B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 1 — HT 2026

Solutions to all problems in Sections A and C

Section A: Problems 1, 2 and 3

1. Find the stable, unstable and center subspaces E^s, E^u and E^c of the linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$$

with matrix $M \in \mathbb{R}^{4\times 4}$ given by

(a)

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b)

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

(c)

$$M = \begin{pmatrix} -11 & 0 & 9 & -2 \\ -5 & -12 & 7 & 6 \\ -19 & 0 & 17 & -2 \\ -17 & -8 & 19 & 2 \end{pmatrix}$$

Solution: We denote the eigenvalues and generalized eigenvectors of M by

$$\lambda_j = a_j + \mathrm{i}\, b_j \qquad \text{and} \qquad \mathbf{w}_j = \mathbf{u}_j + \mathrm{i}\, \mathbf{v}_j,$$

where $a_j, b_j \in \mathbb{R}$, $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{R}^4$, for $j = 1, 2, 3, 4$.

(a) The matrix M is diagonalizable (semi-simple). It has four different eigenvalues and eigenvectors given by

$$a_1 = a_2 = -\frac{1}{2}, \qquad a_3 = a_4 = \frac{1}{2}, \qquad b_1 = b_3 = \frac{\sqrt{3}}{2}, \qquad b_2 = b_4 = -\frac{\sqrt{3}}{2},$$

Radek Erban: erban@maths.ox.ac.uk

$$\mathbf{u}_1 = \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_1 = -\mathbf{v}_2 = \sqrt{3} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\mathbf{u}_3 = \mathbf{u}_4 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_3 = -\mathbf{v}_4 = \sqrt{3} \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, we have

$$E^s = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad E^u = \text{span}\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad E^c = \emptyset.$$

(b) The matrix M is diagonalizable (semi-simple). It has four different eigenvalues and eigenvectors given by

$$\lambda_1 = -2, \qquad \lambda_2 = 2, \qquad \lambda_3 = \mathrm{i}, \qquad \lambda_4 = -\mathrm{i},$$

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \mathbf{u}_4 = \begin{pmatrix} -5 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = -\mathbf{v}_4 = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, we have

$$E^s = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad E^u = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}, \quad E^c = \text{span}\left\{ \begin{pmatrix} -5 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(c) The characteristic polynomial of matrix M is $(\lambda - 8)(\lambda + 4)^3$. Consequently, matrix M has two eigenvalues: $\lambda_1 = 8$ (with multiplicity 1) and $\lambda_2 = -4$ (with algebraic multiplicity 3 and geometric multiplicity 1). The corresponding eigenvectors are

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{1}$$

The generalized eigenvectors corresponding to $\lambda_2 = -4$ are

$$\mathbf{w}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w}_4 = \begin{pmatrix} 4 \\ 1 \\ 4 \\ 0 \end{pmatrix}. \tag{2}$$

They satisfy the equations $(M - \lambda_2 I)\mathbf{w}_3 = 2\mathbf{w}_2$ and $(M - \lambda_2 I)\mathbf{w}_4 = 8\mathbf{w}_3$, which implies $(M - \lambda_2 I)^2\mathbf{w}_3 = \mathbf{0}$ and $(M - \lambda_2 I)^3\mathbf{w}_4 = \mathbf{0}$. Consequently, we have

$$E^s = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 4 \\ 0 \end{pmatrix} \right\}, \qquad E^u = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}, \qquad E^c = \emptyset.$$

2. Consider the linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$$

with matrix $M \in \mathbb{R}^{4\times 4}$ given as in Question 1(c), i.e.

$$M = \begin{pmatrix} -11 & 0 & 9 & -2 \\ -5 & -12 & 7 & 6 \\ -19 & 0 & 17 & -2 \\ -17 & -8 & 19 & 2 \end{pmatrix}$$

and the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0.$$

(a) Assume that $\mathbf{x}_0 \in E^u$ where E^u is the unstable subspace calculated in part 1(c). Assume that $\mathbf{x}_0 \neq \mathbf{0}$. Show that

$$\lim_{t\to\infty} \| \mathbf{x}(t) \| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$$

(b) Assume that $\mathbf{x}_0 \in E^S$ where E^S is the stable subspace calculated in part 1(c). Assume that $\mathbf{x}_0 \neq \mathbf{0}$. Show that

