C2.6 Introduction to Schemes

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Feedback and corrections are welcome!

References

2018-2019 Course Lecture Notes by Prof. Damian Rössler ← on course page

Ravi Vakil, The Rising Sea, Foundations of Algebraic Geometry ← online

http://stacks.math.columbia.edu ← search defns, theorems, proofs in algebra & alg. geometry

Qing Liu, Algebraic Geometry and Arithmetic Curves, OUP 2002 ← modern book, seems rather nice

Eisenbud & Harris, The Geometry of Schemes, Springer GTM 197 ← classic

George R. Kempf, Algebraic Varieties, LMS Lecture notes 172

Classic books by: Mumford (Red Book of Varieties & Schemes)
Hartshorne (Algebraic Geometry)
Shafarevich (Basic Algebraic Geometry 2)

My C3.4 Algebraic geometry notes (see C2.6 course webpage ↙ or my website) try to fill the gap between classical algebraic geometry (C3.4) and C2.6

For the brave, you can look at the original works by the masters in French:
Grothendieck, "Élements de géométrie algébrique" series on www.numdam.org
Serre, "Faisceaux Algébriques Cohérents", Annals of Math. 1955.

Prerequisites

(for more advanced algebra see books by Matsumura, Weibel, Eisenbud)

Commutative algebra (e.g. Atiyah – MacDonald, Introduction to Comm. Alg.)

Category theory — or willingness to read things up as necessary

Homological algebra — or willingness to read things up as necessary

Expectations

That you read the notes regularly after each class.
(This is a 16-lecture course, 2 lectures/week across 8 weeks.)
Not everything can be covered in detail in class, so you need to be willing to look things up as necessary.

Conventions

Diagrams commute unless we say otherwise
Ring means commutative ring with unit 1
Ring homomorphisms are by definition unital i.e. 1 maps to 1

Arrows:
$\hookrightarrow$ means injective
$\twoheadrightarrow$ means surjective

CONTENTS

0.1 Classical Algebraic Geometry : Affine varieties

$R = k[x_1, \dots, x_n]$ polynomial ring over algebraically closed field k

$I \subseteq R$ ideal

$X = \mathbb{V}(I) = \{a \in k^n : f(a) = 0 \ \forall f \in I\}$ affine variety

The topological space

Affine space: $\mathbb{A}^n = k^n$ with Zariski topology:
- closed sets: $\mathbb{V}(I)$
- open sets: $U_I = \mathbb{A}^n \setminus \mathbb{V}(I) = \bigcup_{f \in I} D_f$
- basis of open sets: $D_f = \{a \in k^n : f(a) \neq 0\}$, $f \in R$

$X \subseteq \mathbb{A}^n$ subspace topology: $X \cap U_I$

The functions on it

$R \cong \mathrm{Hom}(\mathbb{A}^n, \mathbb{A}^1)$, $f \longmapsto (a \xmapsto{ev_f} f(a))$ ← The functions on $\mathbb{A}^n$ are polynomial functions.

$\mathbb{I}(X) = \{f \in R : f(X) = 0\}$ ← The functions on $\mathbb{A}^n$ vanishing on X

Remark $\mathbb{V}(\mathbb{I}(X)) = X$ for affine varieties X

Coordinate ring : $\boxed{k[X] = R/\mathbb{I}(X)}$ ← The functions on X are polynomials in the coordinates

Key facts: 1) Hilbert's basis theorem : R Noetherian, so $k[X]$ Noetherian

2) Hilbert's weak nullstellensatz : maximal ideals of R (and of $k[X]$) are $m_a = \mathbb{I}(\{a\}) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$, so correspond to points: $\{a\} = \mathbb{V}(m_a)$

3) Hilbert's Nullstellensatz : $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$ (radical of I $\{f : \exists N, f^N \in I\}$) | Hence: $\mathbb{I}\mathbb{V}I = I$ if I is radical

Lemma There are enough functions to separate points

Pf $a \neq b \in X \subseteq \mathbb{A}^n \Rightarrow$ some coordinate $a_i \neq b_i \Rightarrow x_i \in k[X]$ separates a, b. □

Morphisms between affine varieties

$\mathrm{Hom}(\mathbb{A}^n, \mathbb{A}^m) \cong R^m$ ← polynomial maps $a \mapsto (f_1(a), \dots, f_m(a))$

$\mathrm{Hom}(X,Y) = \{$restriction of a polynomial map $\mathbb{A}^n \to \mathbb{A}^m$ s.t. $X \to Y\}$

Facts: 1) $k[X] \cong \mathrm{Hom}(X, \mathbb{A}^1)$ ← "values of functions are enough to determine the abstract function"

2) $\mathrm{Hom}(X,Y) \cong \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X])$

$$(F : X \to Y) \mapsto \begin{pmatrix} F^* : \mathrm{Hom}(Y, \mathbb{A}^1) \to \mathrm{Hom}(X, \mathbb{A}^1) \\ f \mapsto F^*f = f \circ F \end{pmatrix}$$ ← | "pullback"

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, dashed, "F^*f"'] & Y \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

Equivalence of categories

$\{$affine varieties$\} \longleftrightarrow \{$finitely generated reduced k-algebras & homs of k-algs.$\}$

$X \longmapsto k[X]$

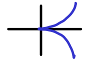
$(F : X \to Y) \longmapsto F^*$

reduced ↑ no nilpotents (f nilpotent if $f^N = 0$ some N)

| Recall: R/J reduced $\Leftrightarrow$ J radical

Note: $\mathbb{I}(X)$ is radical

Remark The "same" (up to isomorphism) X can be embedded in various $\mathbb{A}^n$.

E.g. cuspidal cubic $\mathbb{V}(y^2 - x^3) =$  $\subseteq \mathbb{A}^2_{x,y}$ is $\cong \mathbb{V}(y^2 - x^3, z - x) \subseteq \mathbb{A}^3_{x,y,z}$

0.2 Why schemes?

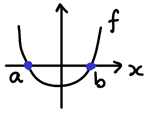
Some reasons:

1) Why always have spaces embedded in $\mathbb{A}^n$? (extrinsic)

Can you make sense of X without reference to $\mathbb{A}^n$? (intrinsic)

2) Why not let R be any ring?

3) When you deform varieties, nilpotents arise naturally and should not be ignored:

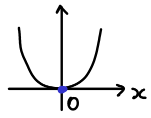


$f = (x-a)\cdot(x-b)$

$X = \mathbb{V}(f) = \{a,b\} \subseteq \mathbb{A}^1$ ← two points

$k[X] \cong k[x]/(x-a) \oplus k[x]/(x-b) \cong k^2$ ← a value at each point

Deform: a, b become 0:



$f = (x-0)\cdot(x-0) = x^2$

$X = \mathbb{V}(f) = \{0\} \subseteq \mathbb{A}^1$

$k[X] \cong k[x]/\sqrt{(x^2)} = k[x]/(x) \cong k$

$\mathbb{I}(\mathbb{V}(x^2)) = \sqrt{(x^2)}$ by Hilbert Nullstell.

notice $k[X]$ is the reduced ring, not $k[x]/(x^2)$

We lost information: classically you cannot tell $x=0$ apart from $x^2=0$

In the theory of schemes, the key role is not played by the topological space.

The Key role is played by the ring of functions, or rather, the sheaf of functions $\mathcal{O}$: on each open set $U \subseteq X$ get a ring of functions $\mathcal{O}(U)$.

Example above: $\mathcal{O}(X) = k[x]/(x^2)$ ← we do not reduce the ring of functions

At what cost? Values of functions need not determine the abstract function:

$$\mathcal{O}(X) \ni \alpha + \beta x \longmapsto (\alpha+\beta x : X = \{0\} \longrightarrow \mathbb{A}^1) \in \mathrm{Hom}(X, \mathbb{A}^1)$$

$$0 \longmapsto \alpha$$

do not recover β.

Idea: the abstract "β" remembers that X arose from the collision of two points, so β records tangential information: $\left.\frac{\partial}{\partial x}\right|_{x=0} (\alpha+\beta x) = \beta$.

0.3 What is a point?

X topological space is reducible if $X = X_1 \cup X_2$ for proper closed $X_i \subseteq X$. ($X_i \neq X$) (and irreducible if not)

Euclidean world (more generally if X Hausdorff): $Y \subseteq X$ irreducible $\Leftrightarrow$ Y = point or $Y = \emptyset$

Classical Alg. Geom.:
- point $a \in X \longleftrightarrow$ max ideal $m_a \subseteq k[X]$
- closed $\emptyset \neq Y \subseteq X$ irreducible $\Longleftrightarrow$ $\mathbb{I}(Y) \subseteq k[X]$ prime ideal

R ring $\Rightarrow$ "points" of R are $\mathrm{Spec}(R) = \{$prime ideals of $R\}$ not just max ideals

Categorically a good choice since functorial:

$\varphi: R \longrightarrow S$ hom of rings $\Rightarrow$ $\varphi^{-1}(\text{prime ideal}) = $ a prime ideal

$\Rightarrow$ $\mathrm{Spec}\, S \xrightarrow[\varphi^{-1}]{} \mathrm{Spec}\, R$

fails for max ideals e.g. $\mathbb{Z} \xrightarrow[\varphi]{\subseteq} \mathbb{Q}$, $\varphi^{-1}(0) = 0$

We were just lucky that homs $k[Y] \to k[X]$ send max ideal → max ideal.

1. DEFINITION OF SCHEMES

1.1 Examples of affine schemes

Motivation: M $n\times n$ matrix over $\mathbb{C}$. Then $\mathbb{C}[x] \twoheadrightarrow \mathbb{C}[M]$, $x \mapsto M$ has Ker $= \langle m_A \rangle$ (min poly)

so $\mathbb{C}[M] \cong \mathbb{C}[x]/\langle m_A \rangle \cong \bigoplus \mathbb{C}[x]/(x-\lambda_i)^{n_i}$

Spec $\mathbb{C}[M] = \{(x-\lambda_i) : \lambda_i$ eigenvalues of $A\}$

Spec(R) some ring R (always: comm. ring with 1)

- As a set: $\mathrm{Spec}(R) = \{$prime ideals $p \subseteq R\}$ ← (prime) Spectrum
- Zariski topology: e.g. $V(R) = \emptyset$, $V(0) = \mathrm{Spec}\, R$

  closed sets: $\boxed{\mathbb{V}(I) = \{\text{prime ideals containing } I\} \subseteq \mathrm{Spec}\, R}$
- sheaf $\mathcal{O}_{\mathrm{Spec}\, R}$ which we construct later. ← spaces of functions

Rmk The global functions are: $\mathcal{O}_{\mathrm{Spec}\,R}(\mathrm{Spec}\, R) = R$. ← so spaces of fns can recover the top. space!

Exercise $V(I) = V(\sqrt{I})$

Key exercise (⇒ axioms for a topology)

$$\boxed{\begin{array}{l} V(I) \cup V(J) = V(I\cdot J) = V(I \cap J) \\ \cap\, V(I_i) = V(\sum I_i) \end{array}}$$

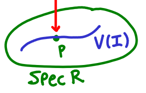
Rmk $(I\cap J)\cdot(I\cap J) \subseteq I\cdot J \subseteq I \cap J$ so $\sqrt{I\cdot J} = \sqrt{I\cap J}$ but $I\cdot J$ and $I \cap J$ may be $\neq$

Key $V(I) = \emptyset \Leftrightarrow I = R \Leftrightarrow 1 \in I$, since any proper ideal $\subseteq$ some max ideal

Topological consequences:
- open sets: $U_I = \mathrm{Spec}\, R \setminus \mathbb{V}(I) = \bigcup_{f\in I} D_f$
- basis of open sets: $D_f = \{p \in \mathrm{Spec}\, R : f \notin p\}$ ($f \in R$)

  $= \{p \in \mathrm{Spec}\, R : f(p) \neq 0\}$

Rmk $D_{f^N} = D_f$ for $N \geq 1$, since $f^N \in p \Leftrightarrow f \in p$



local ring R_p, residue field $K(p) = R_p/p\cdot R_p$

"value of $f \in R$ at p":

$$\begin{array}{ccccccc} R & \longrightarrow & R/p & \hookrightarrow & K(p) = \mathrm{Frac}(R/p) & \cong & R_p/p\cdot R_p \\ f & \longmapsto & & & f(p) & & \end{array}$$

localisation of R at p

target field depends on p!

Rmk: p prime $\Updownarrow$ R/p is integral domain

Remark $\boxed{f(p) = 0 \Leftrightarrow f \in p}$

Examples 1) $R = k[X]$ ← affine variety $X \subseteq \mathbb{A}^n$

$$\begin{array}{ccc} \mathrm{Spec}\, R & \underset{\mathbb{I}}{\overset{\mathbb{V}}{\rightleftarrows}} \text{(bijection)} & \{\text{irreducible subvarieties } Y \subseteq X\} \\ \cup\!\! \mid & & \cup\!\!\mid \\ \mathrm{Specm}\, R = \{\text{max ideals}\} & \longleftrightarrow & X \end{array}$$

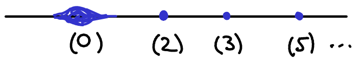
← and Zariski topologies agree

Value of $f \in R$ at m_a: ($m_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$)

$$\begin{array}{ccccc} m_a & \longrightarrow & R/m_a & \cong & k \\ f & \longmapsto & & & f(a) \end{array}$$

← in this case the target field does not depend on the point

2) $\mathrm{Spec}\, \mathbb{Z} = \{0\} \cup \{(p) : p \in \mathbb{N} \text{ prime}\}$



value of $f \in \mathbb{Z}$ at (0):

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathrm{Frac}(\mathbb{Z}/0) = \mathbb{Q} \\ f & \longmapsto & f \end{array}$$

so lost no information.

$\mathbb{V}((0)) = \{$prime ideals containing $(0)\} = \mathrm{Spec}\, \mathbb{Z}$ so the point (0) is dense!

$\mathbb{V}((p)) = \{(p)\}$ are "closed points". Value of $f \in \mathbb{Z}$: $f((p)) = (f \in \mathbb{Z}/p) = (f \bmod p)$

In general Prime ideals p with $\mathbb{V}(p) = \mathrm{Spec}\, R$ are called generic points

prime ideals p with $\mathbb{V}(p) = \{p\}$ are called closed points

Exercise $\{$closed points$\} = \{$max ideals of $R\}$

Exercises · $\mathfrak{a}$ prime ideal $\Rightarrow$ $\mathfrak{a}$ radical ($\mathfrak{a}=\sqrt{\mathfrak{a}}$) ← [recall radical of $\mathfrak{a}$ $\sqrt{\mathfrak{a}}=\{f\in R : f^N\in\mathfrak{a} \text{ some } N\} = \bigcap_{p\in V(\mathfrak{a})} p$; $\sqrt{\mathfrak{a}}\supseteq \text{Nilradical}(R) = \{\text{nilpotent elements of } R\} = \bigcap_{p\in \operatorname{Spec} R} p$]

· For $\mathfrak{a},\mathfrak{b}$ radical, $\mathfrak{a}\subseteq\mathfrak{b}\iff V(\mathfrak{a})\supseteq V(\mathfrak{b})$ ← order reversing!

Cor $\boxed{V(I)\subseteq V(J)\Leftrightarrow\sqrt{I}\supseteq\sqrt{J}}$

Pf $V(I)=V(\sqrt{I})$, so: $\Leftrightarrow V(\sqrt{I})\subseteq V(\sqrt{J})\Leftrightarrow \sqrt{I}\supseteq\sqrt{J}$ by exercise. □

Cor $\boxed{V(\mathfrak{a})=V(\mathfrak{b}) \iff \sqrt{\mathfrak{a}}=\sqrt{\mathfrak{b}}}$

$\Rightarrow$ $\boxed{\{\text{closed sets of } \operatorname{Spec} R\} \overset{1:1}{\longleftrightarrow} \{\text{radical ideals of } R\}}$ order-reversing correspondence

Proposition $f\in R$ vanishes at all $p\in\operatorname{Spec}R \iff f$ nilpotent ← [immediate from

Covering Trick $\boxed{\operatorname{Spec}R=\bigcup D_{f_i} \iff 1\in\langle \text{all } f_i\rangle \iff \langle\text{all } f_i\rangle = R}$

Pf $\operatorname{Spec}R\setminus\bigcup D_{f_i} = \bigcap V(f_i) = V(\langle \text{all } f_i\rangle)$, now use previous Key. □

Theorem Spec R is quasi-compact ← (quasi-compact = compact = open covers have finite subcovers

Pf $\operatorname{Spec}R=\bigcup_i U_i$. As $U_i=\bigcup_j D_{f_{ij}}$, WLOG $U_i = D_{f_i}$.

