), \qquad y = y_0(s), \qquad u = u_0(s), \qquad \frac{\partial u}{\partial x} = p_0(s), \qquad \frac{\partial u}{\partial y} = q_0(s). \tag{6.7}$$

A necessary condition for the solution $u(x, y)$ to be uniquely defined in a neighbourhood of Γ is for the second derivatives of u to be determined on Γ. Now, if we differentiate the initial data along Γ and use the chain rule, we obtain

$$\frac{\mathrm{d}p_0}{\mathrm{d}s} = \frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\partial^2 u}{\partial x^2} + \frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial^2 u}{\partial x \partial y}, \qquad \frac{\mathrm{d}q_0}{\mathrm{d}s} = \frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\partial^2 u}{\partial x \partial y} + \frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial^2 u}{\partial y^2}. \tag{6.8}$$

Along with the PDE (6.1), this gives us a system of three equations for the three second partial derivatives of u, and the determinant of this system is

$$\begin{vmatrix} a & 2b & c \\ \dfrac{\mathrm{d}x_0}{\mathrm{d}s} & \dfrac{\mathrm{d}y_0}{\mathrm{d}s} & 0 \\ 0 & \dfrac{\mathrm{d}x_0}{\mathrm{d}s} & \dfrac{\mathrm{d}y_0}{\mathrm{d}s} \end{vmatrix} = a\left(\frac{\mathrm{d}y_0}{\mathrm{d}s}\right)^2 - 2b\frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\mathrm{d}y_0}{\mathrm{d}s} + c\left(\frac{\mathrm{d}x_0}{\mathrm{d}s}\right)^2. \tag{6.9}$$

A necessary condition for the Cauchy data (6.7) to determine u locally is, therefore,

$$a\left(\frac{\mathrm{d}y_0}{\mathrm{d}s}\right)^2 - 2b\frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\mathrm{d}y_0}{\mathrm{d}s} + c\left(\frac{\mathrm{d}x_0}{\mathrm{d}s}\right)^2 \neq 0. \tag{6.10}$$

6.3 Characteristics

Definition

As for first-order equations, we define *characteristics* to be curves in the (x, y) plane on which Cauchy data do not determine a unique solution. If such a curve is parametrised by $x = x(\tau)$, $y = y(\tau)$ then, from (6.10), we have

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - 2b\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + c\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = 0, \tag{6.11}$$

so the slopes of the characteristics satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda \qquad \text{where} \quad a\lambda^2 - 2b\lambda + c = 0 \tag{6.12}$$

Characteristics may also be defined as curves across which the second derivatives of u may be discontinuous, with lower derivatives continuous. It is left as an exercise to show that this alternative definition leads to the same equation (6.12) for the characteristic slopes.

Classification

The PDE (6.1) is classified according to the number of distinct real characteristics it possesses, and this is determined by the sign of the *discriminant* $b^2 - ac$.

1. $\underline{b^2 > ac}$ implies that there are two distinct real characteristics and the PDE is *hyperbolic*.
2. $\underline{b^2 < ac}$ implies that there are two complex conjugate characteristics and the PDE is *elliptic*.
3. $\underline{b^2 = ac}$ implies that there is one repeated real characteristics and the PDE is *parabolic*.

Since a, b and c are functions of x and y, the PDE may change *type* (*i.e.*, hyperbolic, parabolic or elliptic) from one region to another. However, the type of a PDE at any given position is invariant under coordinate transformations.

Example 6.1 *Suppose we transform to new coordinates* (X, Y)*, which are given as functions of* x *and* y*. Then the PDE* (6.1) *is transformed to*

$$A\frac{\partial^2 u}{\partial X^2} + 2B\frac{\partial^2 u}{\partial X \partial Y} + C\frac{\partial^2 u}{\partial Y^2} = F, \tag{6.13}$$

where

$$A = a\left(\frac{\partial X}{\partial x}\right)^2 + 2b\frac{\partial X}{\partial x}\frac{\partial X}{\partial y} + c\left(\frac{\partial X}{\partial y}\right)^2, \tag{6.14a}$$

$$B = a\frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} + b\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right) + c\frac{\partial X}{\partial y}\frac{\partial Y}{\partial y}, \tag{6.14b}$$

$$C = a\left(\frac{\partial Y}{\partial x}\right)^2 + 2b\frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y} + c\left(\frac{\partial Y}{\partial y}\right)^2. \tag{6.14c}$$

It follows that

$$B^2 - AC = \left(b^2 - ac\right)\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right)^2, \tag{6.15}$$

so that the sign of the discriminant is invariant.

6.4 Canonical forms

Hyperbolic equations

For hyperbolic PDEs, the quadratic equation (6.12) has two real distinct roots, say $\lambda_1(x, y)$ and $\lambda_2(x, y)$. Now suppose the differential equations $\mathrm{d}y/\mathrm{d}x = \lambda_i$ have first integrals given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_1(x, y) \qquad \Rightarrow \qquad \xi(x, y) = \text{const}, \tag{6.16a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_2(x, y) \qquad \Rightarrow \qquad \eta(x, y) = \text{const}. \tag{6.16b}$$

Now we change variables from (x, y) to (ξ, η), using the chain rules

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} \tag{6.17}$$

and so forth. The PDE (6.1) is clearly transformed to an equation with an analogous form, that is,

$$\alpha(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} + \beta(\xi,\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \gamma(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} = \phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{6.18}$$

We know that the characteristics are given by $\xi = \text{const}$ and $\eta = \text{const}$, so the roots of the quadratic form

$$\alpha\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2 - 2\beta\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + \gamma\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 = 0 \tag{6.19}$$

must be $\mathrm{d}\xi/\mathrm{d}\tau = 0$ and $\mathrm{d}\eta/\mathrm{d}\tau = 0$. It follows that $\alpha = \gamma = 0$, so that (6.18) takes the form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{6.20}$$

This is the so-called *canonical form* for second-order hyperbolic PDEs.

Example 6.2 *For the PDE*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y), \tag{6.21}$$

the characteristics are given by

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 1 = 0 \qquad \Rightarrow \qquad y \pm x = \mathit{const}. \tag{6.22}$$

So we set $\xi = x - y$, $\eta = x + y$ *and, by changing variables, obtain*

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{4} f\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}\right) = \phi(\xi,\eta), \ \ \mathit{say}. \tag{6.23}$$

This transformed PDE may be integrated directly to give the general solution

$$u = \iint \phi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta + h_1(\xi) + h_2(\eta). \tag{6.24}$$

Elliptic equations

For elliptic equations, the roots of the characteristic equation (6.12) are complex conjugates, say

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda_R(x,y) \pm \mathrm{i}\lambda_I(x,y). \tag{6.25}$$

Suppose that the integrals of these ODEs can be written in the form

$$\xi(x,y) \pm \mathrm{i}\eta(x,y) = \text{const}, \tag{6.26}$$

for some functions ξ and η. Then we use ξ and η as new variables. Again, the transformed equation must be of the form (6.18) but, this time, the roots of

$$\alpha\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2 - 2\beta\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + \gamma\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 = 0 \tag{6.27}$$

are $\mathrm{d}\xi/\mathrm{d}\tau \pm \mathrm{i}\mathrm{d}\eta/\mathrm{d}\tau = 0$, which implies that $\beta = 0$, $\gamma = \alpha$. The canonical form for elliptic PDEs is, therefore,

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{6.28}$$

Example 6.3 *For the PDE*

$$y^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + 2x^2\frac{\partial^2 u}{\partial y^2} = 0, \tag{6.29}$$

the characteristics satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1 \pm \mathrm{i})\frac{x}{y} \qquad \Rightarrow \qquad (1 \pm \mathrm{i})x^2 - y^2 = \mathit{const.} \tag{6.30}$$

If we choose $\xi = x^2$, $\eta = x^2 - y^2$, then the equation transforms to

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = -\frac{1}{2\xi}\frac{\partial u}{\partial \xi} + \left(\frac{\xi + \eta}{2\xi(\xi - \eta)}\right)\frac{\partial u}{\partial \eta}. \tag{6.31}$$

Parabolic equations

For parabolic PDEs, there is one repeated characteristic slope

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}, \tag{6.32}$$

which we suppose has the solution $\eta(x, y) = \text{const}$. Then any convenient linearly independent function $\xi(x, y)$ may be chosen, and the PDE (6.1) transforms to the canonical form

$$\frac{\partial^2 u}{\partial \xi^2} = \phi\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{6.33}$$

7 Semi-linear hyperbolic equations

7.1 Non-Cauchy data

As stated in Section 5, Cauchy data for second-order equations specifies both u and its normal derivative on a plane curve Γ. In general, however, there may be boundaries on which it is appropriate to prescribe just one boundary condition or even none.