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0} \quad \text{and} \quad \lim_{t\to-\infty} \| \mathbf{x}(t) \| = \infty$$

Solution:

(a) If $\mathbf{x}_0 \in E^u$ and $\mathbf{x}_0 \neq \mathbf{0}$, then there exists a constant $\alpha_1 \neq 0$ such that $\mathbf{x}_0 = \alpha_1 \mathbf{w}_1$, where $\mathbf{w}_1$ is given by (1), and the solution of our initial value problem is

$$\mathbf{x}(t) = \alpha_1 \, e^{\lambda_1 t} \, \mathbf{w}_1.$$

Since $\lambda_1 = 8 > 0$ and $\alpha_1 \neq 0$, we have

$$\lim_{t \to \infty} \| \mathbf{x}(t) \| = \infty \quad \text{and} \quad \lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{0}.$$

(b) If $\mathbf{x}_0 \in E^s$, then there exist constants α_2, α_3 and α_4 such that

$$\mathbf{x}_0 = \alpha_2 \, \mathbf{w}_2 + \alpha_3 \, \mathbf{w}_3 + \alpha_4 \, \mathbf{w}_4 \,, \tag{3}$$

where $\mathbf{w}_2$, $\mathbf{w}_3$ and $\mathbf{w}_4$ are given by (1)–(2), and the solution of our initial value problem is

$$\mathbf{x}(t) = (\alpha_2 + 2\alpha_3 t + 8\alpha_4 t^2) \, e^{\lambda_2 t} \, \mathbf{w}_2 \; + \; (\alpha_3 + 8\alpha_4 t) \, e^{\lambda_2 t} \, \mathbf{w}_3 \; + \; \alpha_4 \, e^{\lambda_2 t} \, \mathbf{w}_4 \,. \tag{4}$$

Indeed, substituting $t = 0$ into (4) and using (3), we confirm that the solution (4) satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Moreover, multiplying (4) by M and using $M\mathbf{w}_2 = \lambda_2 \mathbf{w}_2$, $M\mathbf{w}_3 = \lambda_2 \mathbf{w}_3 + 2\mathbf{w}_2$ and $M\mathbf{w}_4 = \lambda_2 \mathbf{w}_4 + 8\mathbf{w}_3$, we get

$$M\mathbf{x}(t) = \lambda_2 \, \mathbf{x}(t) \; + \; 2 \, (\alpha_3 + 8\alpha_4 t) \, e^{\lambda_2 t} \, \mathbf{w}_2 \; + \; 8 \, \alpha_4 \, e^{\lambda_2 t} \, \mathbf{w}_3 \,,$$

which we also obtain by differentiating (4) as $\mathrm{d}\mathbf{x}/\mathrm{d}t$. In particular, we have confirmed that formula (4) is the solution of our initial value problem with $\mathbf{x}_0$ given by (3). Since $\lambda_2 = -4$ and $\mathbf{x}_0 \neq \mathbf{0}$, we conclude that at least one α_j, $j = 2, 3, 4$, is nonzero and equation (4) implies that

$$\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0} \quad \text{and} \quad \lim_{t \to -\infty} \| \mathbf{x}(t) \| = \infty \,.$$

3. Consider the system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 5$ chemical reactions:

$$2X_1 \xrightarrow{k_1} 2X_1 + X_2 \qquad X_1 + X_2 \xrightarrow{k_2} 2X_1 + X_2 \qquad 3X_1 \xrightarrow{k_3} 2X_1$$

$$X_2 \xrightarrow{k_4} \emptyset \qquad 2X_1 + X_2 \xrightarrow{k_5} 2X_1$$

Let $x_1(t)$ and $x_2(t)$ be the concentrations of the chemical species X_1 and X_2, respectively.

(a) Assuming mass action kinetics, write a system of ODEs (reaction rate equations) describing the time evolution of $x_1(t)$ and $x_2(t)$.

(b) Assume the problem has already been non-dimensionalized and choose the values of dimensionless rate constants as

$$k_1 = \mu, \qquad k_2 = 1, \qquad k_3 = 4, \qquad k_4 = 1 \quad \text{and} \quad k_5 = 1,$$

where $\mu > 0$ is a single parameter that we will vary.

Use an analysis of the dynamics on the center manifold to show that

(i) The origin $[x_1, x_2] = [0, 0]$ is an asympotically stable critical point if $\mu \le 4$.