$\overset{\text{Trick}}{\Rightarrow}$ $1=\sum_{\text{finite}} r_i f_i$ ← so finitely many f_i generate R, so those D_{f_i} cover. □

Basic Exercises

1) $\varphi: R\to S$ ring hom $\Rightarrow \alpha:\operatorname{Spec}S\longrightarrow\operatorname{Spec}R$, $p\longmapsto\varphi^{-1}(p)$ is continuous

indeed $\boxed{\alpha^{-1}(D_f)=D_{\varphi f}}$ ← (Hint: $f\notin p\subseteq R\Rightarrow q$ s.t. $\underbrace{\varphi^{-1}q=p}_{\alpha q}$ has $\varphi f\notin q$)

2) Show that $\underline{\operatorname{Spec}(R/I)}$ "is" the subspace $\underline{\mathbb{V}(I)\subseteq\operatorname{Spec}R}$ and the quotient map $\pi: R\to R/I$ induces via (1) the inclusion map on Specs. ← Here "is" means: can be canonically identified with

see my C 3.4 Notes about ideals in R/I and $S^{-1}R$

Example $\boxed{\begin{aligned}\operatorname{Spec}\left(R/(f)\right) &= \{\text{prime ideals of } R \text{ containing } f\}\\ &= \text{the points of } \operatorname{Spec}R \text{ where } f \text{ vanishes}\\ &= V(f)\end{aligned}}$

3) Show that $\underline{\operatorname{Spec}(S^{-1}R)}$ "is" a subspace of Spec R, where $S^{-1}R$ is localisation of R at a multiplicative set $S\subseteq R$, and $R\to S^{-1}R$, $r\mapsto\frac{r}{1}$ induces via (1) the inclusion

means: $1\in S$, $S\cdot S\subseteq S$ (we do not require $0\notin S$)

Example $S=\{1,f,f^2,f^3,\dots\}$, so $S^{-1}R=R_f$, then:

$\boxed{\begin{aligned}\operatorname{Spec}R_f &= \{\text{Prime ideals of } R \text{ not containing } f\}\\ &= \text{the points of } \operatorname{Spec}R \text{ where } f \text{ does not vanish}\\ &= D_f\end{aligned}}$

4) $D_f\cap D_g = D_{fg}$, so $\boxed{\operatorname{Spec}R_f\cap\operatorname{Spec}R_g=\operatorname{Spec}R_{fg}}$ (idea: $f^N=rg\Rightarrow\frac{1}{g}=\frac{r}{f^N}$)

5) $D_f\subseteq D_g\iff V(f)\supseteq V(g)\iff\sqrt{f}\subseteq\sqrt{g}\iff f\in\sqrt{g}\Leftrightarrow f^N\in(g)$ some $N\iff g\in R_f$ invertible

6) $p\subseteq R$ prime ideal $\Rightarrow R_p:=S^{-1}R$ for $S=R\setminus p$, then $\exists!$ closed point $m_p=p.R_p\in\operatorname{Spec}R_p$

so local ring: $\exists!$ max ideal m ($\Leftrightarrow$ elts outside m are invertible)

Also: $m_p\in U\subseteq\operatorname{Spec}R_p$ open $\Rightarrow U=\operatorname{Spec}R_p$.

1.2 Definition of a scheme

RED: WORDS TO BE DEFINED LATER

Def A ringed space is

- a topological space X ← the points
- with a sheaf of rings $\mathcal{O}_X$ on X ← the functions

Locally ringed space if also:

- all stalks $\mathcal{O}_{X,x}$ are local rings ← the germs of functions near point x

(so $\exists$ unique maximal ideal $m_{X,x}\subseteq\mathcal{O}_{X,x}$ and $\exists$ residue field at x: $k(x)=\mathcal{O}_{X,x}/m_{X,x}$ ← the "value" of a function at x lives here

IDEA

Def An affine scheme is a locally ringed space isomorphic to $(\operatorname{Spec}R,\mathcal{O}_{\operatorname{Spec}R})$ for some ring R.

Def A scheme is a locally ringed space which is locally isomorphic to an affine scheme.

means:

$\forall x\in X$ $\left.\begin{array}{l}\exists \text{ some open neighbourhood } x\in U\subseteq X\\ \exists \text{ some ring } R \text{ depending on } x\end{array}\right\}$ s.t. $(U,\mathcal{O}_X|_U)\cong(\operatorname{Spec}R,\mathcal{O}_{\operatorname{Spec}R})$

1.3 Pre-sheaves

Ab = category of abelian groups and group homs ← if use category $\mathcal{C}$ get (pre)sheaves with values in $\mathcal{C}$ e.g. $\mathcal{C}$ = Rings get presheaf of rings

X = any topological space

Top X = category with objects: open sets $U\subseteq X$
morphs: inclusion maps ← ($\operatorname{Mor}(U,V)=\begin{cases}\emptyset \text{ if } U\not\subseteq V\\ \{\text{incl}\} \text{ if } U\subseteq V\end{cases}$

Def A presheaf (of abelian groups) on X is a contravariant functor

$F:\operatorname{Top}X\longrightarrow\mathrm{Ab}$

So:
- $\forall$ open $U\subseteq X$ have an abelian group $F(U)$ ← elements called sections (over U)
- $\forall$ inclusion $U\to V$ have a "restriction" group hom $\boxed{\begin{array}{c}F(V)\to F(U)\\ s\longmapsto s|_U\end{array}}$
- $F(id:U\to U): F(U)\xrightarrow{id}F(U)$ so $s|_U=s$ for $s\in F(U)$.
- $U\subseteq V\subseteq W\Rightarrow$ $\begin{tikzcd} F(W) \arrow[r] \arrow[rr, bend right] & F(V) \arrow[r] & F(U)\end{tikzcd}$ so: $(s|_V)|_U=s|_U$ for $s\in F(W)$.

Example X topological space, $F(U)=\{\text{continuous functions } U\to\mathbb{R}\}$ with obvious restrictions

Morphism of pre-sheaves = natural transformation of such functors: $\varphi:F\to G$

So: $\forall$ open $U\subseteq X$ have $\varphi_U:F(U)\longrightarrow G(U)$ group hom

$\forall$ inclusion $U\to V$ have

$$\begin{tikzcd} F(U) \arrow[r, "\varphi_U"] & G(U) \\ F(V) \arrow[u] \arrow[r, "\varphi_V"] & G(V) \arrow[u] \end{tikzcd}$$

← restriction homs

so the homs "are compatible with restrictions"

Sub pre-sheaf $F\subseteq G$ means $F(U)\subseteq G(U)$ subgp, compatibly with restrictions ← i.e. this diagram with φ_U = inclusion

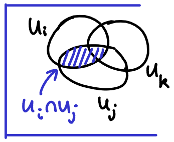
1.4 Sheaves

Def Pre-sheaf F is a sheaf on X if it satisfies the local-to-global condition:

If U_i open, $s_i \in F(U_i)$ agreeing on overlaps:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \in F(U_i \cap U_j)$$

Then $\exists$ unique $s \in F(\bigcup U_i)$ with $s|_{U_i} = s_i$.



idea: can uniquely extend!

Consequences

- two sections $s, t \in F(U)$ equal $\Leftrightarrow$ they equal locally: $s|_{U_i} = t|_{U_i}$, $U = \cup U_i$
- you can build sections by defining local sections, compatibly on overlaps.
- exact sequence: (for $U = \cup U_i$)

$$0 \longrightarrow F(U) \longrightarrow \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \cap U_j)$$

$$s \longmapsto (s_i) \qquad (s_i) \longmapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})$$

- $F(\emptyset) = 0$ (Hint. consider empty covering of $\emptyset$)

Examples

1) Sheaf of continuous real functions: $F(U) = \{\text{continuous maps } U \to \mathbb{R}\}$
2) Skyscraper sheaf at $p \in X$ for group A: $F(U) = \begin{cases} 0 & \text{if } p \notin U \\ A & \text{if } p \in U \end{cases}$
3) Presheaf of constant functions for group A:

$$F(U) = \begin{cases} A & \text{if } U \neq \emptyset \\ 0 & \text{if } U = \emptyset \end{cases}$$

$\leftarrow$ (so $f \in F(U)$ is a constant function $f : U \to A$, $f \equiv a \in A$)
$\leftarrow$ (only want one function on $\emptyset$)

4) Sheaf of locally constant functions for group A. So $f \in F(U)$ means $f : U \to A$ such that $\forall x \in U$, $\exists$ open $x \in V \subseteq U$ with $f|_V : V \to A$ constant.
Warning: it implies f constant on connected components but converse can fail. $\leftarrow$ (e.g. consider $\mathbb{Q}$ with usual Euclidean topology)

Exercise (3) is not a sheaf if $X = 2$ points with discrete topology, $A \neq 0$.

Write $Ab(X)$ = category of sheaves on X and morphs of sheaves
$\nwarrow$ $Sh(X)$ if work with category of Sets instead of Ab
$\|$ (morphs of presheaves)

1.5 Stalks

Def stalk at x of presheaf F is the abelian group

$$\boxed{F_x = \varinjlim_{x \in U} F(U)}$$

$\leftarrow$ direct limit over restriction maps induced by inclusions.

Explicitly:
An element of F_x is determined by $s \in F(U)$ some $U \ni x$ open,
identify $s \sim t$ for $t \in F(V)$ $\Leftrightarrow$ $s|_W = t|_W$ some $U \cap V \supseteq W \ni x$ open

Rmk
- natural map $F(U) \longrightarrow F_x$, $s \longmapsto s_x$ = equivalence class of s. (for $x \in U$)
or write: $s|_x$
- morph $\varphi : F \to G$ then get $\varphi_x : F_x \to G_x$
or write: $\varphi|_x$
($\varphi_x(s_x) = \varphi_U(s)|_x$ if $s \in F(U)$)

Exercise $\varphi, \psi : F \to G$ morphs of sheaves, if all $\varphi_x = \psi_x : F_x \longrightarrow G_x$ then $\varphi = \psi$.

$\swarrow$ Hint.
$$\begin{matrix} \varphi_U(s)|_W & = & \psi_U(s)|_W \\ \| & & \| \\ \varphi_W(s|_W) & & \psi_W(s|_W) \end{matrix}$$
Then use local-to-global

Facts For sheaves F, G $\swarrow$ in category $Ab(X)$

$F \to G$ monomorphism	$\Leftrightarrow$	$F_x \to G_x$	injective	$\forall x$
$F \to G$ epimorphism	$\Leftrightarrow$	$F_x \to G_x$	surjective	$\forall x$
$F \to G$ isomorphism	$\Leftrightarrow$	$F_x \to G_x$	iso	$\forall x$

recall from category theory
mono: $\left.\begin{matrix} H \rightrightarrows F \to G \\ \text{composites equal} \end{matrix}\right\} \Rightarrow H \rightrightarrows F$ equal
epi: $F \to G \rightrightarrows H \Rightarrow G \rightrightarrows H$

Warning mono $\Leftrightarrow$ $F(U) \to G(U)$ inj. $\forall U$, but fails for epi: $F(U) \to G(U)$ need not be surj.

Exercise $F_x \xrightarrow{\varphi_x} G_x$ surj $\Leftrightarrow$ $\forall t \in G(\underset{x}{U})$, $\underset{\exists x \in V \subseteq U}{\exists s \in F(V)}$: $\varphi_V(s) = t|_V \in G(V)$ (but V can depend on t!) $\leftarrow$ see HWK4

Rmk $F \to G$ iso $\Leftrightarrow$ $F(U) \to G(U)$ iso $\forall U$. $\leftarrow$ (Try proving surjectivity by combining the Exercise with injectivity and a local-to-global argument.)
For "$\Rightarrow$": $Ab(U) \to$ Abelian Groups, $F \mapsto F(U)$ is a functor, and functors send isos to isos. For "$\Leftarrow$": $\varinjlim$ functor gives iso on stalks $F_x \cong G_x$. □

1.6 Sheafification

F pre-sheaf $\Rightarrow$ F^+ sheaf (ification):

so $\forall x \in U$, $\exists x \in V \subseteq U$, $t \in F(V)$ $s(y) = t_y \in F_y$ $\forall y \in V$

$$\boxed{F^+(U) = \{ s : U \to \sqcup F_x : \text{locally } s \text{ is a section of } F \}}$$

in fact by definition $s(x) \in F_x$ so $s : U \longrightarrow \bigsqcup_{x \in U} F_x \subseteq \bigsqcup_{x \in X} F_x$

comes with natural morph $F \longrightarrow F^+$ $\leftarrow$ ($s \in F(U) \longmapsto (x \mapsto s_x) \in F^+(U)$)

Exercise: F^+ is a sheaf, $F^+_x = F_x$ and it satisfies:

$$\begin{tikzcd} F^+ \arrow[r, dashed, "\exists!"] & G \\ F \arrow[u] \arrow[ur, "\forall"'] & \end{tikzcd}$$

$\nwarrow$ any sheaf

(Universal property $\forall$ sheaf G on X, $\forall$ presh. morph $F \to G$, $\exists!$ sheaf morph $F^+ \to G$ s.t. diagram commutes)
(determines F^+ uniquely up to unique isomorph)

Hint. In our construction: $F^+_x = F_x \longrightarrow G_x$ so we know locally how sections map but we need to globalise...

Trick:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F^+ \arrow[d] \\ G \arrow[r] & G^+ \end{tikzcd}$$

finally G is sheaf so $G \cong G^+$
(natural iso, using $G_x = G^+_x$ and Facts)

Example (pre-sheaf of constant functions)$^+$ = (sheaf of locally constant functions)

Exercise 1) $F \subseteq G$ sub pre-sheaf, G sheaf $\Rightarrow$ $\exists$ smallest subsheaf $H \subseteq G$ s.t. $F \subseteq H$ $\leftarrow$ Hint mimic definition of F^+
Moreover, $H_x = F_x$.

2) $(DF)(U) = \prod_{x \in U} F_x$ with obvious restriction maps is a sheaf ("sheaf of discontinuous sections")

3) $i : F \longrightarrow DF$ obvious morph, let F^b = presheaf image so $F^b(U) = iF(U) = \prod_{x \in U} F_x$
then $F^b \subseteq DF$ is a sub pre-sheaf and construction (1) gives $H = F^+$.

1.7 Kernels, Cokernels, Images

For $\varphi : F \to G$ morph of sheaves:

- $(\operatorname{Ker} \varphi)(U) = \operatorname{Ker} \varphi_U$ is sheaf $\leftarrow$ ($\varphi_U : F(U) \to G(U)$)
- $\operatorname{Coker} \varphi = (\text{pre-Coker } \varphi)^+$ where $(\text{pre-Coker})(U) = \operatorname{Coker} \varphi_U$
- $\operatorname{Im} \varphi = (\text{pre-Im } \varphi)^+$ where $(\text{pre-Im})(U) = \operatorname{Im} \varphi_U$

Fact Ab(X) is an abelian category

<u>idea</u> it "behaves like" category of abelian gps

<u>Rmk</u> In additive cat;
mono ⟺ $H \to F \to G$ then $H \xrightarrow{0} F$ (where $H \to F \to G$ is $\xrightarrow{0}$)
epi ⟺ $F \to G \to H$ then $G \xrightarrow{0} H$ (where $F \to G \to H$ is $\xrightarrow{0}$)

categorical Ker & Coker, see below

<u>Def</u> <u>abelian category</u> = additive category such that morphisms have Ker, Coker
and i) $\varphi : F \to G$ monomorph is the <u>Ker</u> of its <u>Coker</u>
ii) " epimorph " <u>Coker</u> " <u>Ker</u>

<u>Def</u> <u>additive category</u> means Mor(A, B) abelian gp (so often write Hom(A,B)) s.t.
- Composition of morphisms distributes over addition
- ∃ products A × B
- ∃ zero object 0 (an object that is both <u>initial</u> & <u>terminal</u>)

($\forall$ obj. X, ($\exists!$ morph $0 \to X$) ($\exists!$ morph $X \to 0$)

Functor F of additive/abelian cats is <u>additive</u> if Hom(A,B)→Hom(FA,FB) is gp. hom.

For $\varphi : A \to B$:
<u>Ker φ</u> is a morph Ker φ → A (Ker φ ∈ objects)
s.t.

$$\begin{tikzcd} \forall C \arrow[d, dashed, "\exists!"'] \arrow[rd] \arrow[rrd, "0"] & & \\ \mathrm{Ker}\,\varphi \arrow[r] \arrow[rr, bend right, "0"'] & A \arrow[r, "\varphi"] & B \end{tikzcd}$$

<u>Fact</u> <u>Ker φ</u> is a monomorph.

<u>Coker φ</u> is B → Coker φ (Coker φ ∈ Obj)
s.t.

$$\begin{tikzcd} & \forall C & \\ \mathrm{Coker}\,\varphi \arrow[u, dashed, "\exists!"] & B \arrow[l] \arrow[ul] & A \arrow[l, "\varphi"'] \arrow[ull, "0"'] \arrow[ll, bend left, "0"] \end{tikzcd}$$

<u>Fact</u> <u>Coker φ</u> is an epimorph.

If φ mono, define
the <u>quotient</u> B/A := <u>Coker φ</u>

<u>Im φ</u> = <u>ker</u>(<u>Coker</u> φ)
which is a morph
Im φ → B

<u>Facts</u> ∃! factorization of φ
$A \longrightarrow \mathrm{Im}\,\varphi \longrightarrow B$
Abelian cat ⇒ $A \to \mathrm{Im}\,\varphi$ epi
and = <u>Coker</u>(<u>Ker φ</u>)

<u>Example</u> For abelian gps, (i) says:

$$\mathrm{Ker}\,\pi = \underbrace{A \overset{\varphi}{\underset{\text{inj}}{\hookrightarrow}} B}_{\text{is } \underline{\ker}\ \pi} \xrightarrow[\uparrow \text{ is } \underline{\mathrm{coker}}\,\varphi]{\pi} \underset{\substack{\| \\ \mathrm{coker}\,\varphi}}{B/A}$$

as expected!

I will now stop underlining ker, Coker, Im.

Freyd-Mitchell Thm

<u>Rmk</u> These categorical definitions can be cumbersome to work with. It turns out:
∀ small abelian category $\mathcal{A}$, ∃ a possibly non-commutative ring R with 1
and full faithful exact functor $\mathcal{A} \longrightarrow$ {left R-modules} (in particular preserves ker, coker, and is additive)
⇒ can "pretend" you work with modules.

(Obj($\mathcal{A}$) and Homs are sets not just "class")

<u>Example</u> you just apply the theorem to the <u>small</u> abelian subcategory involved in your diagram/sequence of maps – don't need to use the whole category. Explanation of why the abelian subcat. generated by a small diagram is a small cat: note that Mor(A,B) are ab. groups hence sets. Let C_0 be the (small) full subcat of $\mathcal{A}$ with objects those involved in the small diagram together with the object 0. Let C_1 = (small) full subcat of $\mathcal{A}$ with objects those in C_0, and finite products of objects in C_0, as well as Ker, coker, Im for every morph in C_0 (notice objects are labelled by sets so Obj(C_0) is set).
Continue inductively: C_2 = full subcat of $\mathcal{A}$ get from C_1 by taking finite products, ker, coker, Im. Finally $C = \bigcup_{n \geq 0} C_n$ is the <u>small</u> abelian subcat we wanted.

