B4.1 Functional Analysis 1

Sheet 0 — MT 2025

Warm up / revision sheet

In the questions below the scalar field is assumed to be $\mathbb{R}$ for simplicity, but all results hold when the scalars are complex.

1. Let X be the vector space of real sequences (x_j) and define

$$\|(x_j)\| = \begin{cases} 0 & \text{if } x_j = 0 \text{ for all } j, \\ |x_{j_0}| & \text{if } j_0 = \min\{j \mid x_j \neq 0\}. \end{cases}$$

Show that the triangle inequality fails to hold, so that $\|\cdot\|$ is not a norm.

Solution: Take $x = (1, 3, 0, 0, \dots)$ and $y = (-1, 0, 0, \dots)$. Then $\|x\| = \|y\| = 1$ (with the definition from Q1 of the sheet), but

$$x + y = (0, 3, 0, 0, \dots) \quad \text{so} \quad \|x + y\| = 3.$$

Hence $\|x + y\| \not\leq \|x\| + \|y\|$, so the triangle inequality fails.

2. (a) Let X be a real inner product space and, for each $x \in X$, let $\|x\| = \langle x, x\rangle^{1/2}$. You may assume that $\|\cdot\|$ defines a norm on X. Verify the *Parallelogram Law*: for all $x, y \in X$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Solution:

$$\|x+y\|^2+\|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2+2\|y\|^2.$$

(b) Consider the ∞-norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$ ($n > 2$):

$$\|(x_1, \dots, x_n)\|_\infty = \sup_{1\leq j\leq n} |x_j|.$$

By showing that the Parallelogram Law fails, prove that there is no inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ such that

$$\|x\|_\infty = \langle x, x\rangle^{1/2} \quad \text{for all } x \in \mathbb{R}^n.$$

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Solution: In $\mathbb{R}^n$ with ∞-norm let x and y be defined by $x_j = \delta_{1j}$ respectively $y_j = \delta_{2j}$ for $j = 1, \dots, n$. Then, $\|x\|_\infty = \|y\|_\infty = 1$ and $\|x+y\|_\infty = \|x-y\|_\infty = 1$. So the Parallelogram Law fails for $\|\cdot\|_\infty$ and this norm therefore cannot come from an inner product.

3. Let X be a (real) vector space equipped with a norm $\|\cdot\|$. As usual we define a metric d on X by $d(x,y) = \|x-y\|$. For $x_0 \in X$ and $r > 0$, let the open and closed balls be given by

$$B_r(x_0) = \{x \in X \mid \|x - x_0\| < r\} \quad \text{and ,} \quad \overline{B}_r(x_0) = \{x \in X \mid \|x - x_0\| \leq r\}$$

respectively. *[The terminology was justified in the Metric Spaces course: open balls are open sets and closed balls are closed sets.]*

(a) A subset $C \subseteq X$ is convex if $x, y \in C$ and $0 \leq \lambda \leq 1$ imply $\lambda x + (1-\lambda)y \in C$. Prove that $B_r(x_0)$ and $\overline{B}_r(x_0)$ are convex.

Solution: Take $\lambda \in [0,1]$ and $y, z \in B_r(x_0)$. Then

$$\begin{aligned}\big\|x_0 - (\lambda y + (1-\lambda)z)\big\| &= \|\lambda x_0 - \lambda y + (1-\lambda)x_0 - (1-\lambda)z\| \\ &\leq \|\lambda(x_0 - y)\| + \|(1-\lambda)(x_0 - z)\| \\ &= \lambda\|x_0 - y\| + (1-\lambda)\|x_0 - z\| < r.\end{aligned}$$

Likewise $\overline{B}_r(x_0)$ is convex.

(b) Prove that $\overline{B}_r(x_0)$ is the closure of $B_r(x_0)$.

Solution: We use the characterisation of the closure F of a set F obtained in metric spaces that $x \in \overline{F}$ if and only if there exists a sequence (x_n) so that $x_n \in F$ and $x_n \to x$. Given $x \in \overline{B}_r(x_0)$ we can choose

$$x_n = \Big(1 - \frac{1}{n}\Big)(x - x_0) + x_0 \in B_r(x_0)$$

to get $x_n \to x$ and hence $\overline{B}_r(x_0) \subseteq \overline{B_r(x_0)}$, while conversely if (x_n) is a sequence such that each $x_n \in B_r(x_0)$ and $x_n \to x$ then

$$\|x - x_0\| \leq \|x - x_n\| + \|x_n - x_0\| < \|x - x_n\| + r.$$

Letting $n \to \infty$, we get $\|x - x_0\| \leq r$. Hence $\overline{B_r(x_0)} \subseteq \overline{B}_r(x_0)$.

