Lie Groups

Section C course Hilary 2022

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Example sheet 3

Section A (introductory questions, not for marking, solutions available)

1. Let $\mathfrak{sl}(2,\mathbb{R})$ denote the space of 2×2 real matrices of trace zero.
Show that $\mathfrak{sl}(2,\mathbb{R})$ is a Lie algebra with basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and work out the bracket relations for e, f, h.
By considering subalgebras of this Lie algebra, show that it is not isomorphic to $\mathfrak{su}(2)$.

Solution Since $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, the trace of any commutator of square matrices is zero, so $\mathfrak{sl}(2,\mathbb{R})$ is a Lie algebra when endowed with the matrix commutator. Any traceless 2×2 real matrix is uniquely expressible in the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for real numbers a, b and c, so $\{h, e, f\}$ is a basis for $\mathfrak{sl}(2,\mathbb{R})$. By direct calculation we have

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Recall $\mathfrak{su}(2) \cong (\mathbb{R}^3, \times)$ where $\times$ is the cross product. Any two linearly independent elements of $\mathbb{R}^3$ generate $(\mathbb{R}^3, \times)$ since $v \times w$ is orthogonal to both v and w and is non-zero if v and w are linearly independent. Therefore $\mathfrak{su}(2)$ has no 2-dimensional Lie subalgebras. However $\mathbb{R}\{h, e\}$ is a 2-dimensional Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. Thus $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ cannot be isomorphic.

Section B (questions to be handed in for marking)

2. Let $\varphi : G_1 \to G_2$ be a Lie group homomorphism. Show that

$$\ker \varphi \subseteq G_1$$

is a closed (hence embedded) Lie subgroup with Lie algebra

$$\ker(D_1\varphi) \subset \mathfrak{g}_1.$$

A vector subspace $J \subseteq (V, [\cdot,\cdot])$ of a Lie algebra is called an ***ideal*** *if*

$$[v, j] \in J \text{ for all } v \in V, j \in J.$$

Show that ideals are Lie subalgebras. Show that for a Lie subgroup $H \subseteq G$, with H, G connected,

$$\boxed{H \subseteq G \text{ is a normal subgroup} \Leftrightarrow \mathfrak{h} \subseteq \mathfrak{g} \text{ is an ideal}}$$

(You may find it helpful to first show the identity $ge^Y g^{-1} = e^{\mathrm{Ad}\,(g).Y}$ for $g \in G$ and $Y \in \mathfrak{g}$).
The ***centre*** *of a Lie algebra $(V, [\cdot,\cdot])$ is*

$$Z(V) = \{v \in V : [v, w] = 0 \text{ for all } w \in V\}.$$

For G connected, prove that the centre of the group G is[1]

$$\boxed{Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}))}$$

Deduce that the centre of G is a closed (hence embedded) Lie subgroup of G which is abelian, normal and has Lie algebra

$$\mathrm{Lie}(Z(G)) = Z(\mathfrak{g}).$$

[1]Recall the centre of a group is $Z(G) = \{g \in G : hg = gh \text{ for all } h \in G\} = \{g \in G : hgh^{-1} = g \text{ for all } h \in G\}$.

Finally deduce that, for G connected,

$$\boxed{G \text{ is abelian} \Leftrightarrow \mathfrak{g} \text{ is abelian.}}$$

3. Show that if X, Y belong to the Lie algebra of a Lie group G then

$$[X, Y] = 0 \Rightarrow \exp(X + Y) = \exp(X)\exp(Y).$$

Prove that if G is a connected Lie group with $Z(G) = \{1\}$ then G can be identified with a Lie subgroup of $GL(m, \mathbb{R})$, for some m, so $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(m, \mathbb{R})$.

If $(V, [\cdot, \cdot])$ is a Lie algebra with $Z(V) = \{0\}$, show that V is the Lie algebra of some Lie group.

4. Find all the connected Lie subgroups of $SO(3)$.

5. Show that Lebesgue measure $\mathbf{dx}$ is the bi-invariant Haar measure on $\mathbb{R}^n$ viewed as an additive group.
Find the bi-invariant Haar measure on $(\mathbb{R}_{>0}, \times)$, the multiplicative group of positive reals.

6. Give an example of an *irreducible* representation of S^1 on $\mathbb{R}^2$. Describe what happens to this representation when we complexify it.

Section C (optional extension questions, not to be handed in for marking)

7. (i) Let $\phi : G \to \mathrm{Aut}(V)$ be a representation. If $\alpha : G \to G$ is an automorphism show that $\phi \circ \alpha$ is another representation on the same vector space.

(ii) If $\alpha(g) = hgh^{-1}$ for some $h \in G$ show that the two representations are equivalent.

(iii) Give an example of an automorphism where the two representations are not equivalent [*Think: complex conjugation*].

8. Consider the action of $SO(3)$ on $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth real-valued function.

(i) For $A \in SO(3)$ show that $(Af)(x) = f(A^{-1}x)$ defines an action of $SO(3)$ on the space of all smooth functions.

(ii) If $r^2 = x_1^2 + x_2^2 + x_3^2$ show that $Af = f$.

(iii) Let Δ denote the Laplace operator

$$\Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}.$$

Show that $A\Delta f = \Delta A f$.

(iv) Consider the vector space of functions of the form $f = p$ where $p(x_1, x_2, x_3)$ is a homogeneous polynomial of degree m. Show that this is a finite-dimensional representation V_m of $SO(3)$ and calculate its dimension.

(v) Let $H_m \subseteq V_m$ be the subspace of solutions to $\Delta f = 0$ for $f \in V_m$, the harmonic polynomials of degree m. Show that H_m is a representation space for $SO(3)$ and that $V_2 = H_2 \oplus r^2 H_0$ and $V_3 = H_3 \oplus r^2 H_1$ are decompositions into inequivalent representations.

(vi) Can you generalize this?