1.8 Exactness

A <u>(cochain) complex</u> $F^\bullet = (\dots \to F^{i-1} \xrightarrow{d^{i-1}} F^i \xrightarrow{d^i} F^{i+1} \to \dots)$ in an abelian cat
means composite of two consecutive morphs is zero: $d^{i+1} \circ d^i = 0 \quad \forall i$

<u>(Co)homology</u> $\boxed{H^\bullet(F^\bullet) = \mathrm{Ker}\, d^{i+1} / \mathrm{Im}\, d^i}$ (∃ mono $\mathrm{Im}\, d^i \hookrightarrow \ker d^{i+1}$ and $H^\bullet$ is its coker)

$F^\bullet$ <u>exact</u> means $\mathrm{Im}\, d^i = \mathrm{Ker}\, d^{i+1}$ (⟺ complex with zero homology $H^\bullet = 0$)

Proposition complex $F^\bullet$ in Ab(X) exact ⟺ $F^\bullet_x$ is exact sequence of abelian gps $\forall x \in X$

(Immediate by **Facts** on previous page)

<u>Rmk</u> For SES (short exact sequences) $0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ of sheaves you usually check exactness at level of stalks, but can equivalently check:
i) $0 \to F(U) \to G(U) \to H(U)$ exact ∀ open U
ii) H is smallest subsheaf containing pre-Im β, meaning every section of H can be obtained by gluing local sections of type β(local section of G)

<u>Def</u> A functor of abelian cats is <u>left exact</u> if: $0 \to A \to B \to C \to 0$ exact
$\Rightarrow 0 \to FA \to FB \to FC$ exact
<u>right exact</u> if ⇒ $FA \to FB \to FC \to 0$ exact

(F exact ⟺ F both left & right exact)

<u>Example</u> $\mathrm{Hom}_R(M, \cdot)$ is left exact, $\cdot \otimes_R M$ is right exact, as functors on R-mods (any R-mod M)

1.9 Push-forward (direct image) and inverse image

$f : X \to Y$ continuous ⇒ additive functor $f_* : \mathrm{Ab}\,X \to \mathrm{Ab}\,Y$

<u>Def</u> $F \in \mathrm{Ab}(X)$ gives $f_* F \in \mathrm{Ab}(Y)$:

$$(f_* F)(V) = F(f^{-1}(V))$$

PICTURE IN MIND

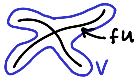
$$\begin{tikzcd} F \arrow[d, dash] & f_*F \arrow[d, dash] \\ X \arrow[r, "f"] & Y \\ f^{-1}V \arrow[u, phantom, sloped, "\subseteq"] & V \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

<u>Exercise</u> $(g \circ f)_* F = g_*(f_* F)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$.

⇒ additive functor $f^{-1} : \mathrm{Ab}\,Y \to \mathrm{Ab}\,X$

<u>Def</u> $F \in \mathrm{Ab}(Y)$ gives $f^{-1}F \in \mathrm{Ab}(X)$ is $(\text{pre-}f^{-1}F)^+$ where

$$(\text{pre-}f^{-1}F)(U) = \varinjlim_{V \supseteq f(U)} F(V)$$



PICTURE IN MIND

$$\begin{tikzcd} f^{-1}F \arrow[d, dash] & F \arrow[d, dash] \\ X \arrow[r, "f"] & Y \supseteq V \\ U \arrow[u, phantom, sloped, "\subseteq"] & fU \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

<u>Exercise</u> $(f^{-1}F)_x = F_{f(x)}$ and $(g \circ f)^{-1} \underset{\text{canonical}}{\cong} f^{-1} \circ g^{-1}$

also follows by by uniqueness up to unique iso of adjoint functors, see next page.

<u>Examples</u> 1) $i : S \to X$ inclusion of an open subset:
$F \in \mathrm{Ab}(S)$ $\quad i_* F : V \longmapsto F(V \cap S)$
$F \in \mathrm{Ab}(X)$ $\quad i^{-1} F : \underset{\text{open}}{U \subseteq S \subseteq X} \longmapsto F(U)$ ← denoted $F|_S$ called <u>restriction</u> of F

2) $i_x :$ point $\to X$, $i_x(\text{point}) = x$
$F \in \mathrm{Ab}(X)$ $\quad i_x^{-1} F = F_x$ ← (more precisely $(i_x^{-1}F)(U) = \begin{cases} F_x & \text{if } U = \{\text{point}\} \\ 0 & \text{if } U = \emptyset \end{cases}$ will not make such remarks again.)

3) $\pi : X \to$ point
$F \in \mathrm{Ab}(X)$ $\quad \pi_* F = \Gamma(X, F) = F(X)$ ← <u>global sections functor</u>

Proposition 1) f_* is left exact ⟵ in particular $\Gamma(X, .)$ is left exact
2) f^{-1} is exact

<u>For f_* : exercise</u>
<u>proof for f^{-1}</u> :

$$\begin{array}{ccccccccc} 0 \to & (f^{-1}A)_x & \to & (f^{-1}B)_x & \to & (f^{-1}C)_x & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & A_{fx} & \to & B_{fx} & \to & C_{fx} & \to 0 \end{array}$$

which by assumption is exact □

<u>Rmk</u> f_* left exact, f^{-1} right exact } would follow by category theory from next proposition

Proposition f^{-1} is the left adjoint functor of f_*, meaning $\exists$ natural iso

$$\mathrm{Mor}(f^{-1}F, G) \simeq \mathrm{Mor}(F, f_*G)$$ which is natural in F and G

Sketch pf

In → direction:

since W=V is allowed

$$\begin{tikzcd} F(V) \arrow[r] & \varinjlim_{W\supseteq fU} F(W) \arrow[r, "\text{given}"] & G(U) \arrow[d, equal] \\ & & G(f^{-1}V) = f_*G(V) \end{tikzcd}$$

← pick $U = f^{-1}V$

Rmk to get a map into a direct limit, you just need a representative element in one of the groups

In ← direction:

$$\begin{tikzcd} F(V) \arrow[r, "\text{given}"] \arrow[d] & G(f^{-1}V) \arrow[d] & \\ \varinjlim_{V\supseteq fU} F(V) \arrow[r] & \varinjlim_{V\supseteq fU} G(f^{-1}V) \arrow[r, "\text{restriction}"] & G(U) \end{tikzcd}$$

← assume $V \supseteq fU$, take $\varinjlim$ over such V

restriction ← notice $f^{-1}V \supseteq U$

Rmk to get map out of a direct limit, need maps out of all groups, compatibly with maps of lim

Now check these two are natural transformations, inverse to each other, and natural in F,G. □

Rmk Another example of adjoint functors, for R-modules, are $\mathrm{Hom}(M,\cdot)$ and $\cdot\otimes M$:

$$\mathrm{Hom}(F\otimes M, G) \cong \mathrm{Hom}(F, \mathrm{Hom}(M,G)) \quad \text{for R-mods } F,G.$$

1.10 Morphisms of ringed spaces

Def $(f,\varphi): (X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ morph of ringed spaces means

often write $\varphi = f^\#$

$X \xrightarrow{f} Y$ continuous map of topological spaces

$f_*\mathcal{O}_X \xleftarrow{\varphi} \mathcal{O}_Y$ morph of sheaves of rings (on Y)

work with Ring(X) instead of Ab(X), so rings & ring homs instead of ab.gps. & gp.hom

(So: $\mathcal{O}_X(f^{-1}V) \underset{\text{ring hom}}{\xleftarrow{\varphi_V}} \mathcal{O}_Y(V)$ for $V \subseteq Y$, compatibly with restrictns)

For a morphism of locally ringed spaces want in addition:

$$\mathcal{O}_{X,x} \xleftarrow{\varphi_x} \mathcal{O}_{Y,fx} \quad \text{is local ring hom}$$

$\varphi: R \to S$ ← local rings is local ring hom if $\varphi(m_R) \subseteq m_S$. Equivalently: $\varphi^{-1}(m_S) = m_R$ since this is prime and contains m_R

(Explanation: $\varphi_V(\underset{\underset{fx}{\mathcal{O}_Y(V)}}{s}) \in \mathcal{O}_X(f^{-1}V)$ is a representative for $\varphi_x(s_{fx})$) with $f^{-1}V \ni x$

(This ensures that germs of functions vanishing at fx map to germs vanishing at x

Rmk Can compose: $(X,\mathcal{O}_X) \xrightarrow{f} (Y,\mathcal{O}_Y) \xrightarrow{g} (Z,\mathcal{O}_Z)$:

$$(g\circ f)_*\mathcal{O}_X = g_*f_*\mathcal{O}_X \xleftarrow{g_*(f^\#)} g_*\mathcal{O}_Y \xleftarrow{g^\#} \mathcal{O}_Z .$$

← g_* is a functor so $g_*(\varphi)$ means: apply g_* to $f_*\mathcal{O}_X \xleftarrow{f^\#} \mathcal{O}_Y$

Rmk Notice in the definition we cannot just talk about a morphism $\mathcal{O}_X \leftarrow \mathcal{O}_Y$ because the sheaves are not defined over the same topological space.

$\Rightarrow$ either need a morph $f_*\mathcal{O}_X \longleftarrow \mathcal{O}_Y$ of sheaves on Y
or a morph $\mathcal{O}_X \longleftarrow f^{-1}\mathcal{O}_Y$ of sheaves on X

By the proposition, this is the same information since $\mathrm{Mor}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X) \simeq \mathrm{Mor}(\mathcal{O}_Y, f_*\mathcal{O}_X)$

(Notice also the map on stalks $\mathcal{O}_{X,x} = (\mathcal{O}_X)_x \longleftarrow (f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,fx}$ is the φ_x above)

Rmk φ local $\Rightarrow$ also get hom on residue fields: $\varphi_x: k(fx) = \mathcal{O}_{Y,fx}/m_{Y,fx} \hookrightarrow \mathcal{O}_{X,x}/m_{X,x} = k(x)$

$\Rightarrow$ field extension $\varphi_x: k(fx) \hookrightarrow k(x)$ ← (in classical algebraic geometry: k alg. closed and x closed point get $\mathrm{id}: k \to k$, $p(fa) \xmapsto{=} (f^*p)(a)$ where $\{p \in k[Y], a \in X\}$

1.11 A sheaf defined on a topological basis

X top. space with a basis B of open subsets ← (means: basic sets cover X, and: $\forall$ basic B_1, B_2, $x \in B_1 \cap B_2$ $\exists$ basic B with $x \in B \subseteq B_1 \cap B_2$)

Def B-sheaf F means

- $F(U) \in Ab$, $\forall$ basic U with homs $F(U) \to F(V)$, $s \mapsto s|_V$ $\forall$ basic $V \subseteq U$ and as usual: $F(U) \xrightarrow{id} F(U)$ and $F(U) \to F(V) \to F(W)$ for $W \subseteq V \subseteq U$

$$\begin{tikzcd} F(U) \arrow[r] \arrow[rr, bend right] & F(V) \arrow[r] & F(W) \end{tikzcd}$$

- local-to-global condition:

$\forall$ basic U with $U = \bigcup U_i$ ← basic

$\forall\, s_i \in F(U_i)$ "agreeing locally on overlaps":

$\forall x \in U_i \cap U_j$ $\exists$ basic $x \in U_k \subseteq U_i \cap U_j$ with

$$s_i|_{U_k} = s_j|_{U_k} \in F(U_k)$$

$\Rightarrow \exists$ unique $s \in F(U)$ with $s|_{U_i} = s_i$.

Rmk stalk $F_x = \varinjlim_{x\in(\text{basic }V)} F(V)$.

Theorem 1) B-sheaf F extends uniquely (up to unique iso) to a sheaf F on X.

(Hence also the stalk is F_x up to canonical iso.

(So F(basic U) and restrictions for basic sets are same up canonical isomorphisms.)

2) B-sheaves F, G then morph F→G on the extended sheaves is uniquely defined by data:

- homs $F(U) \to G(U)$ for basic U, commuting with restrictions (for basic opens)

Proof (1):

Uniqueness Such an extension $\tilde{F}$ is unique (if it exists) because we can canonically identify $\tilde{F}(U)$ for any open U in terms of the B-sheaf data:

$$\tilde{F}(U) \xrightarrow{\text{bijection}} \{ s_V \in F(V) \text{ for } (\text{basic } V) \subseteq U : s_V|_W = s_{V'}|_W \in F(W) \text{ for basic } W \subseteq V\cap V' \}$$

$$s \longmapsto (s_V := s|_V \in \tilde{F}(V) = F(V))$$

Explanation: given s, notice that this holds: $s_V|_W = (s|_V)|_W = s|_W = (s|_{V'})|_W = s_{V'}|_W$.

Conversely, given such $s_V \in F(V) = \tilde{F}(V)$, then $s_V|_{V\cap V'} \in \tilde{F}(V\cap V')$ and $s_{V'}|_{V\cap V'} \in \tilde{F}(V\cap V')$ must equal because their restrictions to a covering of $V\cap V'$ by basic W agree $(= s_W)$.

(and then use sheaf property of $\tilde{F}$)

Existence

$$\boxed{F(U) = \varprojlim_{(\text{basic } V)\subseteq U} F(V)}$$ ← inverse limit over restrictions for basics

$$= \{ (s_V) \in \prod_{(\text{basic } V)\subseteq U} F(V) : s_V|_W = s_W \ \forall\, W \subseteq V \subseteq U \}$$

"compatible families of local sections on basic open sets" basic

with obvious restriction maps (for $U' \subseteq U$ a subset of the (basic V) ⊆ U are ⊆ U')

Notice: $F(\text{basic } U)$ has not changed up to canonical identification:

$$F(U) \xrightarrow{\cong} \varprojlim_{(\text{basic } V) \subseteq U} F(V)$$

$$s \longmapsto (s|_V)$$ which includes $s|_U = s$.

and for stalks:

$$\varinjlim_{x \in (\text{basic } V)} F(V) \xrightarrow{\cong} \varinjlim_{x \in U} F(U)$$ ← includes basic $U=V$ ← (easy check: if sections agree on $x \in W$ then agree on $x \in V \subseteq W$ some basic V.)

Proof (2): by functoriality of $\varprojlim$:

$$\varprojlim_{(\text{basic } V) \subseteq U} F(V) \longrightarrow \varprojlim_{(\text{basic } V) \subseteq U} G(V).$$ $\square$

Rmk Equivalently, it is enough to remember germs around each point:

(alternatively can view $s \in \prod_{x \in U} F_x$)

$$F(U) = \left(\varprojlim_{(\text{basic } V) \subseteq U} F(V) \right) \xrightarrow{\cong} \left\{ s : U \to \bigsqcup_{x \in X} F_x : s(x) \in F_x \text{ which are "locally compatible":} \right\}$$

take germs ↑ ; disjoint union of sets F_x

$\forall x \in U, \exists x \in (\text{basic } V) \subseteq U$
$\exists\, t \in F(V)$
$\exists$ open $x \in W \subseteq V$ } with $t_y = s(y)\ \forall y \in W$

with obvious restriction maps for these (just restrict the map $U \to \bigsqcup F_x$).

Rmk Can simplify
- WLOG W also basic (just pick $x \in \text{basic} \subseteq W$)
- WLOG replace V by W, so $V=W$ basic.

} so: $\forall x \in U\ \exists x \in (\text{basic } V) \subseteq U$ $\exists\, t \in F(V)$ with $t_y = s(y)\ \forall y \in V$

Inverse: have cover $U = \bigcup_{x \in U} (\text{basic } x \in V^x)$ and $t^x \in F(V^x)$ s.t. t^x agree locally (since germs agree) } so ✪ holds so can extend to unique global section.

1.12 Construction of $\mathcal{O}_{\operatorname{Spec} R}$

$X = \operatorname{Spec} R$, we define $\mathcal{O}_X$ first on basic open sets:

Motivation: $\frac{1}{g}$ should be an acceptable function on D_f provided we don't divide by zero!

$$\mathcal{O}_X(D_f) = R \text{ localised at multiplicative set } \{g : g \text{ does not vanish on } D_f\}$$

$$\cong R_f$$ ↖ natural

(Recall exercise: $V(g) \subseteq V(f) \Leftrightarrow D_f \subseteq D_g \Leftrightarrow f^N \in (g) \Leftrightarrow g \in R_f$ invertible)

Rmk $\mathcal{O}_X(X) = \mathcal{O}_X(D_1) = R$.

For $D_f \subseteq D_g$ define natural restriction homs: (which are compatible under composition)

$$\begin{tikzcd} \mathcal{O}_X(D_g) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{O}_X(D_f) \arrow[d, phantom, "\cong" rotate=90] \\ R_g \arrow[r] & R_f \end{tikzcd}$$

← "localise further"

← explicitly: $f^N = rg$ so

$$\frac{x}{g^m} \longmapsto \frac{x r^m}{(rg)^m} = \frac{x r^m}{f^{Nm}}$$

Lemma 1 This is a B-sheaf on X for $B = \{\text{basic open sets } D_f, f \in R\}$

Pf Uniqueness (in ✪): $\alpha, \beta \in R_f = \mathcal{O}_X(D_f)$ and $D_f = \bigcup D_{f_i}$

if $\alpha|_{D_{f_i}} = \beta|_{D_{f_i}}\ \forall i$ then $\alpha = \beta$

Proof By redefining X, R by D_f, R_f we can assume $f=1, R_f = R, D_f = X$.

$\alpha - \beta = 0 \in R_{f_i} \Rightarrow f_i^N \cdot (\alpha - \beta) = 0 \in R$ some $N \in \mathbb{N}$ ← N may depend on i, but WLOG finite subcover D_{f_i} so pick maximal N ← (quasi-compactness)

$$\Rightarrow \underbrace{\langle \text{all } f_i^N \rangle}_{=R} \cdot (\alpha - \beta) = 0$$

recall "Covering Trick" → $= R$ since $X = D_{f_1} \cup \dots \cup D_{f_n} = D_{f_1^N} \cup \dots \cup D_{f_n^N}$ ← (recall $D_f = D_{f^N}$)

$\Rightarrow 1 \cdot (\alpha - \beta) = 0$ so $\alpha = \beta$ $\square$

Existence in ✪: as before WLOG $U = D_f$, R_f become X, R.