(c) Use (i) to show that $(x_1, x_2) \mapsto |x_1|^{1/2} + |x_2|^{1/2}$ does *not* define a norm on $\mathbb{R}^2$.

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Solution: In $\mathbb{R}^2$, consider $x = (1,0)$ and $y = (0,1)$ so $\frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2}, \frac{1}{2})$. The "$p$-norm" formula with $p = 1/2$ would give $\|x + y\| = (2/\sqrt{2})^2 = 2$ so that $x + y$ would not belong to the closed ball centre 0 and radius 1. But x and y do belong to this ball. This contradicts convexity. [A sketch of the set of points $(s,t) \in \mathbb{R}^2$ for which $|s|^{1/2} + |t|^{1/2} \leq 1$ is instructive.]

4. (a) Let X be a real normed space. Let $T : X \to \mathbb{R}$ be a linear map such that $|T(x)| \leq \|x\|$ for all $x \in X$. Prove that T is continuous.

   Solution: Linearity of T implies $|Tx - Ty| = |T(x - y)|$. So $|Tx - Ty| \leq \|x - y\|$, i.e. T is Lipschitz continuous and hence of course continuous.

   (b) Let $X = \ell^p$, $1 \leq p \leq \infty$, equipped with the p-norm $\|x\|_p = (\sum_{j=1}^{\infty} |x_j|^p)^{1/p}$, respectively $\|x\|_\infty = \sup_j |x_j|$. Define $\pi_k : X \to \mathbb{R}$ by $\pi_k((x_j)) = x_k$ (for any $k \geq 1$). Check that each π_k is continuous.

   Solution: Fix k. Note that for any of $1 \leq p \leq \infty$, we have $|x_k| \leq \|(x_j)\|_p$. Therefore π_k is norm-reducing and so continuous by (i).

   (c) Let $X = L^2([0,1])$ and define $T : X \to \mathbb{R}$ by

$$T(f) := \int_0^1 f\,dx.$$

   Check that T is continuous. [*Hint:* Use Hölder's inequality $\|fg\|_{L^1} \leq \|f\|_{L^2}\|g\|_{L^2}$ for every $f, g \in L^2([0,1])$.]

   Solution: We use that the constant function $g \equiv 1$ is an element of $L^2([0,1])$ with $\|g\|_{L^2([0,1])} = \left(\int_0^1 1\,dx\right)^{1/2} = 1$. By Hölder's inequality we thus get that $|T(f)| = \left|\int_0^1 f\right| \leq \|f\|_{L^1} = \|f \cdot g\|_{L^1} \leq \|f\|_{L^2} \cdot \|g\|_{L^2} = \|f\|_{L^2}$ so continuity follows from (i).

   (d) Let X be as in (ii). Let (a_j) be a fixed sequence of real numbers and define

$$Y = \{(x_j) \in X \mid x_{2j} = a_j x_{2j-1} \text{ for all } j \geq 1\}.$$

   Check that Y is a subspace of X and, by writing Y as an intersection of closed sets involving maps π_k, or otherwise, show that Y is closed.

   Solution: Because vector space operations in X are defined coordinatewise, it follows from the Subspace Test (routine calculations!) that Y is a subspace.

   To see that Y is closed, there are different arguments possible:

   Variant 1: We know that a set $F \subset X$ is closed if for any sequence (x_j) with $x_j \in F$ which converges $x_j \to x \in X$, the limit x is again an element of F. Given a sequence

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$(x_j^{(k)}) \subset Y$ which converges to some limit $(x_j^{(k)}) \to (x_j)$ as $k \to \infty$ we must have that also the components converge and hence $x_{2j} = \lim x_{2j}^{(k)} = \lim a_j x_{2j-1}^{(k)} = a_j x_{2j-1}$ so $(x_j) \in Y$.

Variant 2: We recall that for continuous maps the preimage of any closed set is again closed. For each k the map $\rho_k : y \mapsto \pi_{2k}(y) - a_k \pi_{2k-1}(y)$ is continuous, so $\rho_k^{-1}(\{0\})$ is closed. Then

$$Y = \bigcap_k \rho_k^{-1}(\{0\})$$

is an intersection of closed sets and hence is closed.

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