(ii) The origin $[x_1, x_2] = [0, 0]$ is an asymptotically unstable critical point if $\mu > 4$.

(c) Find and classify all critical points and sketch the phase plane in the nonnegative quadrant $\{x_1 \ge 0,\, x_2 \ge 0\}$ for: (i) $\mu \in (0, 4)$; and (ii) $\mu > 4$.

Solution:

(a) Using the definition of mass action kinetics (covered in Lecture 1), we have:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= k_2\, x_1\, x_2 - k_3\, x_1^3 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= k_1\, x_1^2 - k_4\, x_2 - k_5\, x_1^2\, x_2 \end{aligned}$$

(b) Using our values of parameters $k_1 = \mu$, $k_2 = 1$, $k_3 = 4$, $k_4 = k_5 = 1$, we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1\, x_2 - 4\, x_1^3 \tag{5}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu\, x_1^2 - x_2 - x_1^2\, x_2 \tag{6}$$

The origin $[x_1, x_2] = [0, 0]$ is a critical point and we have

$$D\mathbf{f}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Radek Erban: erban@maths.ox.ac.uk

In particular, the linearization has eigenvalues -1 and 0 and the center subspace is spanned by the eigenvector $(1,0)^{\mathrm{T}}$ which corresponds to the 0 eigenvalue. The center manifold is tangent to the center subspace, so it can be locally written as

$$x_2 = c_2\, x_1^2 + c_3\, x_1^3 + c_4\, x_1^4 + \mathcal{O}(x_1^5). \tag{7}$$

Differentiating with respect of t and substituting (5) and (6), we get

$$(\mu - c_2)\, x_1^2 - c_3\, x_1^3 + (c_2(7-2c_2) - c_4)\, x_1^4 + \mathcal{O}(x_1^5) = 0.$$

Consequently, $c_2 = \mu$, $c_3 = 0$ and $c_4 = \mu(7-2\mu)$. Substituting into equation (7), we get

$$x_2 = \mu\, x_1^2 + \mu(7-2\mu)\, x_1^4 + \mathcal{O}(x_1^5).$$

Substituting into equation (5), we get the dynamics on the center manifold as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = (\mu - 4)\, x_1^3 \, + \, \mu(7-2\mu)\, x_1^5 + \mathcal{O}(x_1^6).$$

Consequently, we obtain that the origin is asymptotically stable for $\mu < 4$ and unstable for $\mu > 4$. If $\mu = 4$, we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\,4\, x_1^5 + \mathcal{O}(x_1^6).$$

Thus we conclude that the origin is asymptotically stable for $\mu \le 4$.

(c) The critical points are given as solutions of the system:

$$\begin{aligned} 0 &= x_1\, x_2 \,-\, 4\, x_1^3 \\ 0 &= \mu\, x_1^2 \,-\, x_2 \,-\, x_1^2\, x_2 \end{aligned}$$

Consequently, the first equation implies that we either have $x_1 = 0$ or $x_2 = 4x_1^2$. Substituting into the second equation, we deduce:

(i) If $\mu \in (0,4)$, then the origin is the only critical point which is asymptotically stable.

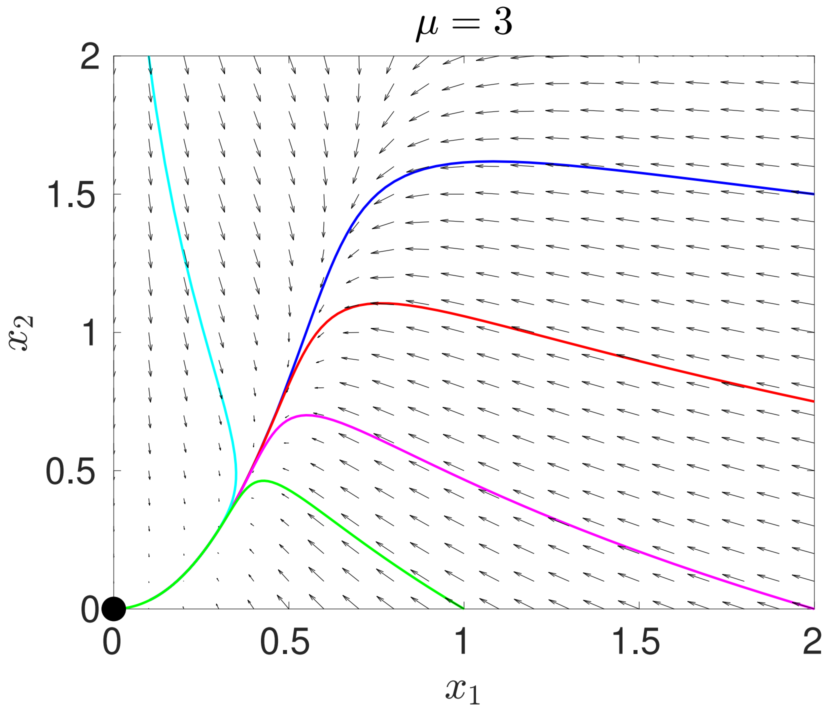
(ii) If $\mu > 4$, then there are three critical points given by