Uniqueness $\Rightarrow$ in ✪ can assume sections $s_i \in \mathcal{O}_X(D_{f_i})$ agree on overlaps $D_{f_i} \cap D_{f_j} = D_{f_i f_j}$

(apply Uniqueness to $D_{f_i f_j}$)

$$s_i|_{D_{f_i f_j}} = s_j|_{D_{f_i f_j}} \in R_{f_i f_j}$$

WLOG $X = D_{f_1} \cup \dots \cup D_{f_n}$ finite cover, $s_i = \frac{g_i}{f_i^{n_i}}$ since $D_{f_i} = D_{f_i^{n_i}}$, WLOG $n_i = 1$, so $s_i = \frac{g_i}{f_i}$

$s_i = s_j$ on $D_{f_i f_j} \Rightarrow (f_i f_j)^N (f_j g_i - f_i g_j) = 0 \in R$ ← N depends on i,j but can pick largest N over finitely many i,j so N works $\forall i,j$

$$\text{rewrite: } \underbrace{(f_j^{N+1})}_{=b_j} \cdot \underbrace{(f_i^N g_i)}_{=a_i} - \underbrace{(f_i^{N+1})}_{=b_i} \cdot \underbrace{(f_j^N g_j)}_{=a_j} = 0$$

notice $s_i = \frac{a_i}{b_i}$, $D_{f_i} = D_{b_i}$ so WLOG $N=0$! so $f_j g_i = f_i g_j$

"Covering Trick": $X = D_{f_1} \cup \dots \cup D_{f_n}$ so $1 = \sum r_i f_i$ ← ("partition of unity" trick)

$$1 \cdot g_j = \left(\sum_i r_i f_i\right) g_j = \sum_i r_i (f_i g_j) = \sum_i r_i (f_j g_i) = f_j \left(\sum r_i g_i\right)$$

$$\Rightarrow s_j = \frac{g_j}{f_j} = \frac{\sum r_i g_i}{1} \in R_{f_j}\ \forall j$$ so we globalised the $s_j \in \mathcal{O}_X(D_{f_j})$ to $\sum r_i g_i \in \mathcal{O}_X(X) = R$ $\square$

Corollary $\mathcal{O}_X$ extends uniquely to a sheaf on $X = \operatorname{Spec} R$ called <u>structure sheaf</u> (or <u>sheaf of regular functions</u>)

$$\text{stalk } \mathcal{O}_{X,p} := \varinjlim_{D_f \ni p} \mathcal{O}_X(D_f)$$

← Messy unpacking of definitions: we identify $\frac{r}{f^m} \in R_f \cong \mathcal{O}_X(D_f)$ and $\frac{s}{g^n} \in R_g \cong \mathcal{O}_X(D_g)$ iff $\frac{r}{f^m} = \frac{s}{g^n} \in R_h$ some $h \in R$ with $p \in D_h \subseteq D_f \cap D_g = D_{fg}$ (iff $h^N(rg^n - sf^m) = 0 \in R$ some N)

Lemma 2

$$\begin{tikzcd} \mathcal{O}_{X,p} \arrow[r, phantom, "\cong"] & R_p \\ \mathcal{O}_X(X) \arrow[u, "\text{rest.}"] \arrow[r, phantom, "\cong"] & R \arrow[u, "\text{localise}"'] \end{tikzcd}$$

Pf $\varinjlim_{D_f \ni p} \mathcal{O}_X(D_f) \cong \varinjlim_{f \notin p} R_f \cong R_p$. $\square$

↖ straightforward algebra exercise ← (Recall in R_p you invert all elements $f \notin p$)

$$\Rightarrow \mathcal{O}_X(U) = \{(s_f) \in \prod_{D_f \subseteq U} R_f \;:\; s_f|_{D_g} = s_g \quad \forall D_g \subseteq D_f\}$$

$$\cong \{s : U \to \bigsqcup_{p\in X} R_p \;:\; s(p) \in R_p \text{ which are locally compatible:}$$

$\forall p \in U$, $\exists$ open nbhd $p \in \underset{\substack{\uparrow \\ \text{some } f\in R}}{D_f} \subseteq U$, $\exists\, t \in \underset{\substack{\| \\ \frac{r}{f^m}}}{\mathcal{O}_X(D_f)}$ $\cong R_f$ with $s(x) = t_x$ $\}$ $\forall x \in D_f$

$t_x = \frac{r}{f^m} \in \mathcal{O}_{X,x}$ ← is image via natural $\mathcal{O}_X(D_f) \to \mathcal{O}_{X,x}$

with the obvious restriction maps.

Rmk • could assume $t = \frac{r}{f}$ since can replace D_f with D_{f^m} $(= D_f)$.

• could just ask $s(x) = t_x$ on a smaller open $p \in V \subseteq D_f$.

Comparison with classical algebraic geometry

recall $X \subseteq k^n$, k = alg. closed field, $k[X] = k[x_1,\dots,x_n]/\mathbb{I}(X)$ } so look at scheme: $X = \operatorname{Spec} k[X]$ } But classically just study closed points: $X = \operatorname{Specm} k[X] \subseteq k^n$, $\underset{\substack{\| \\ \langle x_1-a_1,\dots,x_n-a_n\rangle}}{m_a} \leftrightarrow \overset{\in}{a}$

- X affine variety, $p \in U \subseteq X$ open nbhd

$f : U \longrightarrow k$ is *regular* at p if $\exists$ open nbhd $p \in W \subseteq U$ with

$f = \frac{g}{h}$ on W, $g, h \in k[X]$, $h(w) \neq 0$ $\forall w \in W$

Rmk In fact can assume $W = D_h$ basic open (if $f = \frac{g}{h^N}$, replace D_h by $D_{h^N} = D_h$)

$\mathcal{O}_X(U)$ = k-algebra of functions $U \to k$ regular at all $p \in U$

$\mathcal{O}_{X,p}$ = k-algebra of germs of functions near p, regular at p

(so pairs (U,f) with $p \in U \subseteq X$ open, $f : U \to k$ regular at p and identify $(U,f) \sim (V,g) \Leftrightarrow f|_W = g|_W$ on some open $p \in W \subseteq U \cap V$)

Theorem $\mathcal{O}_X(X) \cong k[X]$ ← (**Rmk** This theorem is not obvious in C3.4 course. $X = \operatorname{Spec} k[X]$ so by Lemma 1 get $\mathcal{O}_X(X) = k[X]$)

- $X \subseteq \mathbb{A}^n$ affine variety

$f \in R = k[x_1,\dots,x_n]$ polynomial

$V(f) = \{f = 0\} \subseteq X$ hypersurface

$D_f = \{f \neq 0\} \subseteq X$ open, but identifiable

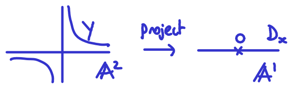
with affine variety $Y = V(zf-1) \subseteq \mathbb{A}^{n+1}$ $(D_f \to Y,\ a \longmapsto (a, \frac{1}{a}))$

and $k[Y] = k[X]/(zf-1) \cong k[X]_f$ via $z \leftrightarrow \frac{1}{f}$

example: $D_x = \mathbb{A}^1 \setminus 0 \cong V(zx-1) = Y \subseteq \mathbb{A}^2$

$t \leftrightarrow (t, t^{-1})$

$k[Y] = k[x]_x = k[x, x^{-1}]$



fact $\mathcal{O}_X(D_f) \cong k[X]_f$

local ring → $\mathcal{O}_{X,p} \cong k[X]_{m_p}$ ← where $m_p = \mathbb{I}(p) = \{f \in k[X] : f(p) = 0\}$ is max ideal corresponding to p.

$\cup\!\!\mid$

$m_{X,p} = m_p \cdot k[X]_{m_p}$ = germs of functions near p vanishing at p

residue field $k(p) = \mathcal{O}_{X,p}/m_{X,p} \cong k$, $\frac{g}{h} \longmapsto \frac{g(p)}{h(p)}$ ← (for $p \in X$ closed point, otherwise more complicated e.g. $\mathbb{A}^1_k = \operatorname{Spec} k[x]$:
• $0 \in \mathbb{A}^1_k$ is closed point $(x) \subseteq k[x]$, $k((x)) = k$.
• $(0) \subseteq k[x]$ not closed point, $k((0)) = k(x)$.

Morphs:

$\alpha : X \to Y$ $\Rightarrow$ $\alpha^\# : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\alpha^{-1}U)$, $\alpha^\#(f : U \to k) = (\alpha^\#(f) = f\circ\alpha : \alpha^{-1}U \to k)$
(morph of aff. vars.) (usual pullback on functions in classical alg. geom)

1.13 Morphisms between Specs

$\varphi : R \to S$ hom of rings $\Rightarrow$ $\boxed{\begin{array}{l}\operatorname{Spec}(\varphi) : \overset{Y}{\overset{\|}{\operatorname{Spec} S}} \longrightarrow \overset{X}{\overset{\|}{\operatorname{Spec} R}} \\ \qquad\qquad p \longmapsto \varphi^{-1}(p)\end{array}}$

Example $\varphi : R \to R_f$, $r \mapsto \frac{r}{1}$ localisation

$\operatorname{Spec} R \longleftarrow \operatorname{Spec} R_f$ is an "inclusion" with image $= D_f$.

$\alpha = \operatorname{Spec}(\varphi) : Y \longrightarrow X$, $p \longmapsto \varphi^{-1}(p)$

Lemma $\alpha^{-1}(D_f) = D_{\varphi(f)}$

Pf $\alpha^{-1}\{q \in X : f \notin q\} = \{p \in Y : \overbrace{\varphi^{-1}(p) = q}^{\text{automatically true!}} \text{ some } q \in X,\ f \notin \varphi^{-1}(p)\}$
$= \{p \in Y : \varphi(f) \notin p\}$. □

Claim $\exists$ $\boxed{\varphi^\# : \mathcal{O}_X \longrightarrow \alpha_*\mathcal{O}_Y \text{ such that } \varphi^\#_X : \mathcal{O}_X(X) = R \xrightarrow{\varphi} S = \alpha_*\mathcal{O}_Y(X)}$

Pf Enough to build $\varphi^\#$ on basic opens, compatibly with restrictions (By Theorem on B-sheaves)

$$\begin{tikzcd}
\varphi^\# : \mathcal{O}_X(D_f) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \alpha_*\mathcal{O}_Y(D_f) = \mathcal{O}_Y(\alpha^{-1}D_f) = \mathcal{O}_Y(D_{\varphi(f)}) \arrow[d, phantom, "\cong" rotate=90] \\
R_f \arrow[r, "\text{natural hom}"] & S_{\varphi(f)} \\
\frac{r}{f^n} \arrow[r, mapsto] & \frac{\varphi(r)}{\varphi(f^n)} = \frac{\varphi(r)}{\varphi(f)^n}
\end{tikzcd}$$

Easy check: compatible with restriction maps for $D_g \subseteq D_f$. □

Claim $\mathcal{O}_{X,p}$ is local and $\varphi^\#_p$ is local

Pf Lemma 2: $\mathcal{O}_{X,p} \cong R_p$ so local with max ideal $m_p = p \cdot R_p$.

For $p \in Y$,
$$\begin{tikzcd}
\varphi^\#_p : \mathcal{O}_{X,\alpha p} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{O}_{Y,p} \arrow[d, phantom, "\cong" rotate=90] \\
R_{\varphi^{-1}p} \arrow[r] & S_p .
\end{tikzcd}$$
□

(easy exercise: this is local. Hint: $\varphi(r) \notin p \Rightarrow r \notin \varphi^{-1}p$)

is direct limit of maps hence: natural map: $\frac{r}{t} \longmapsto \frac{\varphi(r)}{\varphi(t)}$
$t \notin \varphi^{-1}p$ so $\varphi(t) \notin p$

$\Rightarrow$ **Theorem** (ring R) $\longrightarrow$ locally ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

(ring hom $R \xrightarrow{\varphi} S$) $\longrightarrow$ $((\operatorname{Spec}\varphi, \varphi^\#) : (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}))$

contravariant functor $\boxed{\operatorname{Spec} : \text{Rings} \longrightarrow \text{Locally Ringed Spaces}}$ (easy to check)

Claim The functor is fully faithful ← i.e. surj & inj. (so iso) on morphism spaces

Pf Given a hom of loc. ringed spaces $(f, f^\#) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ $\quad X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$

$$\begin{tikzcd}
\text{Let } \varphi := f^\#_X : R \cong \mathcal{O}_X(X) \arrow[r, "f^\#_X"] \arrow[d, "\ell_{fp}"'] & f_*\mathcal{O}_Y(X) = \mathcal{O}_Y(Y) \cong S \arrow[d, "\ell_p"] \\
R_{fp} \cong \mathcal{O}_{X,fp} \arrow[r, "f^\#_p"] & \mathcal{O}_{Y,p} \cong S_p \supseteq m_p = p \cdot S_p
\end{tikzcd}$$

ring hom. ← localisation maps (Lemma 2) for $\mathcal{O}_X, \mathcal{O}_Y$

$$\Rightarrow \varphi^{-1}(p) = \varphi^{-1}(\underbrace{\ell_p^{-1}(m_p)}_{p}) \underset{\substack{\uparrow \\ \text{diagram}}}{=} \ell_{fp}^{-1}(\underbrace{f_p^{\#\,-1}(m_p)}_{m_{fp} \text{ since } f^\#_p \text{ local ring hom}}) = f(p)$$

$\Rightarrow f(p) = \varphi^{-1}(p)$ so $f = \mathrm{Spec}(\varphi)$ is the map on Specs induced by $\varphi : R \to S$.

Upshot: have two morphs of sheaves $f^\#, \varphi^\# : \mathcal{O}_X \to \mathrm{Spec}(\varphi)_* \mathcal{O}_Y$ and $f^\# = \varphi^\#$ since equal on stalks (by the diagram have $f^\#_p = \varphi_p$) $\square$

$$\begin{tikzcd} \frac{r}{1} \arrow[r, mapsto, "f_p"] & \frac{\varphi(r)}{1} \\ \frac{s}{1} \arrow[r, mapsto] & \frac{\varphi(s)}{1} \\ \frac{r}{s} \arrow[r, mapsto] & \frac{\varphi(r)}{\varphi(s)} \end{tikzcd}$$

because:

$$\boxed{\begin{aligned} \tfrac{\varphi(r)}{1} &= f_p(\tfrac{r}{1}) = f_p(\tfrac{r}{s}\cdot\tfrac{s}{1}) \\ &= f_p(\tfrac{r}{s})\cdot f_p(\tfrac{s}{1}) = f_p(\tfrac{r}{s})\cdot\tfrac{\varphi(s)}{1} \end{aligned}}$$

Def Aff = category of affine schemes (and morphs of locally ringed spaces)

(locally ringed spaces $\cong (\mathrm{Spec}\,R, \mathcal{O}_{\mathrm{Spec}\,R})$ some ring R)

$\Rightarrow$ $\boxed{\mathrm{Spec} : \mathrm{Rings}^{op} \longrightarrow \mathrm{Aff} \text{ is an equivalence of categories.}}$

(op = opposite category = reverse arrows so artificially make Spec covariant)

full, faithful, $\underbrace{\text{essentially surjective}}_{\text{each object in target category is iso to an object in image}}$ functor

1.14 Closed affine subschemes

$X = \mathrm{Spec}\,R$, $I \subseteq R$ ideal (Rmk same as specifying a surj. ring hom $\varphi : R \to S$, then $I = \ker\varphi$, $R/I \cong S$)

$Y = V(I) \cong \mathrm{Spec}(R/I)$ are called closed (affine) subschemes of X

(as top. space, $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ but sheaf remembers I: $\mathcal{O}_Y(Y) = R/I$)

$(p \subseteq R \text{ prime} \supseteq I) \longmapsto p\cdot R \subseteq R/I$

Example $I = m$ max ideal $\Rightarrow$ get a closed point $\{m\} = \mathrm{Spec}\,R/m \hookrightarrow X$.

Rmk $\mathrm{Spec}(R/J)$ is closed subscheme of $\mathrm{Spec}(R/I)$ means $J \supseteq I$

Warning $\Rightarrow V(J) \subseteq V(I)$ $\not\Leftarrow$ $\Updownarrow$ $\sqrt{J} \supseteq \sqrt{I}$

Def $\mathrm{Spec}\,R/I \cap \mathrm{Spec}\,R/J := \mathrm{Spec}(R/I+J)$, $\mathrm{Spec}\,R/I \cup \mathrm{Spec}\,R/J := \mathrm{Spec}\,R/I \cap J$

Define sheaf of ideals $\mathcal{J} = \mathcal{J}_{X/Y}$ on X: (also: ideal sheaf)

$\mathcal{J}(D_f) = I \cdot R_f \subseteq R_f = \mathcal{O}_X(D_f)$ ideal

Classical Alg. Geom: $\mathcal{J}(U)$ are the regular functions vanishing on $Y \cap U$

Notice $\mathcal{O}_Y(D_f) = (R/I)_f \cong R_f/I\cdot R_f = \mathcal{O}_X(D_f)/\mathcal{J}(D_f)$

Note $I\cdot R_f = \ker\left(R_f \to R_f/IR_f\right)$ $\|$ $\mathcal{J}(D_f) = \ker\left(\mathcal{O}_X(D_f) \to \mathcal{O}_X(D_f)/\mathcal{J}(D_f)\right)$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{J} &= \ker(\mathcal{O}_X \longrightarrow j_* \mathcal{O}_Y) \\ \mathcal{O}_Y &= \mathcal{O}_X/\mathcal{J} \end{aligned}}$$

where $j : Y \longrightarrow X$ inclusion.

more precisely this is $j_*\mathcal{O}_Y$

1.15 Closed subschemes

(later in course: sheaves of R-modules and quasi-coherence)

Think of these as the regular functions which "vanish" on Y.

$(X, \mathcal{O}_X)$ scheme, sheaf of ideals $\mathcal{J}$ means $\mathcal{J}(U) \subseteq \mathcal{O}_X(U)$ ideal compatibly with restrictions.

