B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 0 — HT 2026

Revision of some Prelims and Part A material, we will build on.

Please solve before the course starts. Do not hand in.

Solutions will be given in our first lecture in Week 1 (Wednesday, 21/01/2025).

1. Let matrix $M \in \mathbb{R}^{3\times 3}$ and vector $\mathbf{x}_0 \in \mathbb{R}^3$ be given as

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_0 = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

   (a) Let $\mathbf{x}_k \in \mathbb{R}^3$, $k = 0, 1, 2, \ldots$, be a sequence defined iteratively by $\mathbf{x}_{k+1} = M\mathbf{x}_k$. Find $\mathbf{x}_k$.

   (b) Find the solution of the system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x} \qquad \text{with the initial condition} \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

2. Let matrix $M \in \mathbb{R}^{2\times 2}$ be given as

$$M = \begin{pmatrix} 1 & 2 \\ -1 & \mu \end{pmatrix}.$$

   (a) Given $\mathbf{x}_0 \in \mathbb{R}^2$, consider sequence $\mathbf{x}_k \in \mathbb{R}^2$, $k = 0, 1, 2, \ldots$, defined iteratively by $\mathbf{x}_{k+1} = M\mathbf{x}_k$. Show that

$$\lim_{k\to\infty} \| \mathbf{x}_k \| = 0, \qquad \text{for} \quad \mu \in (-2, -1)\,,$$

   and

$$\lim_{k\to\infty} \| \mathbf{x}_k \| = \infty, \qquad \text{for} \quad \mu \in (-1, \infty)\,.$$

   (b) Find and classify the critical point of the planar ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$$

   for each value of the parameter $\mu \in \mathbb{R}$.

3. Let $\mu \in [0, 4]$ be a parameter and $x_0 = 0.7$. Define the sequence $x_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots$, iteratively by

$$x_{k+1} = \mu\, x_k\, (1 - x_k)\,.$$

   Write a computer code which plots the first 200 values of this sequence for the following values of parameter μ:

$$\mu = 0.5, \qquad \mu = 2, \qquad \mu = 3.2, \qquad \mu = 3.5, \qquad \mu = 3.55, \qquad \mu = 3.8, \qquad \mu = 4.$$

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4. Let X be a continuous random variable on interval $[0,1]$ with the probability density function $p\colon(0,1)\to(0,\infty)$ given by

$$p(x) = \frac{1}{\pi\sqrt{x(1-x)}}\,.$$

Let function $F\colon[0,1]\to[0,1]$ be defined by $F(x) = 4\,x\,(1-x)$.

Find the probability density function of random variable $F(X)$.

5. Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous system of ODEs given by:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - \mu\,y + y^2(1-x) - x^3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \mu\,x - xy(1+x) + y - y^3 \end{aligned} \qquad (*)$$

(a) Let $\mu \in (-1,1)$. Show that the ODE system $(*)$ has three critical points given by

$$[0,0], \qquad \left[\sqrt{1-\mu^2},\mu\right] \qquad \left[-\sqrt{1-\mu^2},\mu\right]$$

Classify the critical points and sketch the phase plane.

(b) Write a computer code which calculates five solutions of ODEs $(*)$ for the initial conditions

$$[x_1(0),y_1(0)] = [-2,-2], \qquad [x_2(0),y_2(0)] = [-2,2], \qquad [x_3(0),y_3(0)] = [2,-2],$$

$$[x_4(0),y_4(0)] = [2,2] \qquad \text{and} \qquad [x_5(0),y_5(0)] = [0.1,0.1].$$

Plot trajectories $[x_j(t),y_j(t)]$, $j=1,2,3,4,5$, in the phase plane for

$$\mu = \frac{9}{10} \qquad \text{and} \qquad \mu = 2.$$

(c) Transform the ODE system $(*)$ to polar coordinates, *i.e.* write a system of ODEs for variables $r(t)$ and $\theta(t)$, where

$$x(t) = r(t)\cos\theta(t) \qquad \text{and} \qquad y(t) = r(t)\sin\theta(t).$$

(d) Show that the ODE system $(*)$ has a periodic solution (limit cycle) for all values of parameter μ satisfying $|\mu| > 1$.

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