Example 7.1 *Consider a string of length L and density (i.e. mass per unit length) ρ, stretched to a tension T between the points $x = 0$ and $x = L$. It may be shown that small transverse displacements $u(x, t)$ satisfy the* wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{7.1}$$

where t is time and $c^2 = T/\rho$. Suppose we wish to determine $u(x, t)$ in the region $0 \le x \le L$, $0 \le t \le T$. To do so, we need to specify the initial displacement $u(x, 0) = u_0(x)$ and the initial velocity $\partial u/\partial t(x, 0) = v_0(x)$; these correspond to Cauchy data on the initial curve $t = 0$, $0 \le x \le L$. On the boundary curves $x = 0$ and $x = L$, we give just one *boundary condition, namely that the displacement is zero: $u(0, t) = u(L, t) = 0$. Finally, on the boundary $t = T$ we give no boundary conditions at all. The number of boundary conditions applied on each boundary and the characterstics are illustrated in* Figure 7.1.

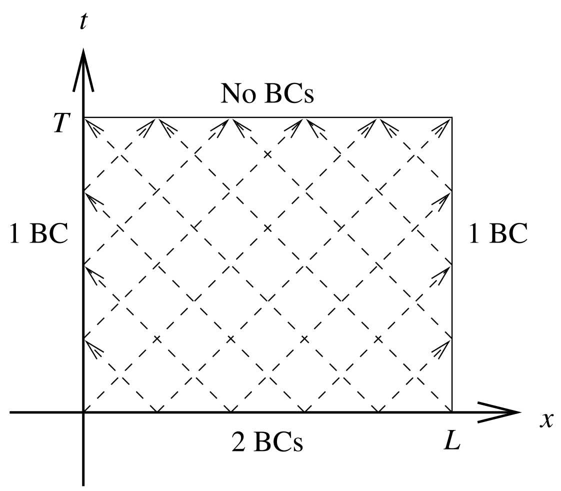

Figure 7.1: Characteristics for Example 7.1, and the number of boundary conditions applied on each boundary.

Example 7.1 illustrates that, in general, the number of boundary conditions needed on any boundary is equal to the number of characteristic families travelling *out* of that boundary. Where two sets of characteristics travel out, two conditions must be given, as in the Cauchy problem. On a boundary with one characteristic family travelling in and one travelling out, just one condition must be given. Finally, no conditions may be given on a boundary where all the characteristics travel in.

7.2 Weak solutions

Weak solutions may be defined in a manner analogous to that used for first-order PDEs. Notice that, unlike quasilinear equations, the semilinear equations considered here do not spontaneously form shocks. Consequently a discontinuous solution will only arise if discontinuous initial data are specified. Furthermore, for semilinear equations, all discontinuities propagate along characteristics.

7.3 Riemann's method

Riemann's method is a way of solving *linear* hyperbolic equations that are in (*i.e.*, if necessary have been transformed into) canonical form

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial x \partial y} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y). \tag{7.2}$$

Suppose Cauchy data are given on a curve Γ in the (x, y) plane, so that u, $\partial u/\partial x$ and $\partial u/\partial y$ are all known on Γ. The condition (6.10) for these data to determine the solution locally translates into the restriction that Γ must be nowehere parallel to either the x- or the y-axis.

We define the *adjoint* of the differential operator $\mathcal{L}$ in (7.2) by

$$\mathcal{L}^\star[v] = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv. \tag{7.3}$$

The point of this definition is to make $v\mathcal{L}[u] - u\mathcal{L}^\star[v]$ be the divergence of something. The general procedure for constructing the adjoint of a differential operator is (i) put all coefficients inside the derivatives; (ii) switch the signs of all odd-order derivatives. By combining (7.2) and (7.3), it is readily shown that

$$v\mathcal{L}[u] - u\mathcal{L}^\star[v] = \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y} + auv\right) + \frac{\partial}{\partial y}\left(-u\frac{\partial v}{\partial x} + buv\right). \tag{7.4}$$

Now we integrate over the region illustrated in Figure 7.2, namely the region bounded by Γ and the lines $x = \xi$ and $y = \eta$, where ξ and η are constants. The intersection between $y = \eta$ and Γ is labelled A, the point where $x = \xi$ intersects Γ is labelled B, and the point (ξ, η) is labelled P. By using (7.4) and Green's theorem, we obtain

$$\iint_D \{R\mathcal{L}[u] - u\mathcal{L}^\star[R]\}\, \mathrm{d}x\mathrm{d}y = \oint_{\partial D} \left\{ R\left(\frac{\partial u}{\partial y} + au\right)\mathrm{d}y + u\left(\frac{\partial R}{\partial x} - bR\right)\mathrm{d}x \right\}, \tag{7.5}$$

where R is called the *Riemann function*. The idea is to choose the properties of R to simplify (7.5) as much as possible.

On the left-hand side of (7.5), we use the fact that $\mathcal{L}[u] = f$ and choose R to satisfy $\mathcal{L}^\star[R] = 0$. On the right-hand side, we note that $\mathrm{d}y = 0$ on AP and that $\mathrm{d}x = 0$ on PB.

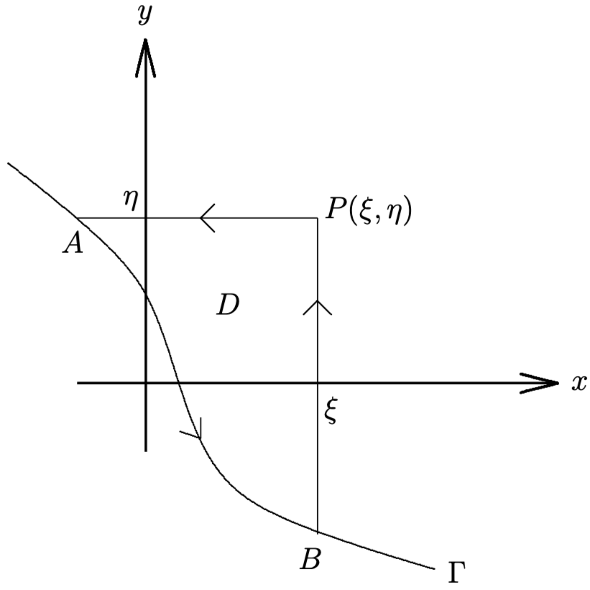

Figure 7.2: The integration region D for Riemann's method.

The latter integral along PB we integrate by parts:

$$\begin{aligned}\iint_D Rf\,\mathrm{d}x\mathrm{d}y &= \int_P^A u\left(\frac{\partial R}{\partial x} - bR\right)\mathrm{d}x + \int_B^P u\left(aR - \frac{\partial R}{\partial y}\right)\mathrm{d}y + [Ru]_B^P \\ &\quad + \int_A^B \left\{R\left(\frac{\partial u}{\partial y} + au\right)\mathrm{d}y + u\left(\frac{\partial R}{\partial x} - bR\right)\mathrm{d}x\right\}.\end{aligned}$$

Now we suppose that $\partial R/\partial x = bR$ on AP and that $\partial R/\partial y = aR$ on BP, to eliminate the first two integrals. If we also choose $R = 1$ at the point P, we end up with

$$u(\xi,\eta) = \iint_D Rf\,\mathrm{d}x\mathrm{d}y + R(B)u(B) - \int_A^B \left\{R\left(\frac{\partial u}{\partial y} + au\right)\mathrm{d}y + u\left(\frac{\partial R}{\partial x} - bR\right)\mathrm{d}x\right\}. \tag{7.6}$$

This gives us the solution at any arbitrary point (ξ,η) in terms of f, u and its derivatives on Γ (*i.e.* the Cauchy data), and the Riemann function R, so if we can find R then the problem is solved in principle.