Def A sheaf of ideals on $X = \mathrm{Spec}\,R$ is quasi-coherent if it arises as $\mathcal{J}$ as above, some ideal $I \subseteq R$ (see 1.14)

on X = scheme " " if $\forall$ affine open U, $\mathcal{J}|_U$ is quasi-coherent.

(later revisit these in Sec. 3.6)

closed subscheme means
- $Y \subseteq X$ closed topological subspace
- $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$ some quasi-coherent sheaf of ideals $\mathcal{J}$ on X,

s.t. $Y \cap (\text{affine open } U) \subseteq U$ is closed affine subscheme for the ideal $\mathcal{J}(U) \subseteq \mathcal{O}_X(U)$.

Rmk $\mathcal{J}$ = Ker of surjection $\mathcal{O}_X \twoheadrightarrow j_*\mathcal{O}_Y$

Rmk $\exists$ 1:1 correspondence $\boxed{\{\text{closed subschemes of } X\} \longleftrightarrow \{\text{quasi-coh. sheaves of ideals on } X\}}$

Can recover $Y \subseteq X$ from $\mathcal{J}$ from the support of $\mathcal{O}_X/\mathcal{J}$:

$Y = \mathrm{Supp}\,\mathcal{O}_X/\mathcal{J} = \{x \in X : (\mathcal{O}_X/\mathcal{J})_x \neq 0\} = \{x \in X : \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$

if $I \subseteq p \subseteq R$ then $I\cdot R_p \neq R_p$ since $IR_p \subseteq \overset{p\cdot R_p}{\overset{\|}{m_p}}$

Example closed point $p \in X$ (so $\overline{\{p\}} = \{p\}$) $\Rightarrow$ pick affine $p \in \mathrm{Spec}\,R \overset{\cdot}{\hookrightarrow} X$ then $p \leftrightarrow \binom{\text{max}}{\text{ideal}} \subseteq R$

$\Rightarrow$ sheaf $\mathcal{J}$ on $\mathrm{Spec}\,R$ $\Rightarrow$ extend $\mathcal{J}$ to X by $\mathcal{J}(V) = \mathcal{O}_X(V)$ if $p \notin V$ (so $\mathcal{O}_Y(V) = 0$)

2. GLOBAL SECTIONS AND THE FUNCTOR OF POINTS

2.0 Points of Spec R (not necessarily closed)

$$\begin{tikzcd} R \arrow[r, "loc"] & R_p \arrow[r, "quotient"] & K(p) = R_p/m_p \\ loc^{-1}(m_p) = p & p\cdot R_p = m_p \arrow[l] & (0) \arrow[l] \end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd} \mathrm{Spec}\,K(p) \arrow[r, hook] \arrow[d, phantom, "\|"] & \mathrm{Spec}\,R_p \arrow[r, hook] & \mathrm{Spec}\,R \\ \{(0)\} & (0) \arrow[r, mapsto] & m_p \arrow[r, mapsto] & p \end{tikzcd}$$

So points of Spec R correspond to the max ideals in the local rings.

2.1 Global sections and basic open sets for locally ringed spaces

$(X, \mathcal{O}_X)$ locally ringed space, $\Gamma(\cdot, \mathcal{O}_X) : \mathrm{Top}(X)^{op} \longrightarrow \mathrm{Rings}$, sections functor

$$\begin{tikzcd} U \arrow[r, mapsto, "\Gamma"] & \mathcal{O}_X(U) \arrow[d, "restrict"] \\ V \arrow[u, "include"', "\cup\!\mid"] \arrow[r, mapsto, "\Gamma"] & \mathcal{O}_X(V) \end{tikzcd}$$

global sections functor: Locally Ringed Spaces$^{op} \longrightarrow$ Rings, $(X, \mathcal{O}_X) \mapsto \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$

$\exists$ canonical map $X \longrightarrow \mathrm{Spec}\,\mathcal{O}_X(X)$, $x \longmapsto \mathrm{res}_x^{-1}(m_{X,x})$ where $\mathrm{res}_x : \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ restricts.

Trick $f \in \mathcal{O}_X(X)$ then $f_x \in \mathcal{O}_{X,x}$ invertible $\Leftrightarrow$ $f(x) \neq 0 \in K(x) = \mathcal{O}_{X,x}/m_x$

image of f via $\mathcal{O}_X(X) \to \mathcal{O}_{X,x} \to K(x)$, $f \longmapsto f_x \longmapsto f(x)$

Pf $f_x \in \mathcal{O}_{X,x} \setminus m_x = \{\text{invertibles of } \mathcal{O}_{X,x}\} \Leftrightarrow f_x \notin m_x$ $\square$

Lemma $f \in \mathcal{O}_X(X) \Rightarrow \boxed{D_f = \{x \in X : f(x) \neq 0 \in K(x)\}}$ is open in X.

$\Leftrightarrow f \notin m_x \Leftrightarrow \binom{f_x \in \mathcal{O}_{X,x}}{\text{invertible}}$

Pf Trick $\Rightarrow \exists g \in \mathcal{O}_{X,x} : f\cdot g = 1$ so $\exists$ open $x \in U \subseteq X$ s.t. $f, g \in \mathcal{O}_X(U)$, $f \cdot g = 1 \in \mathcal{O}_X(U)$

$\Rightarrow x \in U \subseteq D_f$ since $\forall y \in U$, $f_y \cdot g_y = (f\cdot g)_y = 1 \in \mathcal{O}_{X,y}$ so $f_y \in \{\text{invertibles of } \mathcal{O}_{X,y}\}$ so $f(y) \neq 0$, so $y \in D_f$ $\square$

Lemma $f|_{D_f} \in \mathcal{O}_X(D_f)$ is invertible

Pf Lemma $\Rightarrow f$ is locally invertible. If $\begin{smallmatrix} f\cdot h = 1 \text{ on } U \\ f\cdot g = 1 \text{ on } V \end{smallmatrix}$ then $h = g$ on $U \cap V$. So can globalise. $\square$

uniqueness of inverses ($h = h\cdot 1 = hfg = 1\cdot g = g$)

2.2 What it means to be affine

$(X, \mathcal{O}_X)$ (locally ringed space) affine $\Leftrightarrow \exists$ ring R : $\exists X \xrightarrow{\alpha} Y = \mathrm{Spec}\,R$ homeomorph, and $\exists \mathcal{O}_Y \underset{\cong}{\xrightarrow{\varphi}} \alpha_*\mathcal{O}_X$ (local on stalks)

But $\mathcal{O}_Y(Y) = R$ so $R \underset{\cong}{\xrightarrow{\varphi}} \mathcal{O}_X(X)$ so $\mathrm{Spec}\,\mathcal{O}_X(X) \xrightarrow{\cong} Y$. (via $\varphi^{-1}(\cdot)$)

$\overset{\varphi_x\,\text{local}}{\Longrightarrow}$

$$\begin{tikzcd} R \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ \mathcal{O}_{Y,\alpha(x)} = R_{\alpha(x)} \arrow[r, "\varphi_x"] & \mathcal{O}_{X,x} \end{tikzcd}$$

$$\begin{tikzcd} R \supseteq \alpha(x) \arrow[r, "\cong"', "\varphi"] \arrow[d] & \mathrm{res}_x^{-1}(m_x) \subseteq \mathcal{O}_X(X) \arrow[d] \\ \alpha(x)\cdot R_{\alpha(x)} \arrow[r] & m_x \end{tikzcd}$$

so

$$\begin{tikzcd} X \arrow[r, "canonical"] & \mathrm{Spec}\,\mathcal{O}_X(X) \arrow[r, phantom, "\cong"] & Y \\ x \arrow[r, mapsto] & \mathrm{res}_x^{-1}(m_x) \arrow[r, mapsto] & \alpha(x) \end{tikzcd}$$

So a locally ringed space $(X, \mathcal{O}_X)$ is affine precisely if:

- the canonical map $X \longrightarrow \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X)$ is homeomorph
- $\mathcal{O}_X(D_f) \cong (\Gamma(X, \mathcal{O}_X))_f$ $\forall f \in \Gamma(X, \mathcal{O}_X)$ and restrictions are localisations $\leftarrow$ (by Sec. 1.12)

2.3 Functor of points h_Y

MOTIVATION Y set, you recover set Y from $\mathrm{Mor}(\text{point}, Y)$

Y group, " " set " " $\mathrm{Mor}(\mathbb{Z}, Y)$

Functor of points $h_Y : Sch^{op} \longrightarrow Sets$, $\boxed{h_Y(X) = Mor(X,Y)}$ ← (op = opposite category = reverse arrows | Think: "X-valued points of Y")

$\left(\begin{matrix} X \xleftarrow{f} Z \\ \searrow \ \downarrow \\ Y \end{matrix}\right) \to$ on morphs : $h_Y(X \xleftarrow{f} Z) = (Mor(X,Y) \xrightarrow{\cdot \circ f} Mor(Z,Y))$

MOTIVATION

$Y = Spec\ \mathbb{Z}[x]/(x^2+1)$. $\mathbb{C}$-valued points of Y?

$\mathbb{Z}[x]/(x^2+1) \to \mathbb{C}$, $x \mapsto i \Rightarrow$ morph $X = Spec\ \mathbb{C} \longrightarrow Y$ so $\in h_Y(X)$ ← (often write $Y(\mathbb{C})$)

HWK 1 natural transformations

Yoneda lemma $Nat(h_Y, F) \cong F(Y)$ ← ∀ contravariant functor F

[take image of $id_Y \in Mor(Y,Y) = h_Y(Y) \xrightarrow{given} F(Y)$
[Conversely given $\alpha \in F(Y)$, $\varphi \in h_Y(X)$ get $F(\varphi)(\alpha) \in F(X)$

Yoneda embedding $\boxed{h_\bullet : Sch \longrightarrow Sets^{Sch^{op}} \quad Y \mapsto h_Y \text{ is fully faithful}}$ ← (iso on morphisms: $Nat(h_Y, h_W) \cong Mor(Y,W)$)

($Sets^{Sch^{op}}$ = category : {Obj are functors $Sch^{op} \to Sets$; Morph are natural transformations})

UPSHOT ① $\boxed{h_Y \cong h_W \Leftrightarrow Y \cong W}$

② Can now ask which functors $Sch^{op} \to Sets$ are $\cong h_Y$, i.e. represented by a scheme Y.

("tell me who your friends are and I will tell you who you are" $\begin{smallmatrix}\to\\ \to \\ \to\end{smallmatrix} Y$)

Example Will show that $\mathbb{A}^n = Spec\ \mathbb{Z}[x_1, \ldots, x_n]$ represents $Sch^{op} \to Sets$, $X \mapsto \{\text{morphs } \bigoplus_{i=1}^{n} \mathcal{O}_X \to \mathcal{O}_X \text{ which are "}\mathcal{O}_X\text{-linear"}\}$

Example 1 $h_{Spec R}$

scheme or loc.-ringed space.

$Mor(X, Spec R) = Mor_{Sch^{op}}(Spec R, X)$

$$\boxed{\begin{matrix} Y \text{ affine} \\ = Spec\ R \end{matrix} \Longrightarrow \begin{matrix} Mor(X, Spec R) \longrightarrow Hom(R, \Gamma(X, \mathcal{O}_X)) \\ g \longmapsto g_Y^\# \end{matrix} \quad \text{bijective}}$$

$\Rightarrow$ Spec & global sec. are adjoint functors

KEY EXAMPLE
$Y = \mathbb{A}^1 = Spec\ \mathbb{Z}[x]$
$\Downarrow$
$\boxed{\begin{matrix} Mor(X, \mathbb{A}^1) \\ \cong \\ \mathcal{O}_X(X) \end{matrix}}$
(since $\mathbb{Z}[x] \to \mathcal{O}_X(X)$ determined by image of x)

Pf. $\mathcal{O}_Y(Y) \xrightarrow[given]{\varphi} \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{X,x}$ preimage of m_x gives $p \in Spec R = Y$
$\mathcal{O}_Y(Y) = R$ ← $Y = Spec R$; $\mathcal{O}_{X,x} \supseteq m_x$; defines $g : X \longrightarrow Y$, $g(x) = p$

- g is continuous (check $g^{-1}(D_f) = D_{\varphi f}$). ← (see 2.1 for basic opens of locally ringed spaces)
- $\mathcal{O}_Y(D_f) = R_f \xrightarrow{\varphi_f} \mathcal{O}_X(X)_{\varphi f} \to \mathcal{O}_X(D_{\varphi f}) = \mathcal{O}_X(g^{-1} D_f) = g_* \mathcal{O}_X(D_f)$

↑ localise φ

natural map induced by restriction $\mathcal{O}_X(X) \to \mathcal{O}_X(D_{\varphi f})$ since φf invertible in $\mathcal{O}_X(D_{\varphi f})$ see 2.1

[Universal property of localisation: $R_1 \xrightarrow{\varphi} R_2$ and $\varphi(S) \subseteq$ invertibles of $R_2 \Rightarrow \exists!\ R_1 \to S^{-1}R_1 \to R_2$.

Obvious: $\frac{1}{s} \mapsto \varphi(s)^{-1}$

↑ These are compatible with restrictions □

Cor 1 $(X, \mathcal{O}_X)$ scheme $\Longrightarrow$ canonical morph $\boxed{X \longrightarrow Spec\ \Gamma(X, \mathcal{O}_X)}$

(Example 1 for $R = \Gamma(X, \mathcal{O}_X)$ and $id : R \to R$)

Explicitly : on sets $x \longmapsto res^{-1}(m_{X,x}) \subseteq \mathcal{O}_X(X)$

on sheaves over $D_f \subseteq X$: $\mathcal{O}_X(X)_f \xrightarrow{rest} \mathcal{O}_X(D_f)$ ← $\mathcal{O}_X(X) \xrightarrow{restrict} \mathcal{O}_X(D_f)$ now localise at f using that f invertible

Rmk often not useful if X has few global sections (e.g. $\mathbb{P}^n$ only has constants)

Rmk Canonical morph is injective if global sections separate points meaning:

$x \neq y \in X \Rightarrow \exists f \in \Gamma(X, \mathcal{O}X), f(x) \neq f(y)$ (equivalently $\exists f : f(x) = 0, f(y) \neq 0$)

$f(x)=0 \Leftrightarrow f \in m_{X,x}$, $f(y)\neq 0 \Leftrightarrow f \notin m_{X,y}$

Classical algebraic geom. $X \subseteq \mathbb{A}^n$ affine variety $(X = \mathbb{V}(I),\ I \subseteq k[x_1, \ldots, x_n])$

so $\Gamma(X, \mathcal{O}_X) = k[X]$, $\mathcal{O}_X(D_f) = k[X]_f$, $\mathcal{O}_X(U) = \left\{\begin{matrix}\text{regular functions} \\ U \to k\end{matrix}\right\}$, $\mathcal{O}_{X,a} = k[X]_{m_a}$

separates points, and $X \xrightarrow{inj.} \{\text{closed points}\} \subseteq Spec\ k[X]$
$a \longmapsto$ max ideal $m_a \subseteq k[X]$ ($\leftrightarrow$ max ideal of $\mathcal{O}_{X,a}$)

in fact get embedding $\{\text{Category of Affine Varieties}\} \hookrightarrow Sch$

Example 2 $h_Y(Spec R)$

$X = Spec R$, $m \subseteq R$ = local ring $\Rightarrow$ $\boxed{\left\{\begin{matrix} f \in Mor(Spec R, Y) \\ \text{with } f(m) = y \end{matrix}\right\} \xleftrightarrow{1:1} Hom_{\substack{local\\rings}}(\mathcal{O}_{Y,y}, R)}$ via $f \mapsto f_y^\#$

Pf (⇒) (or see ex.4 of HWK 1)

$$\begin{tikzcd} Spec\ R \arrow[r, "f"] & Y \\ m \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & y \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$R = \mathcal{O}_{Spec R, m} \xleftarrow{f_y^\#} \mathcal{O}_{Y,y}$ local hom of rings

(if $m \in U \subseteq Spec R$ open then $U = Spec R$, since $Spec R \setminus U$ closed so if $\neq \emptyset$ then would find another max ideal)

(⇐) Affine case $Y = Spec S$

$\varphi : S_y \longrightarrow R \Rightarrow S \xrightarrow{loc} S_y \to R \Rightarrow Spec R \longrightarrow Spec S = Y$
$\varphi^{-1}(m) = y \cdot S_y \subseteq S_y$, $m \subseteq R$; $S_y = \mathcal{O}_{Y,y}$, $S_y \to R$ given
$m \longmapsto$ (preimage of $\varphi^{-1} m$ via $S \to S_y$) $= y$; $\varphi^{-1}m = y \cdot S_y$

General case

$y \in U \subseteq Y$ open affine, then $\mathcal{O}_{U,y} = \mathcal{O}_{Y,y} \xrightarrow{\varphi} R$ gives $Spec R \to U \subseteq Y$.

Uniqueness: suppose $f : Spec R \to Y$, $m \mapsto y$ gives same φ

pick $y \in V \subseteq Y$ affine open $\Rightarrow f^{-1}(V)$ open $\ni m = \left(\begin{matrix}\text{unique closed} \\ \text{point of } Spec R\end{matrix}\right) \Rightarrow f^{-1}(V) = Spec R$ ← (exercise 6 in 1.1, so trick)

so $f : Spec R \to V \subseteq Y$ so reduce to affine case. □

Cor 2 $x \in X \Rightarrow \cdot \exists$ canonical morph $Spec\ \mathcal{O}_{X,x} \longrightarrow X$.

(By Example 2 for $id : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$)

- Any $Spec R \to X$ factors as $Spec R \to Spec\ \mathcal{O}_{X,x} \to X$ some $x \in X$.
  ↑ local ring ; ↑ induced by a local ring hom

(Notice in proof above we factorised through $S_y \to R$, $S_y = \mathcal{O}_{Y,y}$)

- Any $f : X \to Y$ of schemes, $x \mapsto y$, get

$$\begin{tikzcd} Spec\ \mathcal{O}_{X,x} \arrow[r] \arrow[d, "\text{induced by } f_x^\#"'] & X \arrow[d, "f"] \\ Spec\ \mathcal{O}_{Y,y} \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} m_{X,x} \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto, "f"] \\ m_{Y,y} \arrow[r, mapsto] & y \end{tikzcd}$$

Example Case $X = Spec\ \mathbb{K}$ for field $\mathbb{K}$.

