B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 2 — HT 2026

Solutions to all problems in Sections A and C

Section A: Problems 1, 2 and 3

1. Consider the ODE system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + 2\, x_1^3 - x_1^5$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\, x_2$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) Find and classify all bifurcations of the ODE system. Plot the bifurcation diagram.

(b) Sketch the phase plane for $\mu = -3/4$.

Solution:

(a) The origin $[0, 0]$ is a critical point for all values $\mu \in \mathbb{R}$. Other critical points are of the form $[x_c, 0]$, where x_c is a solution of $\mu = x_c^4 - 2\, x_c^2$. Completing the square, we have $(x_c^2 - 1)^2 = \mu + 1$, which implies:

(i) There is only one critical point $\mathbf{x}_0 = [0, 0]$ for $\mu \in (-\infty, -1)$, which is stable.

(ii) There are five critical points

$$\mathbf{x}_{-2} = \left[-\sqrt{1 + \sqrt{\mu + 1}}, 0 \right], \qquad \mathbf{x}_{-1} = \left[-\sqrt{1 - \sqrt{\mu + 1}}, 0 \right], \qquad \mathbf{x}_0 = [0, 0],$$

$$\mathbf{x}_1 = \left[\sqrt{1 - \sqrt{\mu + 1}}, 0 \right], \qquad \mathbf{x}_2 = \left[\sqrt{1 + \sqrt{\mu + 1}}, 0 \right], \qquad \text{for} \quad \mu \in (-1, 0).$$

Moreover, the critical points $\mathbf{x}_{-2}$, $\mathbf{x}_0$ and $\mathbf{x}_2$ are stable nodes, while the critical points $\mathbf{x}_{-1}$ and $\mathbf{x}_1$ are (unstable) saddles.

(iii) There are three critical points

$$\mathbf{x}_{-2} = \left[-\sqrt{1 + \sqrt{\mu + 1}}, 0 \right], \qquad \mathbf{x}_0 = [0, 0], \qquad \mathbf{x}_2 = \left[\sqrt{1 + \sqrt{\mu + 1}}, 0 \right],$$

for $\mu \in (0, \infty)$. Moreover, the critical points $\mathbf{x}_{-2}$ and $\mathbf{x}_2$ are stable, while the critical point $\mathbf{x}_0$ is unstable.

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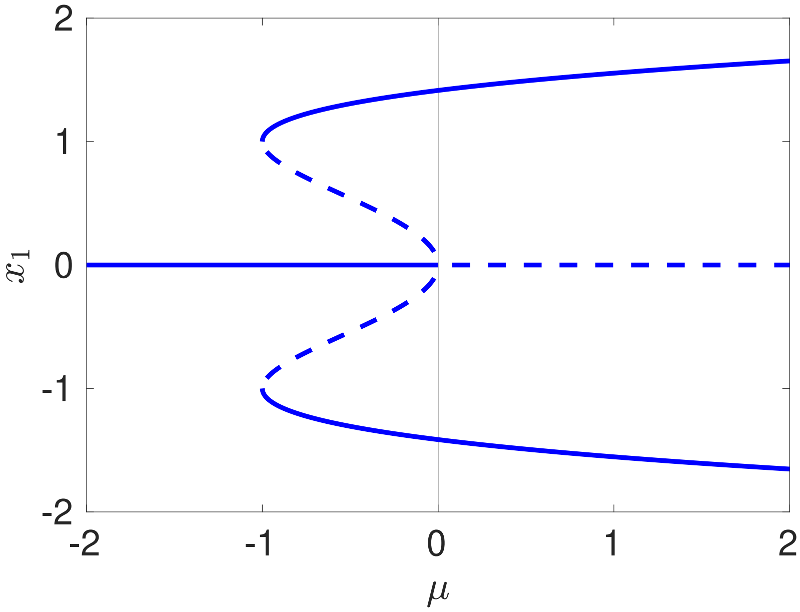
We have a subcritical pitchfork bifurcation at $\mu = 0$. The origin is (locally) stable for $\mu < 0$ and unstable for $\mu > 0$. Two branches of unstable fixed points bifurcate from the origin when $\mu = 0$, as can be seen on the following bifurcation diagram:



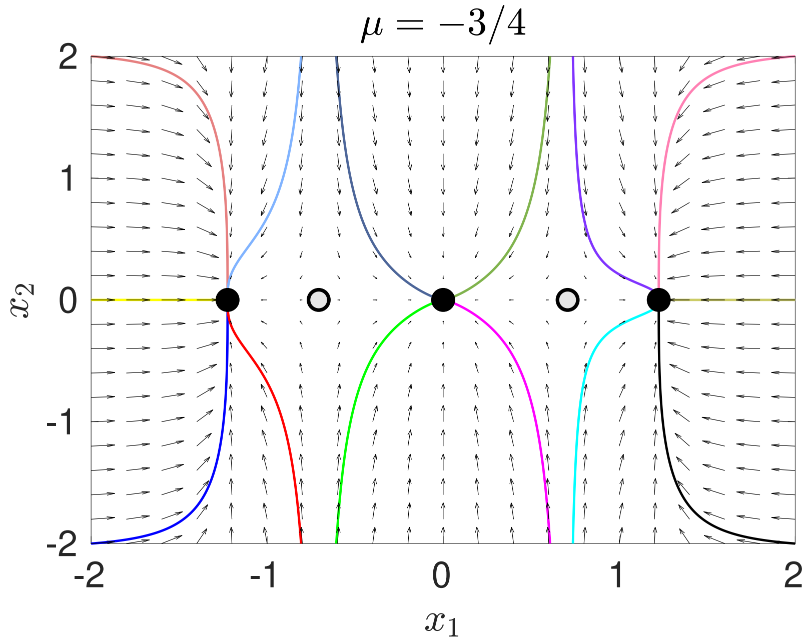
In addition to the subcritical pitchfork bifurcation at $\mu = 0$, we also have a saddle-node bifurcation at $\mu = -1$: stable node $\mathbf{x}_{-2}$ moves towards saddle $\mathbf{x}_{-1}$ as μ approaches -1 from above, and these two critical points collide (mutually annihilate) at $\mu = -1$. We also have a saddle-node bifurcation at $\mu = -1$, where stable node $\mathbf{x}_2$ collides with saddle $\mathbf{x}_1$.

(b) Using $\mu = -3/4$, there are five critical points

$$\mathbf{x}_{-2} = \left[-\sqrt{\frac{3}{2}}, 0\right], \quad \mathbf{x}_{-1} = \left[-\sqrt{\frac{1}{2}}, 0\right], \quad \mathbf{x}_0 = [0,0], \quad \mathbf{x}_1 = \left[\sqrt{\frac{1}{2}}, 0\right], \quad \mathbf{x}_2 = \left[\sqrt{\frac{3}{2}}, 0\right]$$

with saddles at $\mathbf{x}_{-1}$ and $\mathbf{x}_1$ and stable nodes at $\mathbf{x}_{-2}$, $\mathbf{x}_0$ and $\mathbf{x}_2$.

The phase plane is plotted in the figure on the next page, where we visualize stable critical points using filled-in black dots and unstable critical points as empty dots. The figure also includes 14 illustrative trajectories, each starting at the boundary of the plotted box $[-2, 2] \times [-2, 2]$ and converging to one of the stable nodes $\mathbf{x}_{-2}$, $\mathbf{x}_0$ or $\mathbf{x}_2$.



2. Consider the system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 6$ chemical reactions:

$$X_1 \xrightarrow{k_1} X_2 \qquad X_2 \xrightarrow{k_2} X_1 \qquad X_1 \xrightarrow{k_3} \emptyset$$

$$\emptyset \xrightarrow{k_4} X_2 \qquad 2X_1 \xrightarrow{k_5} 3X_1 \qquad 3X_1 \xrightarrow{k_6} 2X_1$$

Let $x_1(t)$ and $x_2(t)$ be the concentrations of the chemical species X_1 and X_2, respectively.

(a) Assuming mass action kinetics, write a system of ODEs (reaction rate equations) describing the time evolution of $x_1(t)$ and $x_2(t)$.

(b) Assume the problem has already been non-dimensionalized and choose the values of dimensionless rate constants as

$$k_1 = 3, \qquad k_2 = 1, \qquad k_3 = 12, \qquad k_4 = \mu, \qquad k_5 = 9 \quad \text{and} \quad k_6 = 2,$$

where $\mu > 0$ is a single parameter that we will vary.

Find and classify all bifurcations of the ODE system.

(c) Plot the bifurcation diagram.

(d) Sketch the phase plane for $\mu = 9/2$.

Solution:

(a) Using the definition of mass action kinetics (covered in Lecture 1), we have:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = k_2\,x_2 - (k_1+k_3)\,x_1 + k_5\,x_1^2 - k_6\,x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = k_4 + k_1\,x_1 - k_2\,x_2$$

(b) Using our values of parameters $k_1 = 3$, $k_2 = 1$, $k_3 = 12$, $k_4 = \mu$, $k_5 = 9$, $k_6 = 2$, we have