To summarise, the properties we require of R are

$$\left.\begin{aligned} \mathcal{L}^\star[R] = \frac{\partial^2 R}{\partial x\partial y} - \frac{\partial}{\partial x}(aR) - \frac{\partial}{\partial y}(bR) + cR = 0, &\qquad x<\xi,\ y<\eta \\ \frac{\partial R}{\partial x} = bR &\qquad y=\eta, \\ \frac{\partial R}{\partial y} = aR &\qquad x=\xi, \\ R = 1 &\qquad (x,y)=(\xi,\eta). \end{aligned}\right\} \tag{7.7}$$

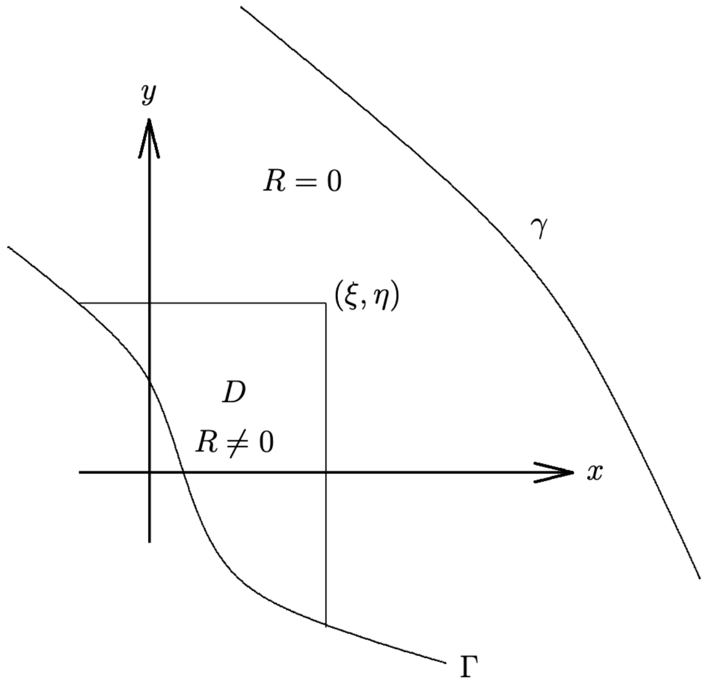

Figure 7.3: Schematic showing the behaviour of the Riemann function.

It may be shown that the problem (7.7) determines $R(x, y; \xi, \eta)$ uniquely. Notice that R is independent of the function f and the boundary data applied to u: it depends only on the orginal differential operator $\mathcal{L}$. Thus, it may be found "once and for all" and then applied to *any* boundary data and *any* right-hand side f. However, finding R explicitly is usually difficult except in simple special cases. It is nevertheless useful for theoretical purposes, for example proving that u depends continuously on its boundary data.

Finally, we note an alternative definition of R in terms of generalised functions. Suppose we form a domain S, in which the solution is to be found, between Γ and a second curve γ. Now we solve the problem

$$\left.\begin{aligned} \mathcal{L}^\star[R] &= \delta(x-\xi)\delta(y-\eta) && \text{in } S, \\ R = \frac{\partial R}{\partial n} &= 0 && \text{on } \gamma, \end{aligned}\right\} \tag{7.8}$$

for R *backwards*, starting from γ, where δ is the Dirac delta function. As illustrated in Figure 7.3, since R and its first derivatives are zero on γ, R is identically zero outside the region D considered previously. Across the lines $x = \xi$ and $y = \eta$, discontinuities in R balance the singularity on the right-hand side of (7.8); in particular,

$$R \sim \mathrm{H}(\xi - x)\,\mathrm{H}(\eta - y) \qquad \text{as} \quad (x, y) \to (\xi, \eta), \tag{7.9}$$

where H is the Heaviside function. It is readily verified that (7.8) implies that R satisfies the properties listed in (7.7).

Example 7.2 *The* inhomogeneous wave equation *in canonical form is*

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial x \partial y} = f(x,y). \tag{7.10}$$

Here, the problem satisfied by R is

$$\left.\begin{aligned} \mathcal{L}^\star[R] = \frac{\partial^2 R}{\partial x \partial y} &= 0, && x < \xi,\ y < \eta \\ \frac{\partial R}{\partial x} &= 0 && y = \eta, \\ \frac{\partial R}{\partial y} &= 0 && x = \xi, \\ R &= 1 && (x,y) = (\xi,\eta). \end{aligned}\right\} \tag{7.11}$$

Notice that R is constant and, therefore, equal to 1 on $x = \xi$ and on $y = \eta$. It follows that the Riemann function is simply given by $R \equiv 1$ in $x < \xi$, $y < \eta$ (or $R = \mathrm{H}(\xi - x)\,\mathrm{H}(\eta - y)$). Thus the solution is

$$u = \iint_D f \,\mathrm{d}x\mathrm{d}y + u(B) - \int_A^B \frac{\partial u}{\partial y}\,\mathrm{d}y, \tag{7.12}$$

which is equivalent to the solution found in Example 6.2.

8 Semi-linear elliptic equations

8.1 Well-posed boundary data

The canonical form for second-order elliptic PDEs is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{8.1}$$

The operator on the left-hand side is referred to as the *Laplacian*, for which the symbols $\nabla^2 u$ or $\triangle u$ are often used as shorthand.

Recall the Cauchy–Kovalevskaya theorem, which states that a unique solution to (8.1) exists in a neighbourhood of an initial curve Γ if u, $\partial u/\partial x$ and $\partial u/\partial y$ are specified analytic functions on Γ and satisfy the condition (6.10). However, the Cauchy problem is *ill posed* for elliptic equations. This manifests itself in several ways, including the following.

- Small changes in the initial data may lead to arbitrarily large changes in the solution. This makes it impossible to compute meaningful solutions numerically.
- Although a unique solution exists in a neighbourhood of Γ, that neighbourhood may be arbitrarily small, in that a singularity may form arbitrarily close to Γ.

It transpires that appropriate boundary data for (8.1) is to give just *one* boundary condition on u everywhere on a closed curve. We show below that this does indeed give a unique solution for Poisson's equation, a special case of (8.1). This should be constrasted with the situation for hyperbolic equations where, as illustrated in Figure 7.1, either 0, 1 or 2 boundary conditions are given at each point on a closed curve, depending on the number of inward-travelling characteristics.

8.2 Uniqueness theorems for Poisson's equation

Poisson's equation is a special case of (8.1), in which the right-hand side f depends only on x and y. Consider the so-called *Dirichlet problem*, namely Poisson's equation for $u(x, y)$ in some domain D, with u given on the boundary of D:

$$\left.\begin{aligned} \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f(x, y) \quad (x, y) \in D, \\ u &= g(x, y) \quad (x, y) \in \partial D. \end{aligned}\right\} \tag{8.2}$$

We will now show that, if a solution of (8.2) exists, then it is unique.

Suppose there exist that two solutions u_1 and u_2 of (8.2). If $\phi = u_1 - u_2$, then ϕ satisfies the homogeneous Dirichlet problem

$$\left.\begin{aligned} \nabla^2 \phi &= 0 \quad (x, y) \in D, \\ \phi &= 0 \quad (x, y) \in \partial D. \end{aligned}\right\} \tag{8.3}$$

Now consider the *Dirichlet integral*

$$\iint_D \boldsymbol{\nabla} \cdot (\phi \boldsymbol{\nabla} \phi) \, \mathrm{d}x\mathrm{d}y = \iint_D \left\{ \frac{\partial}{\partial x} \left(\phi \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\phi \frac{\partial \phi}{\partial y} \right) \right\} \mathrm{d}x\mathrm{d}y. \tag{8.4}$$

This may be written in two ways (i) by expanding out the derivatives on the left-hand side, (ii) by using Green's theorem:

$$\iint_D \left\{ \phi \nabla^2 \phi + |\boldsymbol{\nabla} \phi|^2 \right\} \mathrm{d}x\mathrm{d}y = \oint_{\partial D} \phi \frac{\partial \phi}{\partial n} \, \mathrm{d}s, \tag{8.5}$$

where n and s refer to the outward-pointing normal and arclength respectively on ∂D. Now we simplify by using the properties (8.3) of ϕ:

$$\iint_D |\boldsymbol{\nabla} \phi|^2 \, \mathrm{d}x\mathrm{d}y = 0. \tag{8.6}$$

Since the integrand is non-negative (and assumed to be continuous), it must be identically zero in D. It follows that ϕ is constant in D where, since $\phi = 0$ on ∂D, that constant must be zero. Hence $u_1 \equiv u_2$, so the solution to (8.2), if it exists, is unique.