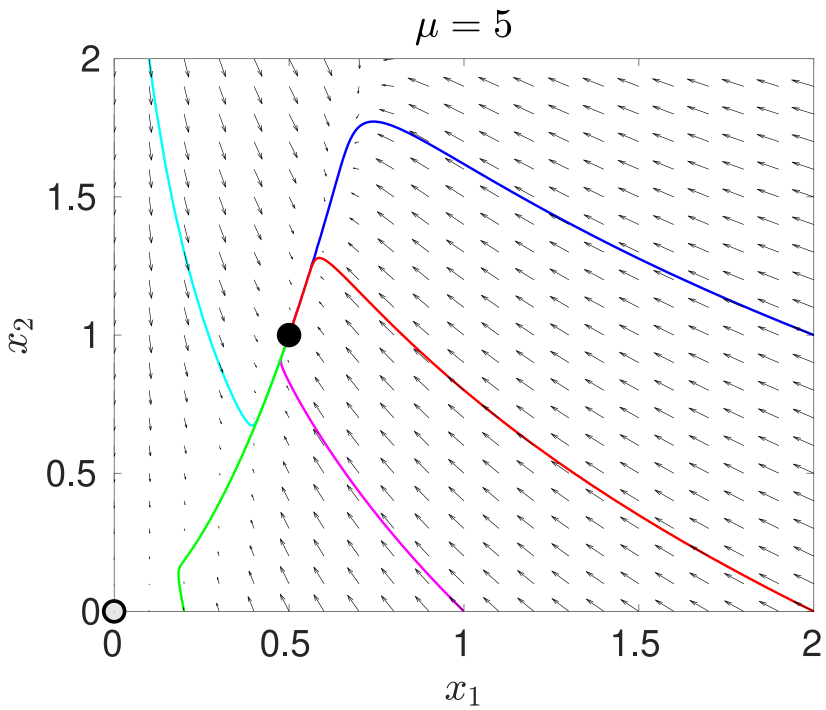
$$[0,0], \qquad \left[\frac{\sqrt{\mu-4}}{2}, \mu - 4\right], \qquad \left[-\frac{\sqrt{\mu-4}}{2}, \mu - 4\right].$$

The first two critical points are in the nonnegative quadrant $\{x_1 \ge 0,\, x_2 \ge 0\}$. The origin $[x_1, x_2] = [0,0]$ is asymptotically unstable and $\left[\frac{\sqrt{\mu-4}}{2}, \mu - 4\right]$ is a stable node for $\mu > 4$.

Radek Erban: erban@maths.ox.ac.uk

The illustrative phase planes in domain $[0,2] \times [0,2]$ are plotted below for $\mu = 3$ and $\mu = 5$. The black dots denote the critical points (filled-in dots are stable and empty dots are unstable). Five illustrative trajectories starting at the boundary of the box are plotted using different colours. They converge to the origin $[0,0]$ for $\mu = 3$, and to the critical point $[1/2, 1]$ for $\mu = 5$.





Radek Erban: erban@maths.ox.ac.uk

Section C: Problem 7

7. Let $g \in C^1(\mathbb{R})$ be a given function satisfying $g(x) \geq 1$ for all $x \in \mathbb{R}$. Consider the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 \qquad \text{with} \qquad x(0) = x_0 \tag{$\bigstar$}$$

and the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{x^2}{g(x)} \qquad \text{with} \qquad x(0) = x_0. \tag{$\blacktriangle$}$$

(a) Find the solution of the initial value problem ($\bigstar$) and the maximum interval $I^{\bigstar}(x_0)$ where the solution is defined for each initial condition $x_0 \in \mathbb{R}$.