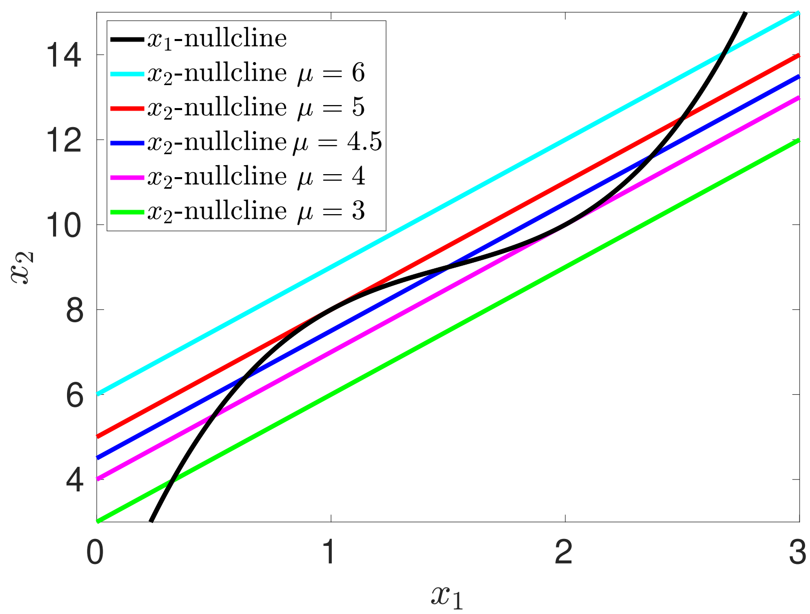
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 - 15\,x_1 + 9\,x_1^2 - 2\,x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu + 3\,x_1 - x_2$$

The nullclines can be written as functions of x_1:

$$x_2 = 15\,x_1 - 9\,x_1^2 + 2\,x_1^3 \qquad (1)$$
$$x_2 = \mu + 3\,x_1 \qquad (2)$$

The x_1-nullcline is independent of μ and is plotted below as the black curve.

The x_2-nullcline is a straight line that depends on μ. We plot x_2-nullcline for five different values of μ below:



The ODE system has two saddle-node bifurcations: one at $\mu = 4$, where the critical point $[2, 10]$ bifurcates into a saddle and a node for $\mu > 4$, and one at $\mu = 5$, where the critical point $[1, 8]$ bifurcates into a saddle and a node for $\mu < 5$.

Both bifurcations can be further analyzed using the the extended center manifold theory. We define new (local) variables by

$$\text{bifurcation at } \mu = 4: \qquad \overline{x}_1 = x_1 - 2, \qquad \overline{x}_2 = x_2 - 10, \qquad \nu = \mu - 4,$$

$$\text{bifurcation at } \mu = 5: \qquad \overline{x}_1 = x_1 - 1, \qquad \overline{x}_2 = x_2 - 8, \qquad \nu = \mu - 5.$$

Then the ODE system can be written in the matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}-3 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \begin{pmatrix}\mp 3\,\overline{x}_1^2 \,-\, 2\,\overline{x}_1^3\\ \nu\end{pmatrix} \tag{3}$$

where the top sign (minus −) corresponds to the local variables used for the bifurcation at $\mu = 4$ and the bottom sign (plus +) corresponds to the local variables used for the analysis of the bifurcation at $\mu = 5$. We define new coordinates by

$$\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

with the inverse transform

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}.$$

Then the system (3) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}-4 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\mp 3\,(y_1+y_2)^2 \,-\, 2\,(y_1+y_2)^3\\ \nu\end{pmatrix}.$$

The extended system is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\\ \nu\end{pmatrix} = \begin{pmatrix}-4 & 0 & -1/4\\ 0 & 0 & 1/4\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ \nu\end{pmatrix} + \frac{\mp 3\,(y_1+y_2)^2 \,-\, 2\,(y_1+y_2)^3}{4}\begin{pmatrix}3\\ 1\\ 0\end{pmatrix}. \tag{4}$$

The corresponding stable and center subspaces are

$$E^s = \operatorname{span}\left\{\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right\}, \qquad E^c = \operatorname{span}\left\{\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}-1/4\\ 0\\ 4\end{pmatrix}\right\}.$$

The extended center manifold is given by

$$y_1 = h(y_2, \nu) = c_{01}\,\nu + c_{20}\,y_2^2 + c_{11}\,\nu y_2 + c_{02}\,\nu^2 + \dots \tag{5}$$

Differentiating with respect of time t, we get

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \frac{\partial h}{\partial y_2}(y_2,\nu)\,\frac{\mathrm{d}y_2}{\mathrm{d}t} + \frac{\partial h}{\partial \nu}(y_2,\nu)\,\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial h}{\partial y_2}(y_2,\nu)\,\frac{\mathrm{d}y_2}{\mathrm{d}t}.$$

Using (4) and (5), we get

$$\left(4c_{01} + \frac{1}{4}\right)\nu + \left(4c_{20} \pm \frac{9}{4}\right) y_2^2 + \left(4c_{11} \pm \frac{9c_{01}}{2} + \frac{c_{20}}{2}\right)\nu\, y_2$$
$$+ \left(4c_{02} \pm \frac{9c_{01}^2}{4} + \frac{c_{11}}{4}\right)\nu^2 \dots = 0\,,$$

where the top sign (plus +) corresponds to the local variables used for the bifurcation at $\mu = 4$ and the bottom sign (minus −) corresponds to the local variables used for the analysis of the bifurcation at $\mu = 5$. This implies

$$c_{01} = -\frac{1}{16}, \qquad c_{20} = \mp\frac{9}{16}, \qquad c_{11} = \pm\frac{9}{64}, \qquad c_{02} = \mp\frac{45}{4096}.$$