R local $\Rightarrow$ residue field $k = R/m$

A local hom $R \xrightarrow{\varphi} \mathbb{K}$ = field factors $R \xrightarrow{quot.} k \longrightarrow \mathbb{K}$

(since $\ker\varphi = \varphi^{-1}(0) = m$) ← (since local hom)

Rmk for a field $\mathbb{K}$ $Spec\ \mathbb{K} = \{(0)\}$

Thus: $\boxed{\left\{\begin{matrix} f \in Mor(Spec\ \mathbb{K}, Y) \\ \text{with } f((0)) = y \end{matrix}\right\} \xleftrightarrow{1:1} Hom(k(y), \mathbb{K})}$ and any $Spec\ \mathbb{K} \to Y$, $(0) \to y$ factors:

$k(y) = \mathcal{O}_{Y,y}/m_{Y,y}$; $Spec\ \mathbb{K} \to Spec\ k(y) \to Y$

$h_Y(Spec\ \mathbb{K})$ ← (also written $Y(\mathbb{K})$)

UPSHOT : Morphs from local rings or fields don't give more information than already know from $Spec\ \mathcal{O}_{X,x} \longrightarrow X$ and $Spec\ k(x) \longrightarrow X$.

Non-examinable :

Rmk $y \in Y$ called $\mathbb{K}$-valued point if $k(y) \cong \mathbb{K}$, then $id_{\mathbb{K}}$ defines a morph $Spec\ \mathbb{K} \to Y$.
(or $\mathbb{K}$-point, or $\mathbb{K}$-rational point)

If Y comes with a morph $Y \xrightarrow{\pi} Spec\ \mathbb{K}$ (hence $\mathcal{O}_Y(U)$ are $\mathbb{K}$-algebras) ← say "Y is a scheme over $\mathbb{K}$" and above require morphs to commute with π, then get $Hom_{\mathbb{K}}(k(y), \mathbb{K})$, and if $k(y) \cong \mathbb{K}$ then $Hom_{\mathbb{K}}(\mathbb{K}, \mathbb{K}) = \{id_{\mathbb{K}}\}$. E.g. $Spec(\mathbb{C})$ has many $\mathbb{C}$-points: one for each automorphism of $\mathbb{C}$ (e.g. $\mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$) but
- if work over $\mathbb{C}$ get only one $\mathbb{C}$-point
- if work over $\mathbb{R}$ get two $\mathbb{C}$-points.

3. PROPERTIES OF SCHEMES

mod = module

3.0 Useful facts from commutative algebra : localisation

R ring, M R-mod, $S \subseteq R$ multiplicative set ← $1 \in S$, $S \cdot S \subseteq S$

$\Longrightarrow$ localisation $S^{-1}M = M \times S \,/\,$relation $(m,s) \sim (n,t) \Leftrightarrow u \cdot (tm - sn) = 0$ (write "$\frac{m}{s}$"; some $u \in S$)

which is an $S^{-1}R$-mod and have R-mod hom $M \to S^{-1}M$, $m \to \frac{m}{1}$ localisation map.

Fact $S^{-1}M \cong M \otimes_R S^{-1}R$ canonically ← (via $\frac{m}{s} \longmapsto m \otimes \frac{1}{s}$ and $\sum \frac{r_i m_i}{s_i} \leftarrow\!\shortmid \sum m_i \otimes \frac{r_i}{s_i}$)

Exercise $\alpha : M \to N$ hom (of R-mods) $\Longrightarrow \exists$ natural $S^{-1}\alpha : S^{-1}M \to S^{-1}N$ ($\frac{m}{s} \longmapsto \frac{\alpha(m)}{s}$)

Fact Localisation R-mods $\to S^{-1}$R-mods is an exact functor.

Cor $S^{-1}(M/N) \cong S^{-1}M / S^{-1}N$

Pf apply S^{-1} to exact sequence $0 \to N \to M \to M/N \to 0$. □

Fact Submods of $S^{-1}M$ have form $S^{-1}N$ for submods $N \subseteq M$ ← (indeed take N = preimage via $M \to S^{-1}M$)

Fact $S^{-1}M = \varinjlim_{f \in S} M_f$ via localisation maps $M_f \to M_g$ whenever $g = fh \in S$, $\frac{m}{f^n} \mapsto \frac{mh^n}{g^n}$ ← (induced by $R_f \to R_g$ via $M \otimes_R R_f \to M \otimes_R R_g$)

(e.g. proof: $\varinjlim M \otimes R_f = M \otimes \varinjlim R_f = M \otimes S^{-1}R$)

Local algebra theorem

multiplicative set $S = R \setminus p$

same results hold if only use max ideals p.

① $x \in M$: $x = 0 \Leftrightarrow x_p = 0 \in M_p \;\; \forall p \in \operatorname{Spec} R$

② $M = 0 \Leftrightarrow M_p = 0 \;\; \forall p \in \operatorname{Spec} R$

③ $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ exact $\Leftrightarrow M_p \xrightarrow{\alpha_p} M_p' \xrightarrow{\beta_p} M_p''$ exact $\forall p \in \operatorname{Spec} R$

④ $f : M \to N$ inj. $\Leftrightarrow f_p : M_p \to N_p$ inj. $\forall p \in \operatorname{Spec} R$
   " surj. " surj "
   " iso. " iso. "

Pf ①⇐ $\operatorname{Ann}(x) = \{r \in R : rx = 0\}$ ideal $\subseteq$ max ideal m (unless $x = 0$) ← $1 \notin \operatorname{Ann} x$ if $x \neq 0$

$x_m = 0 \in M_m \Rightarrow \exists\, r \in R \setminus m$ s.t. $rx = 0 \in M$ ↯ (since $r \notin \operatorname{Ann}(x)$)

② by ①

③⇐ $H := \operatorname{Ker}\beta / \operatorname{Im}\alpha \Rightarrow H_p \cong (\operatorname{Ker}\beta)_p / (\operatorname{Im}\alpha)_p = \operatorname{ker}\beta_p / \operatorname{Im}\beta_p = 0$ now use ② (exact $M_p \xrightarrow{\alpha_p} M_p' \xrightarrow{\beta_p} M_p''$)

(⇒ holds since localisation is exact)

(since $0 \to \operatorname{Ker}\beta \xrightarrow{\text{incl}} M' \xrightarrow{\beta} \operatorname{Im}\beta \to 0$ exact
$\Rightarrow 0 \to (\operatorname{Ker}\beta)_p \to M_p' \xrightarrow{\beta_p} (\operatorname{Im}\beta)_p \to 0$ exact, so $\operatorname{Ker}(\beta_p) = (\operatorname{Ker}\beta)_p$, $\operatorname{Im}(\beta_p) = (\operatorname{Im}\beta)_p$

④ by ③ ← (e.g. inj means $0 \to M \xrightarrow{f} N$ exact) □

RmK $\operatorname{Spec} R = \bigcup D_{f_i}$ ($1 \in \langle \text{all } f_i \rangle$) then above results hold $\Leftrightarrow$ hold when localise at each f_i

Pf $x_i = 0 \in M_{f_i} = M \otimes R_{f_i} \Rightarrow$ localise further at $p \in \operatorname{Spec} R_{f_i}$: $M_{f_i} = M \otimes_R R_{f_i} \to M \otimes_R R_p = M_p$, $0 = x_i \longmapsto x_p$, so 0. □

(Note: every $p \in \operatorname{Spec} R$ is in some $D_{f_i} = \operatorname{Spec} R_{f_i}$)

Recall: $\operatorname{Nil}(R) = \text{nilradical}(R) = \{\text{nilpotent elements}\} = \sqrt{(0)} = \bigcap \{p \in \operatorname{Spec} R\}$ (R ring)

Example $\operatorname{Nil}(R_p) = (\operatorname{Nil}(R))_p$, so by ② : R_p reduced $\forall p \Leftrightarrow R$ reduced ← ($\Leftrightarrow$ no nilpotents $\neq 0$ $\Leftrightarrow \operatorname{Nil}(R) = \{0\}$)

Pf. $\operatorname{Nil}(R_p) \ni \frac{r}{s} \Rightarrow (\frac{r}{s})^n = 0 \in R_p$ some $n \Rightarrow t \cdot r^n = 0$ some $t \notin p \Rightarrow (tr)^n = 0 \Rightarrow tr \in \operatorname{Nil}(R)$
$\Rightarrow \frac{r}{s} = \frac{tr}{ts} \in \operatorname{Nil}(R)_p$. The converse is easy. □

3.1 Noetherian

f.g. = finitely generated

Recall: ring R is Noetherian $\Leftrightarrow$ ideals of R are f.g. $\Leftrightarrow$ submods of f.g. R-mods are f.g. $\Leftrightarrow$ ascending family of ideals in R stabilise ("ascending chain condition" ACC)

($I_1 \subseteq I_2 \subseteq \ldots \Rightarrow I_N = I_{N+1} = \ldots$ some N)

Rmk localisation and quotients preserve Noetherian property

Def An affine open (for the ring R) means an open subset $U \subseteq X$ admitting an isomorphism

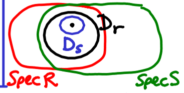
$$(U, \mathcal{O}_X|_U) \cong (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \text{ for some ring } R.$$

← Note: $\mathcal{O}_X(U) \cong R$

Def scheme $(X, \mathcal{O}_X)$ is Noetherian if quasi-compact and locally Noetherian:

Claim The following are equivalent definitions for $(X, \mathcal{O}_X)$ to be locally Noetherian

1) every point has an affine open neighbourhood U with $\mathcal{O}_X(U)$ Noetherian
2) $X = \bigcup U_i$ for open affines U_i with $\mathcal{O}_X(U_i)$ Noetherian
3) given any open affine for a ring R, R must be Noetherian



USEFUL TRICK R,S rings
$p \in \operatorname{Spec} R \cap \operatorname{Spec} S = Y \subseteq X$
$\Rightarrow \exists$ open $p \in D \subseteq Y$ which is basic for both R,S.
Pf $\exists$ basic $p \in D_r \subseteq Y$ for $R \ni r$
$\exists$ basic $p \in D_s \subseteq D_r$ for $S \ni s$
$\Rightarrow s \in S = \Gamma(\operatorname{Spec} S, \mathcal{O}_X)$ ↓ restrict
$s|_{D_r} =: h \in \Gamma(D_r, \mathcal{O}_X) \cong R_r$
$\Rightarrow h = \frac{a}{r^n}$ so $(R_r)_h = R_{ar}$
$\Rightarrow D_s = \{x \in D_r : s(x) \neq 0\}$ (D_s = $\operatorname{Spec} S_s$)
$= \{x \in D_r : h(x) \neq 0\}$
$= \operatorname{Spec} R_{ar} = D_{ar}$

Pf (1)⇔(2) and (3)⇒(1) since schemes are locally affine.

(1)&(2)⇒(3): consider $\operatorname{Spec} R \cong U \subseteq X$

$\forall p \in U$, $\exists$ affine open $p \in V = \operatorname{Spec} S \subseteq X$ with S Noetherian (by (1))

$\Longrightarrow \exists$ basic open $p \in D_g \subseteq U$ for Spec S, some $g \in S$
$= \operatorname{Spec}(S_g)$ and S_g Noeth. (since S Noeth.)

By the USEFUL TRICK, WLOG D_g is basic also for Spec R, say $\operatorname{Spec} R_f$.

Since $\operatorname{Spec} S_g \cong \operatorname{Spec} R_f$ get $S_g \cong R_f$ so Noetherian. Get cover for U,

so need: Algebra Lemma $\left.\begin{array}{l} R_{f_i} \text{ Noeth. } \forall i \\ \langle \text{all } f_i \rangle \ni 1 \end{array}\right\} \Longrightarrow R$ Noeth. ← by "Covering Trick"

proof $I \subseteq R$ ideal (aim: I is f.g.)

$\Longrightarrow I_{f_i} := I \cdot R_{f_i} \subseteq R_{f_i}$ ideal, f.g. since R_{f_i} Noeth., say generators $g_{ij} = \frac{h_{ij}}{f_i^N}$ (some $h_{ij} \in I$)

$\Longrightarrow f_i^N \cdot g_{ij} = \frac{h_{ij}}{1}$ also generate (since $\frac{1}{f_i^N} \in R_{f_i}$)

$\Longrightarrow \bigoplus_{ij} R \xrightarrow[\varphi]{} I$, $e_{ij} \longmapsto h_{ij}$ satisfies φ_{f_i} surjective $\forall f_i$ so φ surj. □

(generator of ij copy of R) (localisation at f_i) ($\varphi_{f_i}(e_{ij}) = \frac{h_{ij}}{1}$ generate) Sec. 3.0

Exercise give an alternative proof of algebra lemma by proving the ACC for R

(Key trick: $I = \bigcap \varphi_i^{-1}(I_{f_i})$ where $\varphi_i : R \to R_{f_i}$ is localisation.
You may need the famous Trick: $\operatorname{Spec} R = D_{f_1^N} \cup \ldots \cup D_{f_n^N}$ so $\sum r_i f_i^N = 1$)

Lemma (Hwk 3 ex 1(v),(vi)) X Noeth. scheme $\Longrightarrow$ every subset of X is quasi-compact.

3.2 Properties that are affine-local

Above we had a property ✯ of affine opens ("ring is Noetherian") satisfying

Affine-local conditions

1) $\operatorname{Spec} R \hookrightarrow X$ ✯ $\Longrightarrow \operatorname{Spec} R_f \hookrightarrow X$ ✯ $\forall f \in R$ ← so property is preserved by localisation

2) $\operatorname{Spec} R = \bigcup D_{f_i}$, $\operatorname{Spec} R_{f_i} \hookrightarrow X$ ✯ $\Longrightarrow \operatorname{Spec} R \hookrightarrow X$ ✯ ← can globalise from basic affines to affine

Claim $X = \bigcup \operatorname{Spec} R_i$ each has $\star$ $\Longrightarrow$ every open affine in X has $\star$ ← "if holds for a cover, it holds $\forall$ affine open"

Pf $\operatorname{Spec} R \hookrightarrow X \Longrightarrow \operatorname{Spec} R = \bigcup_{\text{finite}} D_{f_{ij}}$, $D_{f_{ij}} \subseteq \operatorname{Spec} R_i \overset{(1)}{\Longrightarrow} D_{f_{ij}} \star \overset{(2)}{\Longrightarrow} \operatorname{Spec} R \star$ □ ← (use USEFUL TRICK in 3.1)

Examples of $\star$: "ring is reduced", "ring is Noeth.", "ring is f.g. B-algebra" ← some fixed ring B ("base")

"locally of finite type over B"

so $\exists$ surj. hom of B-alg. $B[x_1, \ldots, x_n] \to$ ring

e.g. field k: Affine vars $X \subseteq \mathbb{A}^n$ loc. finite type$/k$.

3.3 Reduced schemes

$(X\,\mathcal{O}_X)$ reduced if all $\mathcal{O}_X(U)$ reduced rings ($=$ no nilpotents $\neq 0$)

Hwk 1 reduced $\Leftrightarrow$ stalks $\mathcal{O}_{X,x}$ are reduced ←(so "stalk–local property")

$\Leftrightarrow \forall p \in X$ has an open affine neighbourhood for a reduced ring

Rmk $\operatorname{Spec} R$ reduced $\Leftrightarrow$ R reduced (Pf "$\Rightarrow$" $R = \mathcal{O}_X(X)$, "$\Leftarrow$" R reduced $\overset{3.0}{\Leftrightarrow} R_p = \mathcal{O}_{X,p}$ reduced)

Lemma X reduced, $f, g \in \mathcal{O}_X(U)$ take same values $f(x) = g(x) \in k(x) = \mathcal{O}_{X,x}/m_x \Rightarrow f = g$

Pf. Take $f-g$, WLOG $g=0$. On affine, $k(p) \cong \operatorname{Frac}(R/p)$ so $f \in \bigcap p = \text{Nilradical}(R) = \{\text{nilpotents}\} = \{0\}$. □

(Don't confuse this with general fact $\forall$scheme : $f_x = g_x \in \mathcal{O}_{X,x}$ $\forall x \in U \Rightarrow f = g \in \mathcal{O}_X(U)$)

Claim (not that strong a condition e.g. $f, g: \mathbb{C} \to \mathbb{C}$, $f(z) = z$, $g(z) = \bar{z}$ different, but $f^{-1}(0) = g^{-1}(0)$, $\operatorname{Spec} \mathbb{C} = \{0\}$)

X reduced, $f, g: X \to Y$, $f = g$ as topological maps, $f = g$ on open dense set $\Rightarrow f = g$.

Pf enough show $f = g$ locally by sheaf property. WLOG $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} S$ (pick $\operatorname{Spec} S \subseteq f^{-1}(\operatorname{Spec} R)$ $= g^{-1}(\operatorname{Spec} R)$)

Let $s := f^{\#}(r) - g^{\#}(r) \in S$ need show $s = 0$ for each $r \in R$. ←(careful: $f^{\#} - g^{\#}$ is not ring hom)

$\{p \in \operatorname{Spec} S : s(p) = 0 \in k(p)\} = \mathbb{V}(s)$ closed & contains an open dense set, hence $s = 0$ by Lemma □

since $\{p : s_p = 0 \in \mathcal{O}_{X,p}\}$ contains open dense set by assumption

3.4 Irreducible schemes

Def Topological space X is irreducible if X is not a union of 2 proper closed sets : (means $\neq X$)

$X = C_1 \cup C_2 \Longrightarrow X = C_1$ or $X = C_2$ (where C_i closed)

Easy exercise If X irreducible:
- Any non-empty open $U \subseteq X$ is dense and irreducible
- Any two " " U_1, U_2 have $U_1 \cap U_2 \neq \emptyset$ (open, dense, irred)

Hwk 2 $(X, \mathcal{O}_X)$ irreducible $\Leftrightarrow$ all affine opens are irreducible

(Not enough to know it for an affine cover, can you see why?)