(b) Let x_0 be given and denote the orbits corresponding to systems ($\bigstar$) and ($\blacktriangle$) by $\Gamma^{\bigstar}_{x_0}$ and $\Gamma^{\blacktriangle}_{x_0}$, respectively. Show that

$$\Gamma^{\bigstar}_{x_0} = \Gamma^{\blacktriangle}_{x_0}, \qquad \text{for all } \; x_0 \in \mathbb{R},$$

i.e. the ODEs ($\bigstar$) and ($\blacktriangle$) have the same phase portrait.

(c) Find $g(x)$ such that the initial value problem ($\blacktriangle$) has its unique solution on the maximum interval $I^{\blacktriangle}(x_0) = \mathbb{R}$ for each initial condition $x_0 \in \mathbb{R}$.

Solution:

(a) Given the initial condition $x(0) = x_0 \in \mathbb{R}$, the solution of ODE ($\bigstar$) is

$$x(t) = \frac{x_0}{1 - t\,x_0} \qquad \text{for} \quad t \in I^{\bigstar}(x_0), \tag{8}$$

where the maximal interval of existence $I^{\bigstar}(x_0)$ is

$$I^{\bigstar}(x_0) = \begin{cases} \left(-\infty, \dfrac{1}{x_0}\right) & \text{for } \; x_0 > 0, \\ \mathbb{R} & \text{for } \; x_0 = 0, \\ \left(\dfrac{1}{x_0}, \infty\right) & \text{for } \; x_0 < 0. \end{cases}$$

(b) We define the new time τ by

$$\tau \equiv \tau(t) = \int_0^t g(x(s; x_0))\,\mathrm{d}s. \tag{9}$$

Since $g(x) \geq 1$ for all $x \in \mathbb{R}$, we conclude that $\tau(t)$ is a strictly increasing function of t. In particular, its inverse $t(\tau)$ exists and we can use the chain rule to deduce

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{x^2}{g(x)}\,, \tag{10}$$

i.e. $x(t(\tau))$ solves the initial value problem (▲). Using (8), the orbit $\Gamma^{\bigstar}_{x_0} \subset \mathbb{R}$ based on $x_0 \in \mathbb{R}$ is given by

$$\Gamma^{\bigstar}_{x_0} = \big\{ x(t;x_0) \,\big|\, t \in I^{\bigstar}(x_0) \big\} = \left\{ \frac{x_0}{1 - t\,x_0} \,\middle|\, t \in I^{\bigstar}(x_0) \right\} .$$

Using (10), this can be rewritten as

$$\Gamma^{\bigstar}_{x_0} = \big\{ x(t;x_0) \,\big|\, t \in I^{\bigstar}(x_0) \big\} = \big\{ x(t(\tau);x_0) \,\big|\, \tau \in I^{\blacktriangle}(x_0) \big\} = \Gamma^{\blacktriangle}_{x_0}.$$

(c) Let $g(x) = 1 + x^2$. Then the initial value problem (▲) is given as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{x^2}{1+x^2} \qquad \text{with} \qquad x(0) = x_0\,.$$

It has the unique solution $x(\tau) \equiv 0$ for $x_0 = 0$ and

$$x(\tau) = \frac{\tau\,x_0 + x_0^2 - 1 + \sqrt{(\tau\,x_0 + x_0^2 - 1)^2 + 4\,x_0^2}}{2x_0} \qquad \text{for } \; x_0 \neq 0\,, \tag{11}$$

which is defined on the maximum interval of existence $I^{\blacktriangle}(x_0) = \mathbb{R}$ for each initial condition $x_0 \in \mathbb{R}$. Substituting $g(x) = 1 + x^2$ and (8) into our rescaling of time equation (9), we get

$$\tau(t) = t\left(1 + \frac{x_0^2}{1 - t\,x_0}\right). \tag{12}$$

Substituting equation (12) into the solution formula (11), we obtain the solution formula (8). In particular, we confirm that the ODEs (★) and (▲) have the same phase portrait and the trajectories of (▲) are defined for all $\tau \in \mathbb{R}$.

Radek Erban: erban@maths.ox.ac.uk