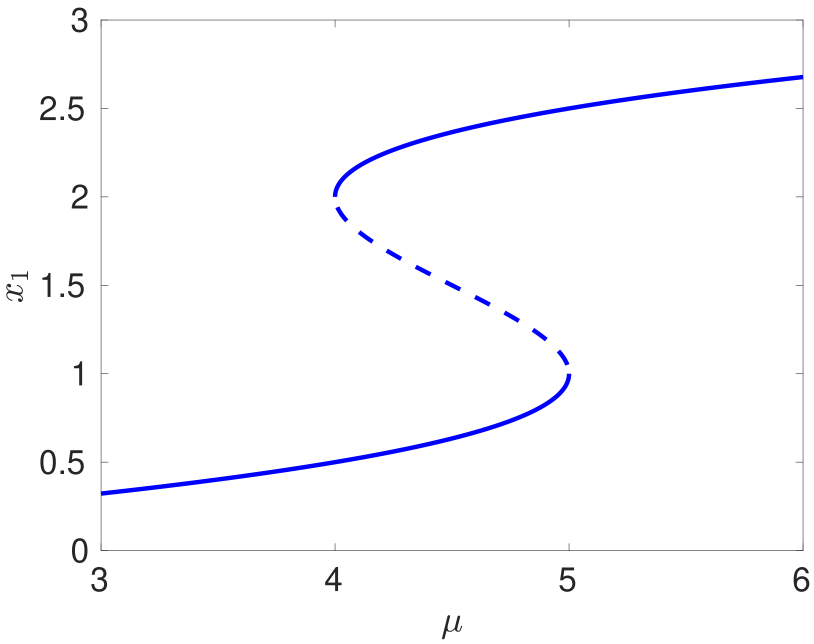
Thus the center manifold is given locally by

$$y_1 = -\frac{1}{16}\,\nu \mp \frac{9}{16}\,y_2^2 + \dots,$$

and we have saddle-node bifurcations at $\mu = 4$ (top signs) and $\mu = 5$ (bottom signs) with the dynamics on the center manifold given by

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = \frac{1}{4}\,\nu \mp \frac{3}{4}\,y_2^2 + \dots$$

(c) The bifurcation diagram is plotted below. The first coordinate of all steady states (x_1) is visualized as a function of parameter μ:



To plot this diagram, we can substitute for x_2 in equation (1) by using equation (2). We get a polynomial equation

$$\mu = 12\,x_1 - 9\,x_1^2 + 2\,x_1^3, \tag{6}$$

which can be solved to obtain all steady states. However, we can also observe that equation (6) defines μ as a function of x_1, so we can simply plot it and swap the axis to obtain the above bifurcation diagram.

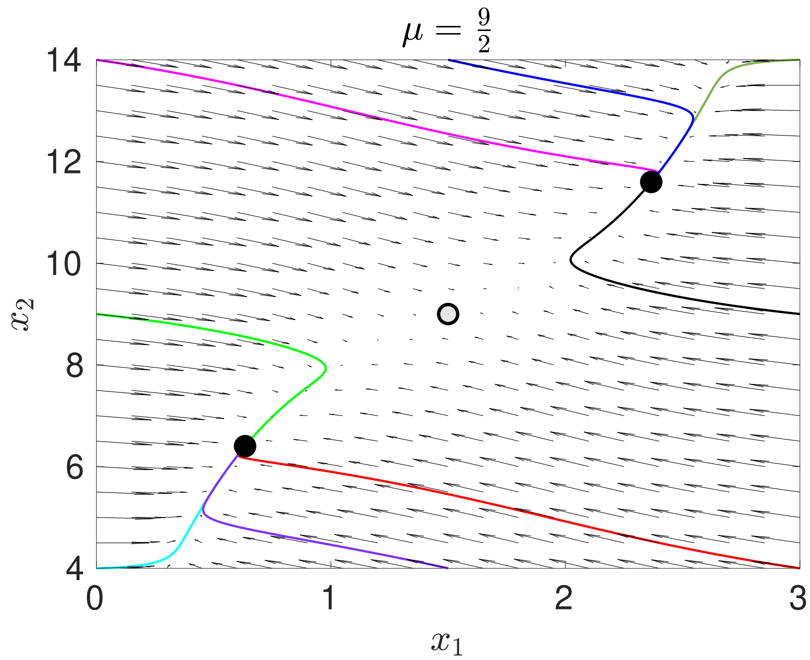
(d) Using $\mu = 9/2$, equation (6) reads as follows

