B4.1 Functional Analysis 1

Sheet 0 — MT 2025

Warm up / revision sheet

In the questions below the scalar field is assumed to be $\mathbb{R}$ for simplicity, but all results hold when the scalars are complex.

1. Let X be the vector space of real sequences (x_j) and define

$$\|(x_j)\| = \begin{cases} 0 & \text{if } x_j = 0 \text{ for all } j, \\ |x_{j_0}| & \text{if } j_0 = \min\{j \mid x_j \neq 0\}. \end{cases}$$

Show that the triangle inequality fails to hold, so that $\|\cdot\|$ is not a norm.

2. (a) Let X be a real inner product space and, for each $x \in X$, let $\|x\| = \langle x, x\rangle^{1/2}$. You may assume that $\|\cdot\|$ defines a norm on X. Verify the *Parallelogram Law*: for all $x, y \in X$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b) Consider the ∞-norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ ($n > 2$):

$$\|(x_1, \dots, x_n)\|_\infty = \sup_{1 \le j \le n} |x_j|.$$

By showing that the Parallelogram Law fails, prove that there is no inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ such that

$$\|x\|_\infty = \langle x, x\rangle^{1/2} \quad \text{for all } x \in \mathbb{R}^n.$$

3. Let X be a (real) vector space equipped with a norm $\|\cdot\|$. As usual we define a metric d on X by $d(x, y) = \|x - y\|$. For $x_0 \in X$ and $r > 0$, let the open and closed balls be given by

$$B_r(x_0) = \{x \in X \mid \|x - x_0\| < r\} \quad \text{and ,} \quad \overline{B}_r(x_0) = \{x \in X \mid \|x - x_0\| \le r\}$$

respectively. *[The terminology was justified in the Metric Spaces course: open balls are open sets and closed balls are closed sets.]*

(a) A subset $C \subseteq X$ is convex if $x, y \in C$ and $0 \le \lambda \le 1$ imply $\lambda x + (1-\lambda)y \in C$. Prove that $B_r(x_0)$ and $\overline{B}_r(x_0)$ are convex.

(b) Prove that $\overline{B}_r(x_0)$ is the closure of $B_r(x_0)$.

(c) Use (i) to show that $(x_1, x_2) \mapsto |x_1|^{1/2} + |x_2|^{1/2}$ does *not* define a norm on $\mathbb{R}^2$.

Stuart White: stuart.white@maths.ox.ac.uk (with thanks to Luc Nuygen and Melanie Rupflin)

4. (a) Let X be a real normed space. Let $T : X \to \mathbb{R}$ be a linear map such that $|T(x)| \leq \|x\|$ for all $x \in X$. Prove that T is continuous.

   (b) Let $X = \ell^p$, $1 \leq p \leq \infty$, equipped with the p-norm $\|x\|_p = (\sum_{j=1}^{\infty} |x_j|^p)^{1/p}$, respectively $\|x\|_\infty = \sup_j |x_j|$. Define $\pi_k : X \to \mathbb{R}$ by $\pi_k((x_j)) = x_k$ (for any $k \geq 1$). Check that each π_k is continuous.

   (c) Let $X = L^2([0,1])$ and define $T : X \to \mathbb{R}$ by

$$T(f) := \int_0^1 f \, dx.$$

   Check that T is continuous. [*Hint:* Use Hölder's inequality $\|fg\|_{L^1} \leq \|f\|_{L^2}\|g\|_{L^2}$ for every $f, g \in L^2([0,1])$.]

   (d) Let X be as in (ii). Let (a_j) be a fixed sequence of real numbers and define

$$Y = \{(x_j) \in X \mid x_{2j} = a_j x_{2j-1} \text{ for all } j \geq 1\}.$$

   Check that Y is a subspace of X and, by writing Y as an intersection of closed sets involving maps π_k, or otherwise, show that Y is closed.

Stuart White: stuart.white@maths.ox.ac.uk (with thanks to Luc Nuygen and Melanie Rupflin)