Lie Groups

Section C course Hilary 2022

kirwan@maths.ox.ac.uk

Example sheet 4

Section A (introductory questions, not for marking, solutions available)

1. Check the following properties hold for a character χ_V associated to a representation V of a compact Lie group G:

(i) $\chi_V(1) = \dim V$;

(ii) χ_V is invariant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$;

(iii) $\chi_V = \chi_W$ for equivalent reps $V \simeq W$;

(iv) $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$;

(v) $\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$;

(vi) $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Solution (i) Let G be a compact Lie group and let V, W be finite dimensional $\mathbb{C}G$-modules.[1] We have $\chi_V(1) = \text{trace}(\text{id}_V) = \dim V$.

(ii) follows from the identity $\text{trace}(PAP^{-1}) = \text{trace}(AP^{-1}P) = \text{trace}(A)$ for (invertible) square matrices A and P.

(iii) V and W are equivalent if and only if $V \cong W$ as $\mathbb{C}G$-modules; that $\chi_V = \chi_W$ for equivalent representations V and W is then immediate from the definitions.

(iv) A basis for $V \oplus W$ is given by taking a union of bases for V and W. It follows immediately that $\chi_{V\oplus W} = \chi_V + \chi_W$.

(v) We may assume without loss of generality that the representations are unitary. Take $g \in G$. Then we may choose bases $\{v_i\}$ and $\{w_j\}$ for V and W respectively consisting of eigenvectors for the multiplication by g map, say

$$gv_i = \lambda_i v_i, \quad gw_j = \mu_j w_j.$$

Then $\{v_i \otimes w_j\}$ forms a basis for $V \otimes W$ and

$$g(v_i \otimes w_j) = \lambda_i \mu_j (v_i \otimes w_j).$$

Therefore

$$\chi_{V\otimes W}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum_i \lambda_i\right)\left(\sum_j \mu_j\right) = \chi_V(g) \cdot \chi_W(g).$$

[1]All modules are assumed to be left modules.

(vi) To show that $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$, use a basis of eigenvectors v_i as in (v). Let $\{v_i^*\}$ be the corresponding dual basis. Then

$$(g \cdot v_i^*)(v_j) = v_i^*(g^{-1}v_j) = v_i^*(\lambda_j^{-1}v_j) = \lambda_i^{-1}\delta_{ij},$$

so $g \cdot v_i^* = \lambda_i^{-1}v_i^*$. By unitarity $\lambda_i^{-1} = \overline{\lambda_i}$. The equalities $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ follow.

Section B (questions to be handed in for marking)

2. Recall that the irreducible representation V_n of $SU(2)$ is given by the space of homogeneous polynomials of degree n in two variables (say z and w) with

$$(A \cdot p)(\mathbf{z}) = p(A^{-1}\mathbf{z}), \quad A \in SU(2),\ p \in V_n, \mathbf{z} = (z, w),$$

and that the map $(z, w) \mapsto (w, -z)$, extended to a complex anti-linear map $J : V_{2n} \to V_{2n}$, defines a real structure on V_{2n}.

Which of the irreducible representations V_n of $SU(2)$ may be regarded as representations of $SO(3)$?

Deduce that for each natural number n we have a real $(2n+1)$-dimensional representation W_n of $SO(3)$.

Show further that the character of W_n is given by

$$\sum_{k=0}^{2n} e^{i(n-k)t}.$$

3. Show that a maximal torus in a compact Lie group is maximal among connected Abelian subgroups.

4. Find the Weyl group of the unitary group $U(n)$.

5. Let B denote the subgroup of $GL(3, \mathbb{C})$ consisting of invertible matrices of the form

$$\begin{pmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{pmatrix} \quad : \quad a, b, c \in \mathbb{C} \text{ and } \alpha, \beta, \gamma \in \mathbb{C}^*.$$

Check that B is indeed a subgroup, and that there is a homomorphism ϕ from B onto the *complex torus* $T_{\mathbb{C}} \cong (\mathbb{C}^*)^3$ of diagonal elements of B. Show $\ker \phi$ may be identified with the subgroup U consisting of elements of B with diagonal entries equal to 1.

Show further that the elements of U with $a = c = 0$ form a normal subgroup of U.

What are the maximal compact connected subgroups of T, B and U? (You need not give detailed proofs).

Section C (optional extension questions, not to be handed in for marking)

6. (i) Let G be a compact Lie group and $C(G)$ the space of complex-valued continuous functions on G. Define a product (the *convolution product*) by

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1})f_2(g)dg$$

where dg denotes the bi-invariant measure. Show that $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$.

(ii) Prove that convolution is commutative if the group is abelian.

(iii) Let $\pi : G \to \mathrm{Aut}(V)$ be a representation of G and $f \in C(G)$ a function. Define $\pi(f) \in \mathrm{End}V$ by

$$\pi(f) = \int_G f(g)\pi(g)dg$$

Show that $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$.

(iv) Use this to give an example of a group where the convolution product is not commutative.

7. Suppose (as in Question 6) that the function f satisfies $f(hgh^{-1}) = f(g)$ for all h. If π is an irreducible representation with character χ show that $\pi(f) = \alpha 1$ where

$$\alpha = \frac{1}{\dim V}\langle f, \bar{\chi}\rangle.$$