Next consider the *Neumann problem*, in which the normal derivative of u, rather than u itself, is specified on ∂D:

$$\left.\begin{aligned} \nabla^2 u &= f(x, y) \quad (x, y) \in D, \\ \frac{\partial u}{\partial n} &= g(x, y) \quad (x, y) \in \partial D. \end{aligned}\right\} \tag{8.7}$$

By Green's theorem,

$$\iint_D \nabla^2 u \, \mathrm{d}x\mathrm{d}y = \oint_{\partial D} \frac{\partial u}{\partial n} \, \mathrm{d}s \qquad \Rightarrow \qquad \iint_D f \, \mathrm{d}x\mathrm{d}y = \oint g \, \mathrm{d}s. \tag{8.8}$$

This is the *solvability condition* for (8.7); if (8.8) is not satisfied, then (8.7) has no solution.

Now suppose that (8.7) *is* satisfied and let u_1 and u_2 be two solutions of (8.7). Then $\phi = u_1 - u_2$ satisfies the homogeneous Neumann problem

$$\left.\begin{aligned} \nabla^2 \phi &= 0 \quad (x, y) \in D, \\ \frac{\partial \phi}{\partial n} &= 0 \quad (x, y) \in \partial D, \end{aligned}\right\} \tag{8.9}$$

and it is straightforward to show (using the same approach as for the Dirichlet problem) that ϕ must therefore be constant. Thus, if the solvability condition (8.8) is satisfied, then the solution to (8.7), if it exists, is unique up to the addition of an arbitrary constant.

This behaviour of the solutions of the Neumann problem is an instance of the *Fredholm alternative*. The homogeneous problem (8.9) has the nontrivial solution $\phi = \text{const}$. Thus there is no solution to the inhomogeneous version (8.7) unless an orthogonality condition (8.8) is satisfied. If so, then the solution to (8.7) is nonunique since an arbitrary multiple of the homogeneous solution may be added.

8.3 Maximum principle

The maximum principle is proved in the course *A1 Differential Equations 1*. It is included here for completeness, but *it is non-examinable*. Suppose u satisfies Poisson's equation in a bounded domain D,

$$\nabla^2 u = f(x,y) \qquad \text{in} \quad D, \tag{8.10}$$

where $f \geq 0$ in D.[3] Then u attains its maximum value on ∂D.

The proof proceeds in two parts. First suppose that f is strictly positive in D. If u has an interior maximum at some point (x_0, y_0) inside D, then the following conditions must be satisfied at (x_0, y_0):

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) \geq \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \qquad \frac{\partial^2 u}{\partial x^2}, \; \frac{\partial^2 u}{\partial y^2} \leq 0. \tag{8.11}$$

But, if f is strictly positive, (8.10) implies that it is impossible for both $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ to be ≤ 0. Hence u cannot have an interior maximum within D, so it must attain its maximum value on ∂D.

This completes the proof for the case where $f > 0$ in D. Now suppose that we only have $f \geq 0$ in D, and consider the function

$$v(x,y) = u(x,y) + \frac{\epsilon}{4}\left(x^2 + y^2\right), \tag{8.12}$$

where ϵ is a positive constant. Then

$$\nabla^2 v = f + \epsilon > 0 \quad \text{in } D \tag{8.13}$$

so, using the result just proved, v attains its maximum value on ∂D. Now, if the maximum value of u on ∂D is M and the maximum value of (x^2+y^2) on ∂D is R^2, then the maximum value of v on ∂D (and thus throughout D) is bounded by $M + (\epsilon/4)R^2$. In other words, the inequality

$$u + \frac{\epsilon}{4}\left(x^2 + y^2\right) = v \leq M + \frac{\epsilon}{4}R^2 \tag{8.14}$$

holds for all $(x,y) \in D$. Letting $\epsilon \to 0$, we see that $u \leq M$ throughout D, *i.e.*, that u attains its maximum value on ∂D.

It clearly follows (by using the above result with u replaced by $-u$) that, if $f \leq 0$ in D, then u attains its *minimum* value on ∂D. In the case $f = 0$, u therefore attains both its maximum and minimum values on ∂V. This is an important property of *Laplace's equation* $\nabla^2 u = 0$: in any closed region D, u is bounded between its maximum and minimum values on ∂D. It may also be used to prove uniqueness of solutions of Poisson's equation. In the homogeneous Dirichlet problem (8.3), ϕ must take its maximum and minimum values on ∂D. But $\phi = 0$ on ∂D and must, therefore, be identically zero.

[3] Actually, the theorem also holds if f depends on u and its derivatives, provided it is non-negative in D.

8.4 Green's functions

Green's functions play the same role for linear elliptic PDEs that Riemann functions play for hyperbolic PDEs. Consider the Dirichlet problem

$$\left.\begin{aligned} \nabla^2 u &= f(x,y) \qquad (x,y) \in D, \\ u &= g(x,y) \qquad (x,y) \in \partial D. \end{aligned}\right\} \tag{8.15}$$

Notice that the Laplacian is a *self-adjoint* operator, so we consider the integral

$$\begin{aligned} \iint_S \left\{ u\nabla^2 G - G\nabla^2 u \right\} \mathrm{d}x\mathrm{d}y &= \iint_S \boldsymbol{\nabla} \cdot \{u\boldsymbol{\nabla} G - G\boldsymbol{\nabla} u\}\, \mathrm{d}x\mathrm{d}y \\ &= \oint_{\partial S} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) \mathrm{d}s, \end{aligned} \tag{8.16}$$

where n and s refer to the outward-pointing normal and arclength on ∂D, and G is the Green's function. If we choose G to satisfy

$$\left.\begin{aligned} \nabla^2 G &= \delta(x-\xi)\delta(y-\eta) \qquad \text{in } D, \\ G &= 0 \qquad \text{on } \partial D, \end{aligned}\right\} \tag{8.17}$$

where δ is the Dirac delta-function, then we have

$$u(\xi,\eta) = \iint_D Gf\, \mathrm{d}x\mathrm{d}y + \oint_{\partial D} g\frac{\partial G}{\partial n}\, \mathrm{d}s. \tag{8.18}$$

If the Green's function G can be found from (8.17), then (8.18) gives the solution at any point $(\xi,\eta) \in D$ in terms of G and the given functions f and g.

Now we have to find a function G whose Laplacian is equal to a delta function. Thus G should satisfy $\nabla^2 G = 0$ when $(x,y) \neq (\xi,\eta)$ and should have some sort of singularity as $(x,y) \to (\xi,\eta)$, and it turns out that the correct singularity is

$$G \sim \frac{1}{2\pi} \log\left|(x,y) - (\xi,\eta)\right| + O(1) \qquad \text{as} \quad (x,y) \to (\xi,\eta). \tag{8.19}$$

It is readily verified that $\log\left|(x,y)-(\xi,\eta)\right|$ satisfies Laplace's equation for all $(x,y) \neq (\xi,\eta)$. Hence

$$\iint_D \phi\nabla^2 G\, \mathrm{d}x\mathrm{d}y = \iint_{S_\epsilon} \phi\nabla^2 G\, \mathrm{d}x\mathrm{d}y, \tag{8.20}$$

where S_ϵ is a disc of radius ϵ centred at (ξ,η) and $\phi(x,y)$ is any suitably differentiable test function. Recalling that $\nabla^2 G$ is a distribution, we evaluate the right-hand side as

$$\iint_{S_\epsilon} \phi\nabla^2 G\, \mathrm{d}x\mathrm{d}y = \iint_{S_\epsilon} G\nabla^2\phi\, \mathrm{d}x\mathrm{d}y + \oint_{\partial S_\epsilon} \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n} \right) \mathrm{d}s. \tag{8.21}$$

Now introducing local polar coordinates $x = \xi + r\cos\theta$, $y = \eta + r\sin\theta$, letting $\epsilon \to 0$ and using the asymptotic behaviour (8.19), we find that

$$\iint_{S_\epsilon} \phi \nabla^2 G \,\mathrm{d}x\mathrm{d}y \sim \int_0^{2\pi}\int_0^{\epsilon} \frac{1}{2\pi}\log(r)\nabla^2\phi\, r\,\mathrm{d}r\mathrm{d}\theta + \int_0^{2\pi} \frac{1}{2\pi r}\phi(\xi,\eta) r\,\mathrm{d}r\mathrm{d}\theta - \int_0^{2\pi} \frac{1}{2\pi}\log(\epsilon)\frac{\partial\phi}{\partial r}\epsilon\,\mathrm{d}\theta. \quad (8.22)$$

The first and third integrals on the right-hand side vanish as $\epsilon \to 0$, so we are left with

$$\iint_{S_\epsilon} \phi\nabla^2 G\,\mathrm{d}x\mathrm{d}y \to \phi(\xi,\eta) \qquad \text{as} \quad \epsilon \to 0, \qquad (8.23)$$

and hence (8.20) reduces to

$$\iint_D \phi\nabla^2 G\,\mathrm{d}x\mathrm{d}y \equiv \phi(\xi,\eta). \qquad (8.24)$$

Thus $\nabla^2 G$ is indeed equal to a delta function.