$$4\,x_1^3 \,-\, 18\,x_1^2 \,+\, 24\,x_1 \,-\, 9 \,=\, 0.$$

The solutions of this equation are $3/2 \pm \sqrt{3}/2$ and $3/2$. Using (2), we conclude that there are three critical points

$$\mathbf{x}_- = \left[\frac{3-\sqrt{3}}{2}, 9 - \frac{3\sqrt{3}}{2}\right], \qquad \mathbf{x}_0 = \left[\frac{3}{2}, 9\right], \qquad \mathbf{x}_+ = \left[\frac{3+\sqrt{3}}{2}, 9 + \frac{3\sqrt{3}}{2}\right],$$

where $\mathbf{x}_-$ and $\mathbf{x}_+$ are stable nodes and $\mathbf{x}_0$ is a saddle. The phase plane is plotted here:



We visualized stable critical points using filled-in black dots and the unstable critical point as an empty dot. The above figure also includes 8 illustrative trajectories, each starting at the boundary of the plotted box $[0, 3] \times [4, 14]$ and converging to one of the stable nodes $\mathbf{x}_-$ or $\mathbf{x}_+$.

3. Let $\mu > 0$ be a parameter. Consider the map

$$x_{k+1} = F(x_k; \mu)$$

where

$$F(x; \mu) = \mu\, x \exp[1 - x].$$

(a) Let $\mu > 0$ be fixed. Find $a(\mu)$ such that $F(x;\mu)$ maps interval $[0, a(\mu)]$ in $[0, a(\mu)]$.

(b) Sketch the graphs of $F(x;\mu)$ and $F(F(x;\mu);\mu)$ on interval $[0, a(\mu)]$ for $\mu = 4$.

(c) Find all fixed points and the values of μ for which the fixed points are stable.

(d) Find a value of μ such that the map has a stable period 2-cycle.

(e) Plot the bifurcation diagram.

Solution:

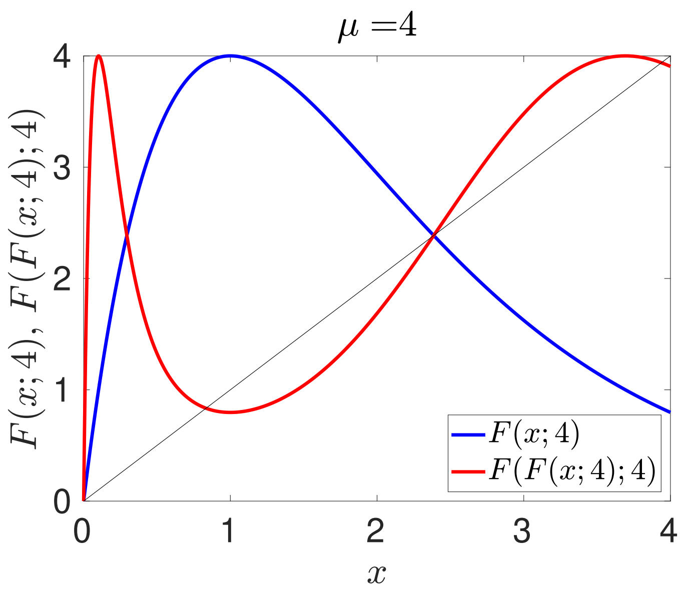
(a) The maximum of function $F(x;\mu)$ is equal to μ, which is achieved at $x = 1$.

In particular, we can choose

$$a(\mu) = \begin{cases} 1 & \text{for} \quad \mu \in (0,1); \\ \mu & \text{for} \quad \mu > 1. \end{cases}$$

Then $F(x;\mu)$ maps interval $[0, a(\mu)]$ in $[0, a(\mu)]$ for all $\mu > 0$.

(b) The graphs of $F(x; 4)$ and $F(F(x; 4); 4)$ are given here:



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Using the simplified notation (introduced in lectures), we have

$$F_\mu(x) = F(x;\mu) = \mu\, x\, \exp[1-x]$$

and

$$F_\mu^{(2)} = F_\mu(F_\mu(x)), \qquad F_\mu^{(3)} = F_\mu(F_\mu(F_\mu(x))), \qquad \dots$$

In particular, graphs plotted in part (b) visualize $F_\mu(x)$ and $F_\mu^{(2)}(x)$ and can be used to find fixed points and 2-cycles.

(c) To find formulas for fixed points, we solve

$$x = F_\mu(x) = \mu\, x\, \exp[1-x].$$

This equation has two solutions

$$x = 0, \qquad \text{and} \qquad x = 1 + \log(\mu).$$

Differentiating, we obtain

$$F_\mu'(x) = \mu\,(1-x)\, \exp[1-x],$$

which implies

$$F_\mu'(0) = \mu\, \exp[1], \qquad F_\mu'(1+\log(\mu)) = -\log(\mu)\,.$$

In particular, the fixed point at $x = 0$ is stable for $\mu \in (0, 1/e]$ and the fixed point at $x = 1 + \log(\mu)$ is stable for $\mu \in [1/e, e)$.