Hwk 1 $\operatorname{Spec} R$ irreducible $\Leftrightarrow$ $\operatorname{Nil}(R)$ prime ideal

$\Leftrightarrow R/\operatorname{Nil}(R)$ integral domain

$\Leftrightarrow \exists!$ generic point, namely $\operatorname{Nil}(R)$

Example $\mathbb{V}(I) = \operatorname{Spec}(R/I) \subseteq \operatorname{Spec} R$ irreducible $\Leftrightarrow \sqrt{I}$ prime ideal.
Since $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ as sets, ($\operatorname{Nil}(R/I) = \sqrt{I}$)
irred. closed subsets of $\operatorname{Spec} R$ are: $\mathbb{V}(p)$ for $p \in \operatorname{Spec} R$. So:
irred. components : if p minimal
(irred. & max w.r.t. $\subseteq$) (w.r.t. $\subseteq$)

Recall $p \in X$ generic point if closure $\bar{p} = X$ (p is dense)

Claim $(X, \mathcal{O}_X)$ irreducible $\Rightarrow \exists!$ generic point y, and $y \in$ every affine open $\neq \emptyset$

Pf affine open $\emptyset \neq U \subseteq X \overset{\text{ex.above}}{\Longrightarrow} U$ irred. $\overset{\text{Hwk 1}}{\Longrightarrow} \exists!$ generic pt $x \in U \Rightarrow \bar{x} \supseteq \bar{U} = X$ ($\bar{x}$ in X closed and $\supseteq U$)

Suppose $y \in X$ generic $\Rightarrow$ if $y \in X \setminus U$ then $\bar{y} \subseteq \overline{X \setminus U} = X \setminus U$ not dense, so $y \in U$, so $y = x$. □

Hwk 2 irreducible $\overset{\Longrightarrow}{\not\Longleftarrow}$ connected ←($X \neq U_1 \cup U_2$ for disjoint open $U_i \neq \emptyset$). Fact $\operatorname{Spec} R$ connected $\Leftrightarrow$ no idempotents $\neq 0, 1$ ← $r \in R$ with $r^2 = r$

Classifies connected components of $\operatorname{Spec} R$ in terms of idempotents

Exercise R Noetherian $\Rightarrow \exists!$ sequence of prime ideals $p_1, \ldots, p_n$ (up to reordering): $\begin{cases} \bigcap p_i = \operatorname{Nil}(R) \\ p_i \not\supseteq \bigcap_{j \neq i} p_j \end{cases}$

(Same Pf. as in C3.4) (in fact they are the minimal prime ideals of R)

$\Rightarrow \exists!$ sequence of irred. closed subsets $C_i = \mathbb{V}(p_i)$ (up to reordering): $\operatorname{Spec} R = \bigcup C_i$, $C_i \not\subseteq \bigcup_{j \neq i} C_j$

(which as top. subspaces are the irreducible components) as topological spaces

Warning: $q = (x^2) \subseteq k[x] = R \Rightarrow p = \operatorname{Nil}(R/q) = (x)$, $C = \operatorname{Spec}(R/p) = \{0\} = \operatorname{Spec}(R/q)$ as top. spaces, not as schemes

Non-examinable (see C3.4 Notes on Lasker–Noether theorem)

To recover the scheme $\operatorname{Spec}(R) = \bigcup \mathbb{V}(q_i)$, $\mathbb{V}(q_i) \not\subseteq \bigcup_{j \neq i} \mathbb{V}(q_j)$ (Noeth.) ← (so "irredundant": can't omit q_i)

need primary decomposition ←(like "unique factorization" but for ideals)

$\{0\} = q_1 \cap q_2 \cap \cdots \cap q_n \cap \cdots \cap q_m$ where q_i are primary ideals s.t. $q_i \not\supseteq \bigcap_{j \neq i} q_j$

$q \subseteq R$ primary ideal if zero divisors of R/q are nilpotent

(Equivalently: $ab \in q \Rightarrow a \in q$ or $b^N \in q$ some N ($\Leftrightarrow$ if $a, b \notin q$ then $a, b \in \sqrt{q}$)

Example p^N is primary if p prime ideal, e.g. $(3^4) \subseteq \mathbb{Z}$

Example $(18) = (2 \cdot 3^2) = (2) \cap (3^2) \subseteq \mathbb{Z}$ is primary decomposition.

Rmk $p = \sqrt{q}$ is prime ideal ("associated prime ideal") and is smallest prime ideal containing q.
So: $ab \in q, a \notin q \Rightarrow b \in p$

The q_i are not unique, but the $p_i = \sqrt{q_i}$ are unique (up to reordering)

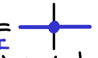
(the p_i are precisely the prime ideals arising as radicals of annihilators of elts of R)

The $\mathbb{V}(q_i)$ are called primary components : not unique as schemes, but are unique topologically. ($\mathbb{V}(q_i) = \mathbb{V}(p_i)$ as closed sets)

- WLOG $p_1 = \sqrt{q_1}, \ldots, p_n = \sqrt{q_n}$ are as in previous exercise: the minimal prime ideals
  (so $\operatorname{Nil}(R) = p_1 \cap \cdots \cap p_n$, which is the primary decomposition for $R/\operatorname{Nil}(R)$)
  give the isolated components $\mathbb{V}(q_i)$ (as top. subspace $= \mathbb{V}(p_i)$ irreducible comp.). These $q_1, \ldots, q_n$ are unique.
- The other $q_{n+1}, \ldots, q_m$ give rise to the embedded components $\mathbb{V}(q_j)$, $j \geq n+1$ (not unique).

(Note $p_j \supseteq p_i$ some i, so $\mathbb{V}(p_j) \subseteq \mathbb{V}(p_i) \subseteq \mathbb{V}(q_i)$ are closed subschemes, but $\mathbb{V}(q_j) \not\subseteq \mathbb{V}(p_i)$ as scheme)

Rmk Can apply above to R/I to get $\sqrt{I} = p_1 \cap \cdots \cap p_n$, $I = q_1 \cap \cdots \cap q_n \cap \cdots \cap q_m$, etc.

Example $I = (y^2, xy) \subseteq k[x,y] = R$, $X = \operatorname{Spec}(k[x,y]/I) =$  $\times$ $/\!/$ ← as top. space

$\sqrt{I} = q_1$, $I = q_1 \cap q_2$ for $q_1 = (y)$, $p_1 = (y)$ min prime, ← annihilator of $x \in R/I$; $\mathbb{V}(q_1)$ is isolated, irreducible

Think: functions vanishing on x-axis in $\mathbb{A}^2$, and " order 2 at 0. $q_2 = (x,y)^2$, $p_2 = (x,y)$ embedded prime, $\mathbb{V}(q_2) =$ "fattened origin" is embedded

not unique, e.g. could also pick (y^2, x). ← notice $p_2 \supseteq p_1$ so not minimal.

multiplicity $= l =$ max length of finite length ideals in $\mathcal{O}_{X,p_2}$ (max length of chain of ideals $I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_l = 0$)
In example: $I_0 = (\bar{x}) \supseteq (\bar{0}) = I_1$ ($/(y^2, xy)$)

3.5 Integral schemes

$(X, \mathcal{O}_X)$ integral if all $\mathcal{O}_X(U)$ ID ←(integral domain $=$ no zero divisors $\neq 0$)

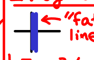
Hwk 2 $\Leftrightarrow \mathcal{O}_X(U)$ ID $\forall$ affine open U

Fact Localisation, Direct limits $\varinjlim$ $\}$ preserve ID property

Cor X integral $\Rightarrow \mathcal{O}_{X,x}$ ID (but not $\Leftarrow$)

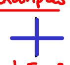
Hwk 2 $\boxed{X \text{ integral} \Leftrightarrow \text{reduced and irreducible}}$

2 Key Non-examples



"fat line"

$k[x,y]/(x^2)$ not reduced



$k[x,y]/(xy) \cong k[x] \oplus k[y]$ reducible : union of two axes

nonexaminable fact if X is locally Noeth: X integral $\Leftrightarrow \begin{cases} \cdot \text{ connected} \\ \cdot\ X = \bigcup \operatorname{Spec} R_i \text{, } R_i \text{ integral} \end{cases}$

$\operatorname{Spec} R$ integral $\Leftrightarrow$ R integral domain ← Example All irreducible affine varieties $X \subseteq \mathbb{A}^n$ ($\operatorname{Spec} k[X]$)

Claim $(X,\mathcal{O}_X)$ integral $\Rightarrow$ restrictions $\mathcal{O}_X(U)\longrightarrow\mathcal{O}_X(V)$ are injective (for $V\neq\emptyset$)

$\Rightarrow$ • all sections can be compared in $\mathcal{O}_{X,y}$ $\leftarrow$ y = generic point

• $K(y)\cong\mathcal{O}_{X,y}\cong \mathrm{Frac}\,\mathcal{O}_X(U)$ via restriction (any $U\neq\emptyset$) $\leftarrow$ called function field $K(X)$

Pf $\mathcal{O}_X(U)\to\mathcal{O}_X(V)\to\mathcal{O}_{X,y}$ so enough show $s_y=0\Rightarrow s=0$.

If show $s=0$ on every open affine $\subseteq U$ then $s_x=0$ all $x\in U$ so $s=0\in\mathcal{O}_X(U)$.

$\Rightarrow$ WLOG $U=\mathrm{Spec}\,R$, $y=\mathrm{Nil}(R)=\{0\}$ (since R is ID), so $\mathcal{O}_X(U)\longrightarrow\mathcal{O}_{X,y}$ becomes $R\hookrightarrow R_{(0)}=\mathrm{Frac}\,R$, $r\longmapsto\frac{r}{1}$ inj. since R is ID. Thus $s_y=0\Rightarrow s=0$. □

Classical Alg. Geometry $X\subseteq\mathbb{A}^n$ irred. affine var (so $\mathrm{Spec}\,k[X]$) $\Rightarrow$

$$\begin{tikzcd}
\mathcal{O}_X(X) \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_X(D_f) \arrow[r] \arrow[d, equal] \arrow[rr, bend left] & \mathcal{O}_{X,p} \arrow[d, equal] & \mathcal{O}_{X,(0)} \arrow[d, equal] \arrow[r, equal] & k(X) \\
k[X] \subseteq & k[X]_f \subseteq & k[X]_p \subseteq & \mathrm{Frac}\,k[X] \arrow[ur, equal] &
\end{tikzcd}$$

3.6 Properties of morphisms

$\leftarrow$ all properties we list are preserved when compose such morphs

A morph of schemes $f: X\longrightarrow Y$ is: (will suppress $f^\#,\mathcal{O}_X,\mathcal{O}_Y$ from notation)

① **affine**: equivalent conditions:
- f^{-1}(affine open) is $\boxed{\text{affine}}$
- $\exists$ affine open cover V_i of Y, $f^{-1}(V_i)$ $\boxed{\text{affine}}$
- $\forall$ affine open cover V_i of Y, $f^{-1}(V_i)$ $\boxed{\text{affine}}$

② **quasi-compact**: replace $\boxed{\text{affine}}$ by $\boxed{\text{quasi-compact}}$

③ **locally of finite type**: • $\forall$ affine opens $U\subseteq X$, $V\subseteq Y$ with $f(U)\subseteq V$,
$f^\#:\mathcal{O}_Y(V)\longrightarrow\mathcal{O}_X(U)$ finite type

(meaning: $\mathcal{O}_Y(V)\xrightarrow{f^\#_V}\mathcal{O}_X(f^{-1}V)\xrightarrow{\text{rest}}\mathcal{O}_X(U)$)

(Rings: $A\to B$ finite type means B f.g. as A-alg., i.e. $\exists$ surj $A[x_1,\dots,x_n]\longrightarrow B$ of A-algs)

$\Updownarrow$

• $\exists$ open affine covers $Y=\bigcup V_i$, $f^{-1}(V_i)=\bigcup U_{ij}$
$f^\#:\mathcal{O}_Y(V_i)\to\mathcal{O}_X(U_{ij})$ finite type

④ **finite type**: ② + ③ : quasi-compact & locally finite type ($\Leftrightarrow$ this holds for finite # of U_{ij} for each i)

⑤ **closed immersion**: iso onto a closed subscheme. (see 1.14, 1.15)

Explicitly: $f: X\xrightarrow{\text{homeo}} f(X)\overset{\text{closed}}{\subseteq} Y$
$f^\#:\mathcal{O}_Y\longrightarrow f_*\mathcal{O}_X$ surjective (so ideal sheaf $\mathcal{J}=\ker f^\#$) $\leftarrow$ automatically quasi-coherent.

Idea: functions on X are restrictions of functions of Y

$\Updownarrow$

• $\forall$ aff. open $U=\mathrm{Spec}\,R\subseteq Y$ $\exists$ ideal $I\subseteq R$ s.t. $f^{-1}(U)\cong\mathrm{Spec}(R/I)$

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, "\cong"] \arrow[d, "f"] & \mathrm{Spec}(R/I) \arrow[d] \\
U \arrow[r, "\cong"] & \mathrm{Spec}\,R
\end{tikzcd}$$

$\Updownarrow$

• $\exists$ aff. cover $Y=\bigcup\mathrm{Spec}\,R_i$, ideals $I_i\subseteq R_i$, $f^{-1}(\mathrm{Spec}\,R_i)=\mathrm{Spec}(R_i/I_i)$

RmK Can specify an ideal $I\subseteq R$ by a surjective ring hom $R\twoheadrightarrow S$ (get $I=\ker$) Conversely given I consider $S=R/I$

Example $X=Y_{red}\subseteq Y$ closed subscheme: $X=Y$ as topological space and (reduction of Y: it's reduced) sheaf of ideals $\mathcal{J}(U)=\{s\in\mathcal{O}_Y(U): s(p)=0\in K(p),\forall p\in U\}$ (so $\mathcal{O}_X=\mathcal{O}_Y/\mathcal{J}$)

Note locally: on $U=\mathrm{Spec}\,R$, $\mathcal{J}(U)=\{s\in R: s\in\bigcap p=\mathrm{Nil}(R)=\{\text{nilpotents}\}\}$, so locally $\mathcal{J}$ agrees with $\mathrm{Nil}(\mathcal{O}_Y)$, indeed $\mathcal{J}$ is the sheafification of $\mathrm{Nil}(\mathcal{O}_Y)$ $\leftarrow$ need not be sheaf, e.g. $Y=\bigsqcup_n Y_n$, $Y_n=\mathrm{Spec}(\mathbb{Z}/2^n)$
$2\in\mathcal{O}_Y(Y)$, $2\notin\mathrm{Nil}(\mathcal{O}_Y(Y))$ but $2\in\mathrm{Nil}(\mathcal{O}_Y(Y_n))$, $2\in\mathcal{J}(X)$

⑥ **open immersion**: iso onto an open subscheme $\leftarrow$ $U\overset{\text{open}}{\subseteq}Y$, $\mathcal{O}_U=\mathcal{O}_Y|_U$

Explicitly: $f: X\xrightarrow{\text{homeo}} f(X)\overset{\text{open}}{\subseteq} Y$ (idea: functions on X are the same as " " Y locally)
$f^\#:\mathcal{O}_Y\longrightarrow f_*\mathcal{O}_X$ iso ($\Leftrightarrow$ iso on stalks $f^\#_x:\mathcal{O}_{Y,fx}\longrightarrow\mathcal{O}_{X,x}$)

⑦ **flat**: all $\mathcal{O}_{Y,fx}\longrightarrow\mathcal{O}_{X,x}$ are flat ring homs

Not intuitively clear, but ensures that fibers of f vary in a controlled way:
Many invariants of fibers like dimension, do not change unless you "expected" it!
It is weaker than saying the fibers are locally iso e.g. it allows two points to collide as vary fiber.

Algebra: R-mod M is flat if $M\otimes_R\cdot$ is exact functor on R-mods
$\varphi:R\to S$ flat ring hom means S flat R-mod (using $r\cdot s=\varphi(r)s$)

Basic facts

1) $M\otimes_R\cdot$ always right exact, so M flat R-mod $\Leftrightarrow$ $N_1\hookrightarrow N_2$ implies $M\otimes_R N_1\hookrightarrow M\otimes_R N_2$ ($\hookrightarrow$: injective (hom of R-mods))

Fact Enough to check $M\otimes_R I\hookrightarrow M\otimes_R R$ $\forall$ f.g. ideal $I\overset{\subseteq}{\hookrightarrow} R$.

2) M free $\Rightarrow$ M flat (Pf. $M\cong\bigoplus_{i\in I}R\Rightarrow M\otimes N\cong\bigoplus_{i\in I}N$. □)

Example $\prod_{\text{infinite}}\mathbb{Z}$ is not free $\mathbb{Z}$-mod, but it is flat. An abelian gp is flat $\mathbb{Z}$-mod $\Leftrightarrow$ torsion free (so no elts $\neq0$ of finite order)

Non-example $\mathbb{Z}/n$ is not flat $\mathbb{Z}$-mod: $\mathbb{Z}\overset{\cdot n}{\hookrightarrow}\mathbb{Z}$ then $\cdot\otimes\mathbb{Z}/n$ get $\mathbb{Z}/n\xrightarrow{\cdot0}\mathbb{Z}/n$ not inj.