An alternative statement of (8.17) is, therefore,

$$\left.\begin{aligned} \nabla^2 G &= 0 && \text{in} \quad D\backslash(\xi,\eta), \\ G &= 0 && \text{on} \quad \partial D, \\ G &\sim \frac{1}{2\pi}\log\big|(x,y)-(\xi,\eta)\big| && \text{as} \quad (x,y)\to(\xi,\eta). \end{aligned}\right\} \qquad (8.25)$$

The trick to obtaining G is thus to find a function that satisfies Laplace's equation inside D and is equal to $-1/(2\pi)\log\big|(x,y)-(\xi,\eta)\big|$ on ∂D.

Example 8.1 Green's function for a half-plane

For the problem

$$\left.\begin{aligned} \nabla^2 u &= f(x,y) && y>0, \\ u &= g(x) && y = 0, \\ u &\to 0 && y\to\infty \quad , \end{aligned}\right\} \qquad (8.26)$$

the Green's function satisfies

$$\left.\begin{aligned} \nabla^2 G &= 0 && y>0,\ (x,y)\neq(\xi,\eta), \\ G &= 0 && y = 0, \\ G &\sim \frac{1}{2\pi}\log\big|(x,y)-(\xi,\eta)\big|, && (x,y)\to(\xi,\eta). \end{aligned}\right\} \qquad (8.27)$$

This problem is readily solved using the method of images. *Consider an image singularity at the point* $(\xi,-\eta)$, *that is the function* $1/(2\pi)\log\big|(x,y)-(\xi,-\eta)\big|$. *This function clearly satisfies Laplace's equation away from the singularity (which is outside the half-plane in which* G *is to* *be defined). It is also equal to* $1/(2\pi)\log\big|(x,y)-(\xi,\eta)\big|$ *on the line* $y=0$*. Hence the Green's function for this problem is*

$$\begin{aligned} G(x,y;\xi,\eta) &= \frac{1}{2\pi}\log\big|(x,y)-(\xi,\eta)\big| - \frac{1}{2\pi}\log\big|(x,y)-(\xi,-\eta)\big| \\ &= \frac{1}{4\pi}\log\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right). \end{aligned} \tag{8.28}$$

Example 8.2 Green's function for a circle
For a Dirichlet problem on a circular disc of radius a*, the Green's function satisfies*

$$\left.\begin{aligned} \nabla^2 G &= 0 && x^2+y^2<a^2,\ (x,y)\neq(\xi,\eta), \\ G &= 0 && x^2+y^2=a^2, \\ G &\sim \frac{1}{2\pi}\log\big|(x,y)-(\xi,\eta)\big|, && (x,y)\to(\xi,\eta). \end{aligned}\right\} \tag{8.29}$$

The point (ξ,η) *inside the disc has a corresponding* image point (ξ',η') *outside the disc, defined by*

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \frac{a^2}{\xi^2+\eta^2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}. \tag{8.30}$$

This image point has the property that, for any point (x,y) *on the circle* $x^2+y^2=a^2$*, the ratio*

$$\frac{\big|(x,y)-(\xi,\eta)\big|}{\big|(x,y)-(\xi',\eta')\big|}$$

is constant, and equal to $\big|(\xi,\eta)\big|/a$*. The Green's function is therefore given by*

$$\begin{aligned} G(x,y;\xi,\eta) &= \frac{1}{2\pi}\log\big|(x,y)-(\xi,\eta)\big| - \frac{1}{2\pi}\log\left(\frac{\big|(x,y)-(\xi',\eta')\big|\,\big|(\xi,\eta)\big|}{a}\right) \\ &= \frac{1}{4\pi}\log\left(\frac{\big((x-\xi^2)+(y-\eta)^2\big)a^2}{(x^2+y^2)(\xi^2+\eta^2)-2a^2x\xi-2a^2y\eta+a^4}\right). \end{aligned} \tag{8.31}$$

Notice that, unlike the Riemann function for hyperbolic PDEs, the Green's function is dependent on the shape of the region D. Unless D is a simple shape, it is usually difficult to find G explicitly. One useful approach is to use *conformal mapping* to transform D into a simple domain such as a disc — see below

It is also possible to define a Green's function for Neumann problems, but this is more difficult since the solvability condition (8.8) must be enforced. This may be achieved by defining G to satisfy

$$\left.\begin{aligned} \nabla^2 G &= \delta(x-\xi)\delta(y-\eta) - C && \text{in } D, \\ \frac{\partial G}{\partial n} &= 0 && \text{on } \partial D, \end{aligned}\right\} \tag{8.32}$$

where C is chosen to satisfy the solvability condition. Integration of (8.32) over D leads to the condition $C = 1/A$, where A is the area of D. Then, by an argument analogous to that used for the Dirichlet problem, we obtain the solution as

$$u(\xi, \eta) = \bar{u} + \iint_D Gf \,\mathrm{d}x\mathrm{d}y - \oint_{\partial D} gG \,\mathrm{d}s. \tag{8.33}$$

Here, $\bar{u}$ is the average value of u,

$$\bar{u} = \frac{1}{A} \iint_D u \,\mathrm{d}x\mathrm{d}y, \tag{8.34}$$

which remains indeterminate (recall that the solution of the Neumann problem is only determined up to the addition of an arbitrary constant).

8.5 Complex variable methods

There is a close connection between Laplace's equation and functions of a complex variable $z = x + \mathrm{i}y$. If we transform variables from (x, y) to $(z, \bar{z})$, where

$$z = x + \mathrm{i}y, \qquad \bar{z} = x - \mathrm{i}y, \tag{8.35}$$

then we find

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0, \tag{8.36}$$

and it follows that the general solution is

$$u = h_1(z) + h_2(\bar{z}), \tag{8.37}$$

for some arbitrary functions h_1 and h_2. If we require u to be a real-valued function of x and y, then we must have $h_2 = \overline{h_1}$, so the general real-valued solution of Laplace's equation is

$$u = \Re\big[f(z)\big] \tag{8.38}$$

(where $f = 2h_1$). Solutions of Laplace's equation may sometimes by found by spotting a function $f(z)$ whose real part is equal to a given function on a given curve.

Example 8.3 *Find a function $u(x, y)$ that satisfies Laplace's equation in $y > 0$ and is equal to $|x|$ on $y = 0$.*

The complex-valued function

$$f(z) = z + \frac{2\mathrm{i}}{\pi} z \log z \tag{8.39}$$

may be split into its real and imaginary parts as follows:

$$f(z) = \left\{ \left(1 - \frac{2\theta}{\pi}\right) x - \frac{2y}{\pi} \log r \right\} + \mathrm{i} \left\{ \left(1 - \frac{2\theta}{\pi}\right) y + \frac{2x}{\pi} \log r \right\}, \tag{8.40}$$