If $\mu = 1/e$, then there is one stable fixed point at $x = 1 + \log(\mu) = 0$.

(d) To find 2-cycles, we have to solve:

$$x = F_\mu^{(2)}(x) = F_\mu(F_\mu(x)) = \mu^2\, x\, \exp[2 - x - \mu\, x\, \exp[1-x]]$$

Since $x \neq 0$, we have

$$x + \mu\, x\, \exp[1-x] = 2(1+\log(\mu)).$$

This equation is solved by the fixed point $x = 1 + \log(\mu)$, but it also has two other solutions for $\mu > e$ giving a period 2-cycle, which is stable until $\mu \approx 4.6$, so we can choose, for example, $\mu = 3$ or $\mu = 4$.

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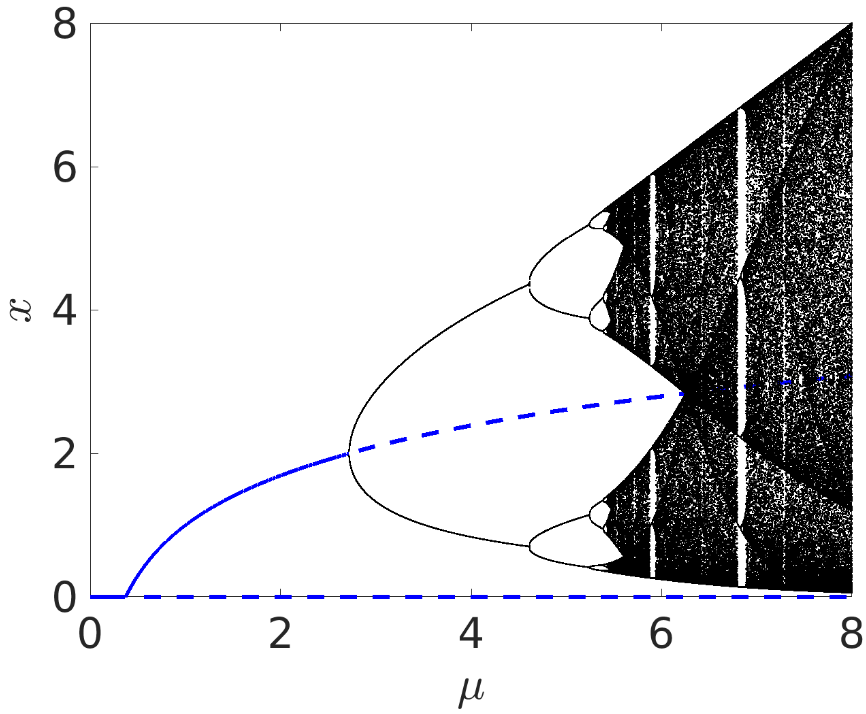
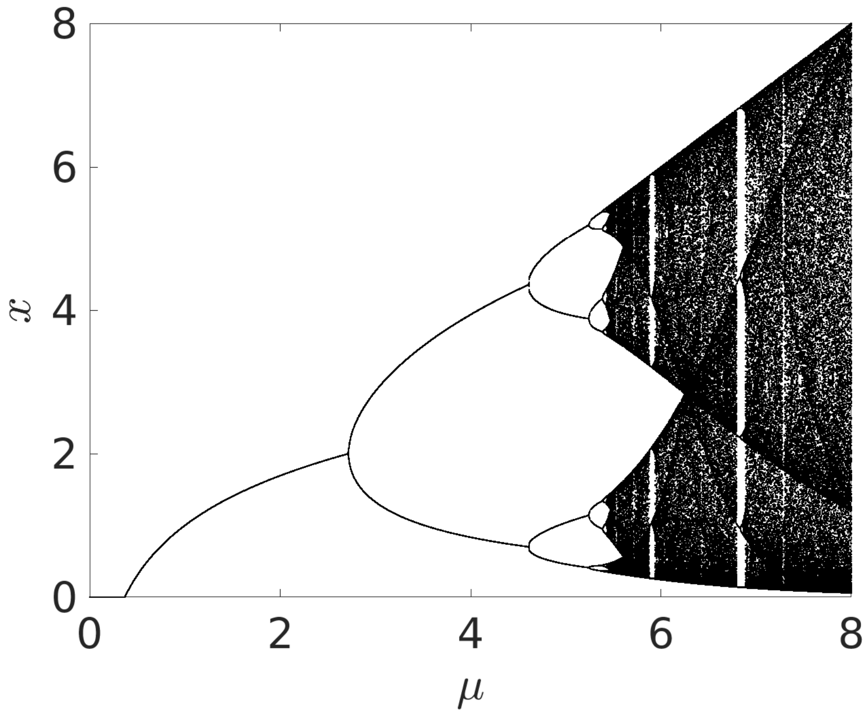
(e) To plot the bifurcation diagram, we can visualize the information derived in parts (c) and (d), and continue numerically:



or we can numerically compute the whole bifurcation diagram:



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Section C: Problem 7

7. Let $x_0 \in [-1,1]$ and $F : [-1,1] \to [-1,1]$.
Define sequence $x_k \in [-1,1]$, $k = 0,1,2,\dots$, iteratively by

$$x_{k+1} = F(x_k)\,.$$

(a) Let $F(x) = 2\,x^2 - 1$, *i.e.* we have

$$x_{k+1} = 2\,x_k^2 \; - \; 1\,.$$

(i) Find maxima and minima of F in interval $[-1,1]$ and verify that $F([-1,1]) \subset [-1,1]$.

(ii) Let $h(y) = \cos(\pi y)$ and define function $G : [0,1] \to [0,1]$ by $G = h^{-1} \circ F \circ h$. Find $G(y) = h^{-1}(F(h(y)))$ for $y \in [0,1]$ as a piecewise defined function.

(iii) Define the sequence $y_k \in [0,1]$, $k = 0,1,2,\dots$, iteratively by $y_{k+1} = G(y_k)$. Find a relation between x_k and y_k.

(iv) Find the invariant distribution $p(x)$, defined for $x \in [-1,1]$, and satisfying: If the random variable X is distributed according to $p(x)$, then the random variable $F(X)$ is also distributed according to $p(x)$.

(v) Write a computer code which plots a histogram of first 10^6 points in the orbit of $x_0 = 0.7$ obtained by $x_{k+1} = F(x_k)$. Plot the invariant distribution $p(x)$ (obtained in part (iv)) in the same figure for comparison.

(b) Let $F(x) = x(4x^2 - 3)$, *i.e.* we have

$$x_{k+1} = x_k\,(4\,x_k^2 - 3)\,.$$

Answer questions (i), (ii), (iii), (iv) and (v) for this map.

(c) Let $F(x) = 8x^2(x^2 - 1) + 1$, *i.e.* we have

$$x_{k+1} = 8\,x_k^2\,(x_k^2 - 1) + 1\,.$$

Answer questions (i), (ii), (iii), (iv) and (v) for this map.

Solution:

(a) Let $F(x) = 2x^2 - 1$. Then $F'(x) = 4x$.

(i) Since $F'(x) = 4x$, the minimum is at $x = 0$ and is equal to -1. The maxima are at the boundaries of the interval $[-1,1]$ and $F(\pm 1) = 1$. Therefore, $F([-1,1]) \subset [-1,1]$.

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(ii) Since $h(y) = \cos(\pi y)$ for $y \in [0,1]$, we have $h^{-1}(z) = (\arccos z)/\pi$ for $z \in [-1,1]$, which implies

$$G(y) = h^{-1}(F(h(y))) = \frac{1}{\pi}\arccos\left(2\cos^2(\pi y) - 1\right) = \frac{1}{\pi}\arccos\left(\cos(2\pi y)\right).$$

Since $\arccos : [-1,1] \to [0,\pi]$, we conclude

$$G(y) = \begin{cases} 2y & \text{for} \quad y \in [0,1/2] \\ 2(1-y) & \text{for} \quad y \in [1/2,1]. \end{cases}$$

(iii) Let $x_0 = h(y_0)$. Then, using $G = h^{-1} \circ F \circ h$, we have

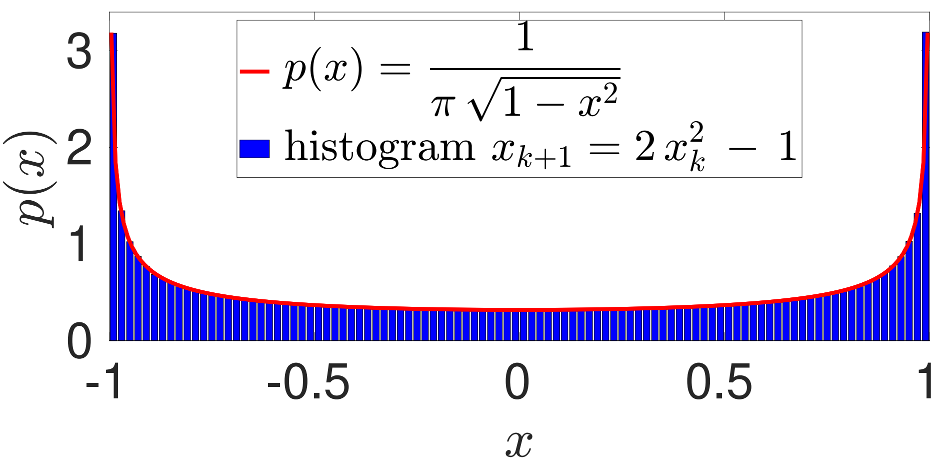
$$\begin{aligned} y_0 &= h^{-1}(x_0) \\ y_1 &= G(y_0) = h^{-1}(F(h(y_0))) = h^{-1}(F(x_0)) = h^{-1}(x_1) \\ y_2 &= G(y_1) = h^{-1}(F(h(y_1))) = h^{-1}(F(x_1)) = h^{-1}(x_2) \\ \vdots &= \vdots \end{aligned}$$

In particular, we have $y_k = h^{-1}(x_k)$ by induction.