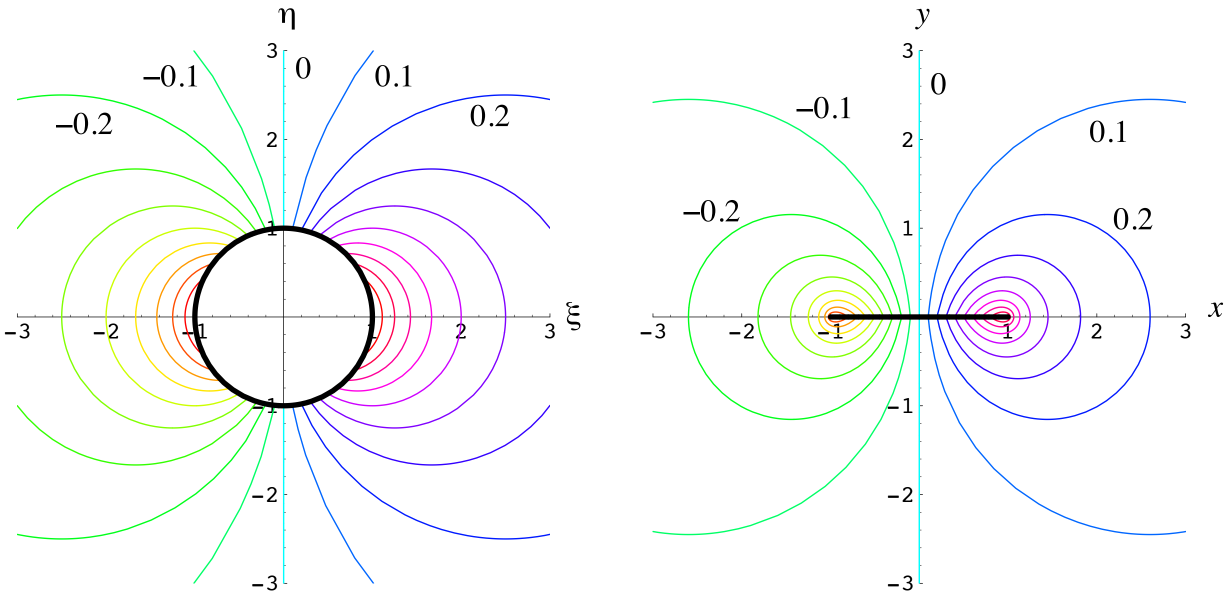

Figure 8.1: Contour plots of the solution u given by (8.44) versus $\xi = \Re(\zeta)$ and $\eta = \Im(\zeta)$ (left plot); x and y (right plot). The contour values are $u = -0.9, -0.8, \cdots, 0.8, 0.9$.

where (r, θ) are plane polar coordinates, i.e. $x = r\cos\theta$, $y = r\sin\theta$. Notice that, on $y = 0$, $\Re(f)$ is equal to x when $\theta = 0$ and $-x$ when $\theta = \pi$; in other words, $\Re(f)$ is equal to $|x|$ when $y = 0$. A suitable solution is, therefore,

$$u(x, y) = \Re\big[f(z)\big] = \left(1 - \frac{2}{\pi}\tan^{-1}(y/x)\right)x - \frac{y}{\pi}\log\left(x^2 + y^2\right). \tag{8.41}$$

Another useful technique from complex variable theory is *conformal mapping*. This is a mapping from one complex variable $z = x + \mathrm{i}y$ to another $\zeta = \xi + \mathrm{i}\eta$, given by a functional relation $\zeta = g(z)$, where g is analytic and $g'(z) \neq 0$. Conformal mapping may be used to transform a complicated solution domain to a simple one (such as a half-plane or a disc). This technique works because Laplace's equation is invariant under conformal mapping; see a textbook on complex analysis, *e.g.*, Priestley[4] for more details.

Example 8.4 *Consider the solution of Laplace's equation for $u(x, y)$ with $u \to 0$ as $x^2 + y^2 \to \infty$ and $u = x$ on the line segment $y = 0$, $-1 \leq x \leq 1$. This line segment is the image of the unit circle $|\zeta| = 1$ under the conformal map*

$$x + \mathrm{i}y = z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \qquad \Rightarrow \qquad \zeta = z + \sqrt{z^2 - 1}. \tag{8.42}$$

The boundary conditions are mapped to

$$u = \Re(\zeta) \quad \text{on } |\zeta| = 1, \qquad\qquad u \to 0 \quad \text{as } \zeta \to \infty, \tag{8.43}$$

[4] H. A. Priestley, 1990 *Introduction to Complex Analysis, revised edition.* Oxford University Press.

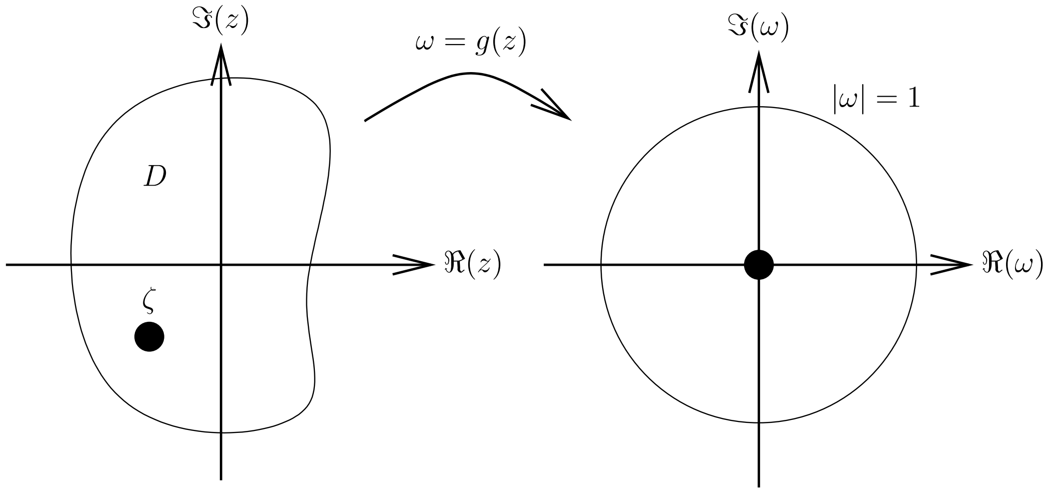

Figure 8.2: Schematic of the conformal transformation $\omega = g(z)$ mapping D to the unit disc and ζ to the origin.

and the solution may then easily be spotted. On the unit circle, we have $\bar{\zeta} = \zeta^{-1}$ *and, hence,* $\Re(\zeta^{-1}) = \Re(\zeta)$*. A suitable solution for* u *is thus*

$$u = \Re\left(\zeta^{-1}\right) = \Re\left(z - \sqrt{z^2 - 1}\right). \tag{8.44}$$

Contour plots of u *in the* ζ*- and* z*-planes are shown in* Figure 8.1.

Conformal mapping may be used to find the Green's function for any domain that can be conformally mapped onto the unit disc. If we write $G = \Re\big[f(z)\big]$, then the problem (8.25) implies the following properties for f:

$$\left.\begin{aligned} \Re(f) &= 0 && \text{on } \partial D, \\ f &\sim \frac{1}{2\pi}\log(z - \zeta), && \text{as } z \to \zeta, \end{aligned}\right\} \tag{8.45}$$

where $\zeta = \xi + \mathrm{i}\eta$. Now suppose we write $\omega = g(z)$ where the conformal map g transforms D to the unit disc and the point ζ to the origin, as illustrated in Figure 8.2. In the ω-plane, (8.45) becomes

$$\left.\begin{aligned} \Re(f) &= 0 && \text{on } |\omega| = 1, \\ f &\sim \frac{1}{2\pi}\log\omega, && \text{as } \omega \to 0, \end{aligned}\right\} \tag{8.46}$$

and the solution is simply $f = (1/2\pi)\log\omega$. Hence, if we can find the appropriate conformal mapping g, then the Green's function is

$$G = \frac{1}{2\pi}\log\big|g(z)\big|. \tag{8.47}$$

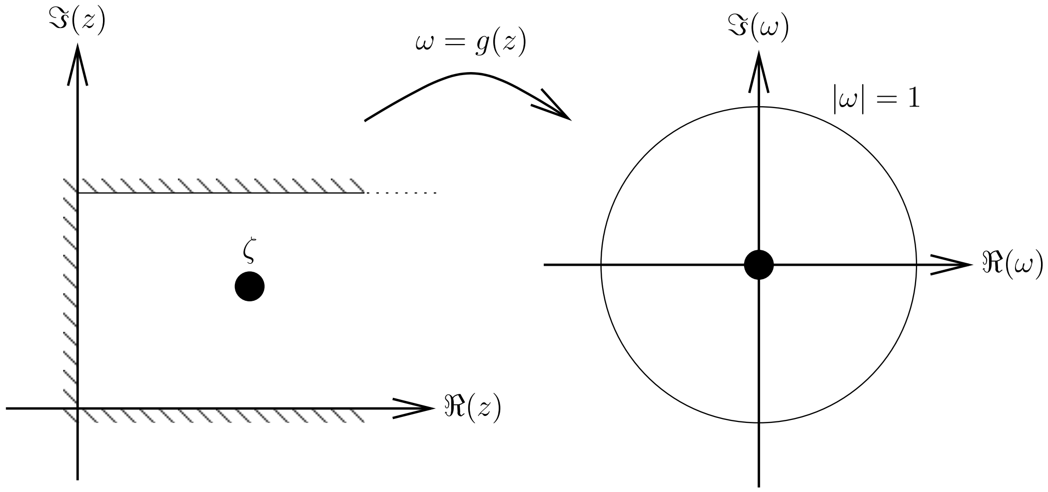

Figure 8.3: Schematic of the conformal transformation $\omega = g(z)$ mapping the strip $x > 0$, $0 < y < a$ to the unit disc and ζ to the origin.

Example 8.5 Green's function for a circle (again)
The Möbius transformation

$$\omega = g(z) = \frac{a(z-\zeta)}{\bar{\zeta}z - a^2} \tag{8.48}$$

maps the disc $|z| < a$ *onto the unit disc* $|\omega| < 1$ *and maps the point* $z = \zeta$ *to* $\omega = 0$. *The Green's function for a disc of radius* a *is, therefore,*

$$G = \frac{1}{2\pi} \log \left| \frac{a(z-\zeta)}{\bar{\zeta}z - a^2} \right|, \tag{8.49}$$

and it is straightforward to show that this reproduces (8.31).

Example 8.6 Green's function for a strip
The semi-infinite strip $x > 0$, $0 < y < a$ *is mapped to the unit disc (as illustrated in* Figure 8.3*) by the transformation*

$$\omega = g(z) = \frac{\cosh(\pi z/a) - \cosh(\pi\zeta/a)}{\cosh(\pi z/a) - \cosh(\pi\bar{\zeta}/a)}, \tag{8.50}$$

which also maps $z = \zeta$ *to* $\omega = 0$. *The Green's function for this shape is then* $G = 1/(2\pi)\log|g|$.

9 Semi-linear parabolic equations

9.1 General properties

In parabolic PDEs one independent variable usually represents time, so we use x and t as independent variables instead of x and y. The canonical form for second-order parabolic equations is then

$$\frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) \tag{9.1}$$

and specific examples include the *heat equation*, or *diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{9.2a}$$

the *reaction-diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x, t, u), \tag{9.2b}$$

and the *reaction-convection-diffusion equation*

$$\frac{\partial u}{\partial t} + q(x, t, u)\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + f(x, t, u). \tag{9.2c}$$

Another example is the Black–Scholes equation in mathematical finance. All these equations have the repeated characteristic $t = \text{const}$. They also have the general property that any initial singularities in u at $t = 0$ are instantaneously smoothed out when t is positive. Conversely, *backward* diffusion equations such as

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \tag{9.2d}$$

are *ill posed* when solved forwards in time, with smooth initial data giving rise to finite-time singularities in general.

9.2 Well posed boundary data

Typical boundary data for a diffusion equation are:

- an initial condition for u at $t = 0$;
- one boundary condition on each of two curves C_1 and C_2 in the (x, t)-plane that are nowhere parallel to the x-axis.

Example 9.1 *The* heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{9.3}$$

is a simple model for the temperature $u(x,t)$ in a uniform bar of conductive material, where x is position and t is time. Suppose the bar is of length L, its initial temperature is $u_0(x)$ and its ends are kept at zero temperature. Then the boundary and initial conditions are

$$u = u_0(x) \qquad t = 0, \tag{9.4a}$$

$$u = 0 \qquad x = 0, \tag{9.4b}$$

$$u = 0 \qquad x = L. \tag{9.4c}$$

If, instead of being held at constant temperature, an end is insulated, then the Dirichlet boundary condition $u = 0$ is replaced by the Neumann boundary condition $\partial u/\partial x = 0$. Also, the boundary conditions at $x = 0$ and $x = L$ may in general be given on moving boundaries, say $x = x_1(t)$ and $x = x_2(t)$.

9.3 Uniqueness theorem

We will now show that the problem

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(x,t) && t > 0,\ x_1(t) < x < x_2(t),\\
u &= u_0(x) && t = 0,\ x_1(t) < x < x_2(t),\\
u &= g_1(t) && t > 0,\ x = x_1(t),\\
u &= g_2(t) && t > 0,\ x = x_2(t),
\end{aligned}\right\} \tag{9.5}$$

has at most one solution. Suppose there are two solutions u_1 and u_2; then $\phi = u_1 - u_2$ satisfies the homogeneous problem

$$\left.\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{\partial^2 \phi}{\partial x^2} && t > 0,\ x_1(t) < x < x_2(t),\\
\phi &= 0 && t = 0,\ x_1(t) < x < x_2(t),\\
\phi &= 0 && t > 0,\ x = x_1(t),\\
\phi &= 0 && t > 0,\ x = x_2(t).
\end{aligned}\right\} \tag{9.6}$$

Now, Leibnitz' rule gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} \phi^2 \,\mathrm{d}x &= \int_{x_1(t)}^{x_2(t)} 2\phi\frac{\partial \phi}{\partial t}\,\mathrm{d}x + \frac{\mathrm{d}x_2}{\mathrm{d}t}\phi^2(x_2,t) - \frac{\mathrm{d}x_1}{\mathrm{d}t}\phi^2(x_1,t)\\
&= \int_{x_1(t)}^{x_2(t)} 2\phi\frac{\partial^2 \phi}{\partial x^2}\,\mathrm{d}x = \left[2\phi\frac{\partial \phi}{\partial x}\right]_{x_1}^{x_2} - \int_{x_1(t)}^{x_2(t)} 2\left(\frac{\partial \phi}{\partial x}\right)^2 \mathrm{d}x\\
&= -\int_{x_1(t)}^{x_2(t)} 2\left(\frac{\partial \phi}{\partial x}\right)^2 \mathrm{d}x \le 0.
\end{aligned} \tag{9.7}$$

So the integral

$$\int_{x_1(t)}^{x_2(t)} \phi^2 \,\mathrm{d}x$$

is clearly non-negative, is zero when $t = 0$ and is a non-increasing function of t. From this we can conclude that it must be identically zero and, hence, that $\phi \equiv 0$.

9.4 Maximum principle

The maximum principle is proved in the course *A1 Differential Equations 1*. It is included here for completeness, but *it is non-examinable*. Suppose u satisfies the inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{9.8}$$

in a region D bounded by the lines $t = 0$, $t = \tau$ and two non-intersecting smooth curves C_1 and C_2 that are nowhere parallel to the x-axis. Suppose also that $f \leq 0$ in D. Then u takes its maximum value either on $t = 0$ or on one of the curves C_1 or C_2.

The proof is similar to that for Poisson's equation. We start by supposing that f is strictly negative in D. At an internal maximum inside D, u must satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} = 0, \qquad \frac{\partial^2 u}{\partial x^2} \leq 0, \qquad \frac{\partial^2 u}{\partial t^2} \leq 0. \tag{9.9}$$

On the other hand, if u has a maximum at a point on $t = \tau$, then it must satisfy

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} \geq 0, \qquad \frac{\partial^2 u}{\partial x^2} \leq 0 \tag{9.10}$$

there. With f assumed to be negative, both of these lead to contradictions, and it follows that u must take its maximum value somewhere on ∂D but not on $t = \tau$.

Now, if the inequality is not strict, *i.e.* $f \leq 0$, then define

$$v(x,t) = u(x,t) + \frac{\epsilon}{2}x^2, \tag{9.11}$$

where ϵ is a positive constant. Then v satisfies

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(x,t) - \epsilon < 0 \tag{9.12}$$

so that v takes its maximum value on either $t = 0$, C_1 or C_2. Thus, if the maximum value of u over these three portions of ∂D is M, and the maximum value of $|x|$ on C_1 and C_2 is L, then

$$u \leq v \leq \frac{\epsilon}{2}L^2 + M. \tag{9.13}$$

Now we let $\epsilon \to 0$ and conclude that $u \leq M$, *i.e.* u takes its maximum value on $C_1 \cup C_2 \cup \{t = 0\}$.

If $f \geq 0$ in D, then a similar argument shows that u attains its minimum value on either C_1, C_2 or $t = 0$. Thus, for the homogeneous equation in which $f \equiv 0$, u attains both its maximum and its minimum values on $C_1 \cup C_2 \cup \{t = 0\}$. This property may be used in a uniqueness proof just as for Poisson's equation.