(iv) The invariant distribution is

$$p(x) = \frac{1}{\pi\sqrt{1-x^2}}. \tag{7}$$

(v) The blue histogram of first 10^6 points in the orbit of $x_0 = 0.7$ compared with the invariant distribution (red line) given by formula (7):

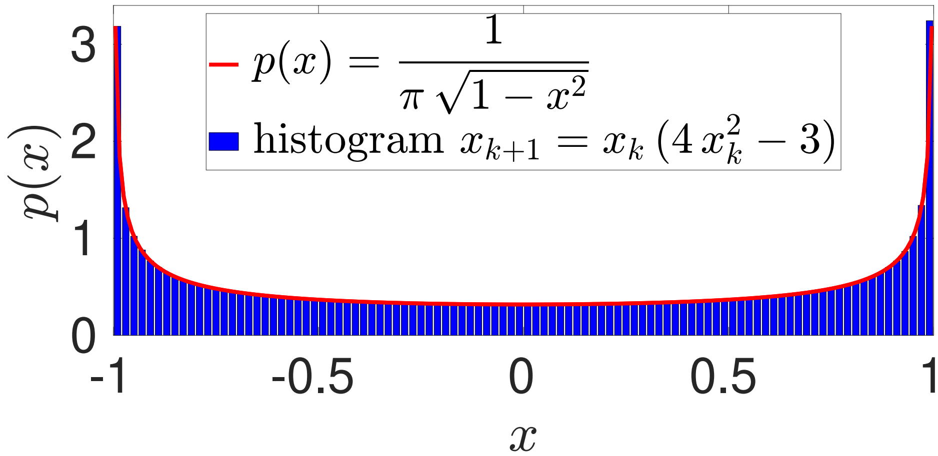


(b) Let $F(x) = x(4x^2 - 3)$. Then $F'(x) = 12x^2 - 3$ and F has maxima at $x = -1/2$ and $x = 1$ where $F(-1/2) = F(1) = 1$ and minima at $x = -1$ and $x = 1/2$ where

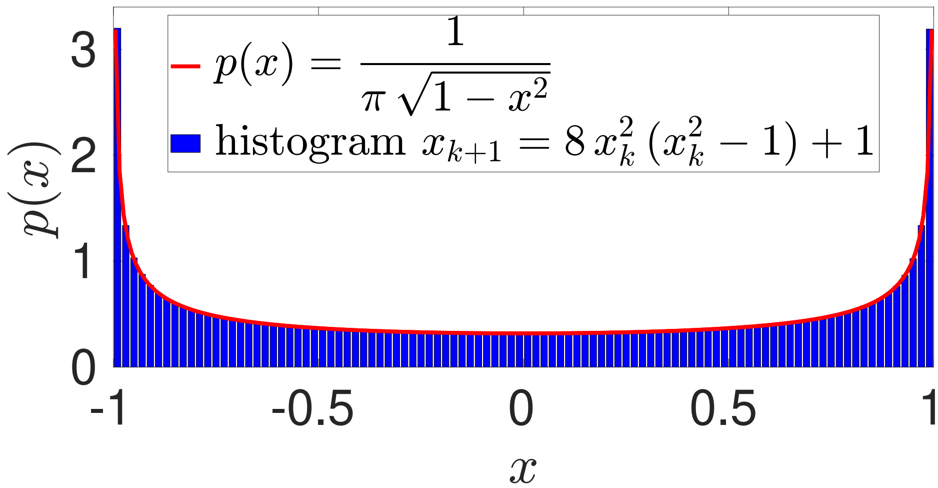
$F(-1) = F(1/2) = -1$. Using $\cos(3z) = 4\cos^3(z) - 3\cos(z)$, we get

$$G(y) = \begin{cases} 3y & \text{for} \quad y \in [0, 1/3] \\ 2 - 3y & \text{for} \quad y \in [1/3, 2/3] \\ 3y - 2 & \text{for} \quad y \in [2/3, 1]. \end{cases}$$

The invariant distribution is again given by (7) and the histogram is:



(c) Let $F(x) = 8x^2(x^2 - 1) + 1$. The invariant distribution is again given by (7) and the histogram is:



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