9.5 Green's functions

Consider the problem

$$\left.\begin{aligned} \mathcal{L}[u] = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad & t > 0,\ x_1(t) < x < x_2(t), \\ u = u_0(x) \qquad & t = 0,\ x_1(t) < x < x_2(t), \\ u = g_1(t) \qquad & t > 0,\ x = x_1(t), \\ u = g_2(t) \qquad & t > 0,\ x = x_2(t). \end{aligned}\right\} \tag{9.14}$$

If the adjoint differential operator is defined by

$$\mathcal{L}^\star[G] = -\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial x^2}, \tag{9.15}$$

then Green's theorem leads to

$$\begin{aligned} \iint_D \left\{G\mathcal{L}[u] - u\mathcal{L}^\star[G]\right\} \mathrm{d}x\mathrm{d}t &= \iint_D \left\{\frac{\partial}{\partial t}(Gu) + \frac{\partial}{\partial x}\left(u\frac{\partial G}{\partial x} - G\frac{\partial u}{\partial x}\right)\right\} \mathrm{d}x\mathrm{d}t \\ &= \oint_{\partial D} \left\{\left(u\frac{\partial G}{\partial x} - G\frac{\partial u}{\partial x}\right) \mathrm{d}t - Gu\,\mathrm{d}x\right\}, \end{aligned} \tag{9.16}$$

where D is the region bounded by $t = 0$, $t = \tau$, $x = x_1(t)$ and $x = x_2(t)$. So, if we choose the *Green's function* G to satisfy

$$\left.\begin{aligned} -\mathcal{L}^\star[G] = \frac{\partial G}{\partial t} + \frac{\partial^2 G}{\partial x^2} = 0 \qquad & \text{in } D, \\ G = 0 \qquad & x = x_1(t), \\ G = 0 \qquad & x = x_2(t), \\ G = \delta(x-\xi) \qquad & t = \tau \end{aligned}\right\} \tag{9.17}$$

where δ is the Dirac delta-function, then equation (9.16) reduces to

$$\begin{aligned} u(\xi,\tau) = \iint_D G(x,t;\xi,\tau)f(x,t)\,\mathrm{d}x\mathrm{d}t + \int_{x_1(0)}^{x_2(0)} G(x,0;\xi,\tau)u_0(x)\,\mathrm{d}x \\ + \int_0^\tau g_1(t)\frac{\partial G}{\partial x}(x_1(t),t;\xi,\tau)\,\mathrm{d}t - \int_0^\tau g_2(t)\frac{\partial G}{\partial x}(x_2(t),t;\xi,\tau)\,\mathrm{d}t. \end{aligned} \tag{9.18}$$

If G can be found from (9.17), then this gives an explicit formula for the solution u at any point $(\xi,\tau) \in D$. Notice that G satisfies a *backward* diffusion equation and thus is solved backwards in time, starting at $t = \tau$. As for elliptic equations, actually finding the Green's function is difficult except in a few simple cases.

Example 9.2 *Suppose the heat equation is to be solved on the whole real line, i.e.,*

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad & t > 0,\ \ -\infty < x < \infty, \\ u = u_0(x) \qquad & t = 0,\ \ -\infty < x < \infty, \\ u \to 0 \qquad & t > 0,\ x \to \pm\infty. \end{aligned}\right\} \tag{9.19}$$

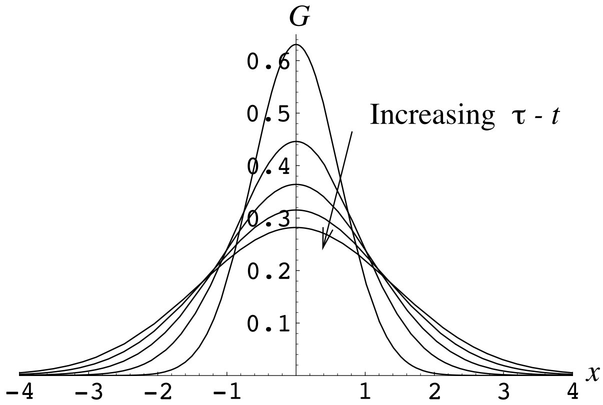

Figure 9.1: The Green's function (9.21) plotted versus x when $\xi = 0$, for $(\tau - t) = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$.

Then the Green's function satisfies

$$\left.\begin{aligned} \frac{\partial G}{\partial \tilde{t}} - \frac{\partial^2 G}{\partial \tilde{x}^2} &= 0 && \tilde{t} > 0,\ -\infty < \tilde{x} < \infty, \\ G &= \delta(\tilde{x}) && \tilde{t} = 0,\ -\infty < \tilde{x} < \infty, \\ G &\to 0 && \tilde{t} > 0,\ \tilde{x} \to \pm\infty, \end{aligned}\right\} \tag{9.20}$$

where $\tilde{x} = x - \xi$ and $\tilde{t} = \tau - t$. This problem corresponds to the heat flow due to an initial "hot spot" concentrated at $x = 0$. The solution may be found, for example, by taking a Fourier transform in x (see Example 1.10*):*

$$G = \frac{1}{2\sqrt{\pi(\tau - t)}} \exp\left(-\frac{(x-\xi)^2}{4(\tau - t)}\right), \tag{9.21}$$

whose behaviour is illustrated in Figure 9.1*. Thus the solution of* (9.19) *is*

$$u(\xi, \tau) = \int_{-\infty}^{\infty} \frac{u_0(x)}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) \mathrm{d}x + \int_0^{\tau}\int_{-\infty}^{\infty} \frac{f(x,t)}{2\sqrt{\pi(\tau - t)}} \exp\left(-\frac{(x-\xi)^2}{4(\tau - t)}\right) \mathrm{d}x\mathrm{d}t. \tag{9.22}$$

Example 9.3 *Next consider the heat equation on a half-line:*

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(x,t) && t > 0,\ 0 < x < \infty, \\ u &= u_0(x) && t = 0,\ 0 < x < \infty, \\ u &= g(t) && t > 0,\ x = 0, \\ u &\to 0 && t > 0,\ x \to \infty, \end{aligned}\right\} \tag{9.23}$$

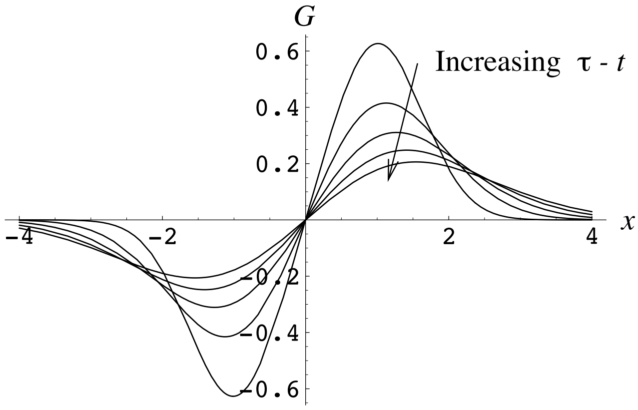

Figure 9.2: The Green's function (9.25) plotted versus x when $\xi = 1$, for $(\tau - t) = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$.

so the Green's function satisfies

$$\left.\begin{aligned}
\frac{\partial G}{\partial t} + \frac{\partial^2 G}{\partial x^2} &= 0 && t < \tau,\ 0 < x < \infty,\\
G &= \delta(x-\xi) && t = \tau,\ 0 < x < \infty,\\
G &= 0 && t < \tau,\ x = 0,\\
G &\to 0 && t < \tau,\ x \to \infty.
\end{aligned}\right\} \tag{9.24}$$

Now, the Green's function (9.21) *found previously satisfies the heat equation and has the right singular behaviour as* $(x,t) \to (\xi,\tau)$*, but it is not zero when* $x = 0$*. However, we can enforce this boundary condition by putting an image singularity at the point* $(-\xi,\tau)$*, that is*

$$G = \frac{1}{2\sqrt{\pi(\tau-t)}} \exp\left(-\frac{(x-\xi)^2}{4(\tau-t)}\right) - \frac{1}{2\sqrt{\pi(\tau-t)}} \exp\left(-\frac{(x+\xi)^2}{4(\tau-t)}\right), \tag{9.25}$$

which is illustrated in Figure 9.2*. This Green's function can be used to reproduce the solution* (1.42), (1.46) *of* Example 1.11.