Fact (Lazard) R-mod M is flat $\Leftrightarrow$ $M=\varinjlim M_i$ some f.g. free R-mods M_i

3) R local, M finite R-mod (so $M=\sum_{\text{finite}}Rm_i$): M flat $\Leftrightarrow$ M free $\leftarrow$ $\mathcal{O}_{Y,f(x)}$ local but $\mathcal{O}_{X,x}$ is rarely finite over it

4) $A\longrightarrow B$ flat, $B\to C$ flat $\Rightarrow$ $A\longrightarrow C$ flat

Pf $N_1\hookrightarrow N_2$ A-mods $\Rightarrow$ $B\otimes_A N_1\hookrightarrow B\otimes_A N_2$ B-mods $\Rightarrow$ $\underbrace{C\otimes_B B}_{=C}\otimes_A N_1\hookrightarrow\underbrace{C\otimes_B B}_{=C}\otimes_A N_2$ □

5) $A\longrightarrow B$ flat $\Rightarrow$ $A_p\longrightarrow B_p=B\otimes_A A_p$ flat $\forall p\in\mathrm{Spec}\,A$

Pf $N_1\hookrightarrow N_2$ A_p-mods $\Rightarrow$ $N_1\hookrightarrow N_2$ A-mods (via $A\to A_p$) $\Rightarrow$ $B\otimes_A N_1\hookrightarrow B\otimes_A N_2$ □ ($=B\otimes_A A_p\otimes_{A_p}N_1=B_p\otimes_{A_p}N_1$)

6) Ring hom $\varphi:A\to B$, multiplicative sets $S\subseteq A$, $T\subseteq B$ with $\varphi(S)\subseteq T$, then
$\psi: S^{-1}B=S^{-1}A\otimes_A B\longrightarrow T^{-1}B$, $\frac{a}{s}\otimes b\longmapsto\frac{\varphi(a)b}{\varphi(s)}$ factorizes as $S^{-1}B\xrightarrow{\cong}(\varphi(S))^{-1}B\xrightarrow{\text{localisation}}T^{-1}B$
$\frac{a}{s}\otimes b\longmapsto\frac{\varphi(a)b}{\varphi(s)}\longrightarrow\frac{\varphi(a)b}{\varphi(s)}$

Since isos of rings and localisation are exact functors, get ψ flat.

Example: $p\subseteq B$ prime ideal, $q=\varphi^{-1}p\subseteq A$ prime ideal, $S=A\setminus q$, $T=B\setminus p$ $\Rightarrow$ $B_q=B\otimes_A A_q\longrightarrow B_p$ flat

Theorem $\boxed{\varphi:A\longrightarrow B \text{ flat ring hom} \Leftrightarrow \varphi^\#:\mathrm{Spec}\,B\longrightarrow\mathrm{Spec}\,A \text{ flat}}$

Pf ($\Rightarrow$) $A\to B$ flat $\Rightarrow$ $A_q\longrightarrow B_q$ flat for $q=\varphi^{-1}p$ ($=\varphi^\# p$) by (5), $B_q\longrightarrow B_p$ flat by (6) $\overset{(4)}{\Longrightarrow}$ $A_q\to B_p$ flat.

($\Leftarrow$) Recall $\mathrm{Ker}(B\otimes_A N_1\xrightarrow{\psi}B\otimes_A N_2)\neq0\overset{\text{(see 3.0)}}{\Longleftrightarrow}\ker\psi_p\neq0$ $\forall p\in\mathrm{Spec}\,B$.

$\mathrm{Ker}(N_1\to N_2)=0\Rightarrow\mathrm{Ker}(A_q\otimes_A N_1\to A_q\otimes_A N_2)=0\underset{\text{flatness}}{\Longrightarrow}\mathrm{Ker}(\overbrace{\underbrace{B_p\otimes_{A_q}A_q\otimes_A N_1}_{=B_p\otimes_A N_1}\to\underbrace{B_p\otimes_{A_q}A_q\otimes_A N_2}_{=B_p\otimes_A N_2}}^{=\psi_p})=0$ □

Motivation: Deformations (see Homework 2 ex. 6)

Flatness $\Rightarrow$ 1-parameter families of schemes have "limits".

Fact
$B=\mathrm{Spec}\,k[t]$
$B^*=B\setminus0=\mathrm{Spec}\,k[t,t^{-1}]$
$X\subseteq\mathbb{A}^n_B$ closed subscheme

$$\begin{tikzcd}
X \arrow[r, phantom, "\subseteq"] \arrow[dr, dotted, "\pi"'] & \mathbb{A}^n_B \arrow[d] \\
 & B
\end{tikzcd}$$

(also $k[[t]]$, $k[t]_{(t)}$ work)

will define later, here $\mathbb{A}^n_B=\mathrm{Spec}\,k[t,x_1,\dots,x_n]$

π flat over 0 $\Leftrightarrow$ fiber X_0 is "limit" $\lim_{b\to0}X_b$

defined rigorously later in 5.1, for now $X_b=\pi^{-1}(b)=\mathrm{Spec}\,K(b)\times_B X$ $=\mathrm{Spec}(K(b)\otimes_{k[t]}R)$ if $X=\mathrm{Spec}\,R$

($\lim_{b\to0}X_b$ means: fiber over 0 of closure of $X^*=\pi^{-1}(B^*)$ (see 5.1: $B^*\times_B X$)

so $\Leftrightarrow$ $\overline{X^*}=X$

Fact Another nice property of flat morphs $f: X \longrightarrow B$, for B, X locally Noeth.:

$$\dim_x f^{-1}(b) = \dim_x X - \dim_b B \quad \text{where } b = f(x)$$

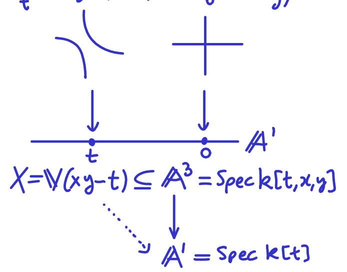
so dimensions of fibers don't "jump" unexpectedly.

$\dim_x X$ = max length d of chain of irreducible closed Z_i: $\{x\} \subseteq Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_d \subseteq U$ minimizing over open $x \in U \subseteq X$

example: $\mathbb{A}^2$ has dim=2
$\{p\} \subseteq \text{line} \subseteq \text{plane}$
$\;\;\| \qquad\quad \| \qquad\quad \|$
$Z_0 \subseteq Z_1 \subseteq Z_2$

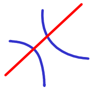
Geometrical motivation (very loosely)

$X_t = \mathbb{V}(xy-t) \subseteq \mathbb{A}^2 \quad X_0 = \mathbb{V}(xy)$

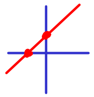


$X = \mathbb{V}(xy-t) \subseteq \mathbb{A}^3 = \operatorname{Spec} k[t,x,y]$

$\mathbb{A}^1 = \operatorname{Spec} k[t]$



how many times does a line in $\mathbb{A}^2$ intersect fiber?



if have a family for which intersection number is constant, it may be easy to calculate for a degenerate fiber

in such theorems you will almost always see the flatness assumption

Remarks about calculating closures of sets in $X = \operatorname{Spec} R$

1) $p \in \operatorname{Spec} R \implies \boxed{\overline{p} = V(p)}$

Pf $p \in V(p) \Rightarrow \overline{p} \subseteq V(p)$ (since $V(p)$ closed)

Converse: $p \in \overline{p} \overset{\text{say}}{=} V(I) \implies I \subseteq p$
$q \in V(p) \implies p \subseteq q$ $\Big\} \Rightarrow I \subseteq p \subseteq q \implies q \in V(I)$ □

Example $X^* = V_*(p_1 \cdot p_2 \cdot \cdots \cdot p_k) \subseteq \mathbb{A}^n_{B^*}$, $p_j \subseteq R[x_1, \ldots, x_n, t, t^{-1}]$ prime ideals
$B^* = \operatorname{Spec} R[t,t^{-1}]$
$= V_*(p_1) \cup \ldots \cup V_*(p_k)$ where $V_*(\cdot)$ is $V(\cdot)$ calculated in $\mathbb{A}^n_{B^*}$
$\Rightarrow \overline{X^*} = V(p_1) \cup \ldots \cup V(p_k) \subseteq \mathbb{A}^n_B$
$= V(p_1 \cdot p_2 \cdots p_k)$
since $p_i \in X^* \subseteq \overline{X^*}$ and $p_i \in \overline{V_*(p_i)} \subseteq V(p_i) = \overline{p_i}$

Recall topology:
X topological space
$Y \subseteq X$ top. subspace
$\overline{Y} = \bigcap_{\substack{C \text{ closed} \\ Y \subseteq C}} C$
so any closed $C \supseteq Y$ satisfies $\overline{Y} \subseteq C$. Also:
$\overline{Y_1 \cup \ldots \cup Y_n} = \overline{Y_1} \cup \ldots \cup \overline{Y_n}$
Pf $Y_i \subseteq Y_1 \cup \ldots \cup Y_n$
$\Rightarrow \overline{Y_i} \subseteq \overline{Y_1 \cup \ldots \cup Y_n}$
converse:
$Y_1 \cup \ldots \cup Y_n \subseteq \underbrace{\overline{Y_1} \cup \ldots \cup \overline{Y_n}}_{\text{closed}}$
$\Rightarrow \overline{Y_1 \cup \ldots \cup Y_n} \subseteq \overline{Y_1} \cup \ldots \cup \overline{Y_n}$.

2) For $\varphi: R \longrightarrow S$ ring hom, $\alpha: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, $\alpha(p) = \varphi^{-1} p$:

Given $C = V(J) \subseteq \operatorname{Spec} S$, $\boxed{\overline{\alpha(C)} = V(\varphi^{-1} J)}$
(J radical)

Pf $J = \sqrt{J} = \bigcap_{\substack{J \subseteq p \\ p \in \operatorname{Spec} S}} p \implies \varphi^{-1} J = \bigcap_{J \subseteq p} \varphi^{-1} p$
(so $p \in C$)
$\alpha(p) = \varphi^{-1} p \in V(\varphi^{-1} J)$ ($\varphi^{-1} J \subseteq \varphi^{-1} p$)
$\alpha C \subseteq V(\varphi^{-1} J)$

since $\alpha(C) \subseteq \overline{\alpha(C)} \overset{\text{say}}{=} V(I)$, $I \subseteq \varphi^{-1} p$
$\varphi^{-1}(p) \in$, $\forall p \in \operatorname{Spec} S$, $J \subseteq p$
$\Rightarrow I \subseteq \varphi^{-1} J$
$\underset{\overline{\alpha C}}{\overset{\|}{V(I)}} \supseteq \underset{\alpha C}{\overset{\cup\!\shortmid}{V(\varphi^{-1} J)}}$. □

Example $S = R_f$ localisation, $f \in R$, if $\varphi: R \hookrightarrow R_f$ injection then $\varphi^{-1} J = R \cap J$
e.g. $X^* = V(J) \subseteq \mathbb{A}^n_{B^*}$ for $B = \operatorname{Spec} R[t]$, $B^* = \operatorname{Spec} R[t, t^{-1}]$
so $\mathbb{A}^n_B = \operatorname{Spec} R[x_1, \ldots, x_n, t]$, $\mathbb{A}^n_{B^*} = R[x_1, \ldots, x_n, t, t^{-1}]$

$\Rightarrow \boxed{\overline{X^*} = V(R[x_1, \ldots, x_n, t] \cap J) \subseteq \mathbb{A}^n_B}$ is the closure

RmK Also know inverse images of closed sets: $\boxed{\alpha^{-1}(V(I)) = V(\langle \varphi I \rangle)}$

Pf $p \in \alpha^{-1}(V(I)) \Leftrightarrow \alpha p = \varphi^{-1}(p) \in V(I) \Leftrightarrow I \subseteq \varphi^{-1}(p) \Leftrightarrow \varphi I \subseteq p \Leftrightarrow p \in V(\langle \varphi I \rangle)$. □

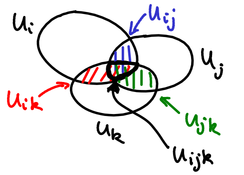
4. GLUING THEOREMS

4.1 Gluing sheaves

$X = \bigcup U_i$ open cover, abbreviate $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$

F_i sheaf on U_i

$\varphi_{ij}: F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ji}}$



Compatibility conditions
1) $\varphi_{ii} = \mathrm{id}$
2) $\varphi_{ji} = \varphi_{ij}^{-1}$
3) $\varphi_{ik}|_{U_{ijk}} = \varphi_{jk} \circ \varphi_{ij}|_{U_{ijk}}$

Example F sheaf on X, $F_i := F|_{U_i}$ (so $F_i(V) = F|_{U_i}(V) = F(U_i \cap V)$, $\forall$ open $V \subseteq U_i$)
φ_{ij} = isos induced by double restrictions (iso of functors $\cdot|_{U_i}|_{U_{ij}} \cong \cdot|_{U_j}|_{U_{ij}}$)

Theorem $\exists$, up to unique iso, a sheaf F on X with isos

$$\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$$

s.t. $\psi_j^{-1} \circ \varphi_{ij} \circ \psi_i|_{U_{ij}}$ is the natural iso $F|_{U_i}|_{U_{ij}} \cong F|_{U_j}|_{U_{ji}}$

$$\begin{tikzcd} F|_{U_i}|_{U_{ij}} \arrow[r, "\psi_i"] \arrow[d, "\cong"'] & F_i|_{U_{ij}} \arrow[d, "\varphi_{ij}"] \\ F|_{U_j}|_{U_{ji}} \arrow[r, "\psi_j"] & F_j|_{U_{ji}} \end{tikzcd}$$

Pf Let $E = \bigsqcup_i \bigsqcup_{x \in U_i} (F_i)_x \Big/$ equivalence relation $(F_i)_x \xrightarrow[\varphi_{ij}]{\cong} (F_j)_x$ for $x \in U_{ij}$
(using conditions)

$F(U) = \{ s: U \to E : s$ is locally a section of some $F_i \}$. □

($\forall x \in U, \exists i, \exists$ open $x \in V_i \overset{\subseteq U}{\subseteq} U_i$, $\exists t \in F_i(V_i)$, $s(y) = t_y \overset{\in (F_i)_y}{} \; \forall y \in V_i$)

Theorem Given sheaves F, G constructed as above from local data F_i, φ_{ij} on U_i; G_i, ψ_{ij}
a morph $f: F \to G$ can be uniquely defined from data:

- morphs $f_i : F_i \longrightarrow G_i$
- compatibility condition: $\psi_{ij} \circ f_i|_{U_{ij}} = f_j|_{U_{ji}} \circ \varphi_{ij}$

so:

$$\begin{tikzcd} F_i|_{U_{ij}} \arrow[r, "\varphi_{ij}"] \arrow[d, "f_i"'] & F_j|_{U_{ji}} \arrow[d, "f_j"] \\ G_i|_{U_{ij}} \arrow[r, "\psi_{ij}"] & G_j|_{U_{ji}} \end{tikzcd}$$

s.t. via identifications $F|_{U_i} \simeq F_i$, $G|_{U_i} \simeq G_i$ recover $f|_{U_i} = f_i$

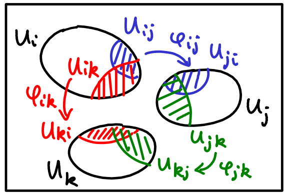
4.2 Gluing schemes

U_i schemes, $U_{ij} \subseteq U_i$ open subschemes ($U_{ii} = U_i$)

$\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}$ isos ← (think "go from U_i to U_j")

gluing conditions
1) $\varphi_{ii} = \mathrm{id}$
2) $\varphi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{ji} \cap U_{jk}$
3) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ when restricted as maps $U_{ij} \cap U_{ik} \to U_k$

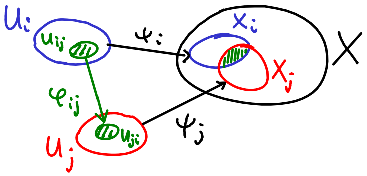
(case $k = i$: $\varphi_{ji}^{-1} = \varphi_{ij}$)



Example if $U_i \subseteq X$ open subschemes, can take $U_{ij} = U_i \cap U_j \subseteq X$ with $\varphi_{ij} = id$

Claim (exercise) $\exists$ unique (up to iso) scheme X with open cover $X = \cup X_i$:

- isos of schemes $U_i \xrightarrow[\psi_i]{\cong} X_i$
- $$\begin{tikzcd} U_{ij} \arrow[r, "\psi_i", "\cong"'] \arrow[d, "\varphi_{ij}"', "\cong"] & X_i \cap X_j \arrow[d, "id"] \\ U_{ji} \arrow[r, "\cong", "\psi_j"'] & X_i \cap X_j \end{tikzcd}$$



Gluing Lemma Suppose we built X as above

$\Rightarrow f : X \to Y$ morph can be uniquely defined from morphs $f_i : X_i \to Y$ s.t.

compatibility condition:
$$\begin{tikzcd} X_i \cap X_j \arrow[r] \arrow[d, "id"'] & X_i \arrow[dr, "f_i"] & \\ X_j \cap X_i \arrow[r] & X_j \arrow[r, "f_j"'] & Y \end{tikzcd} \quad ⊛$$

Pf Continuous map: $f : X \to Y$ defined by $f|_{X_i} = f_i$ (compatibly)

on sheaves need $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ $\leftarrow$(recall get $\mathcal{O}_Y \to f_*\mathcal{O}_X$ by adjunction)

$(f^{-1}\mathcal{O}_Y)|_{X_i} = f|_{X_i}^{-1}\mathcal{O}_Y = f_i^{-1}\mathcal{O}_Y$ $\leftarrow$($X_i \xrightarrow{\psi_i} X$ inclusion, then $\psi_i^{-1} f^{-1}\mathcal{O}_Y = (f \circ \psi_i)^{-1}\mathcal{O}_Y$

$f_i^\# \in Mor(\mathcal{O}_Y, (f_i)_*\mathcal{O}_{X_i}) \cong Mor(f_i^{-1}\mathcal{O}_Y, \mathcal{O}_{X_i})$ and $\mathcal{O}_{X_i} = \mathcal{O}_X|_{X_i}$ since open subsc.

Finally we can glue the $f_i^\# : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_X|_{X_i}$ by ⊛ to get $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. □

Consequence (X, Y schemes) $h_Y|_{Top(X)^{op}} : Top(X)^{op} \longrightarrow Sets$ is a sheaf of sets.
$$U \longmapsto h_Y(U) = Mor(U, Y)$$

4.3 Affine space by gluing (see Homework for projective space)

comes with morph $\mathbb{A}^n_R \to Spec\, R$ via constant polys $R \to R[x_1, .., x_n]$.

Affine n-space over Spec R: $\mathbb{A}^n_R := Spec\, R[x_1, .., x_n]$ $(=: \mathbb{A}^n_{Spec R})$

$$\begin{tikzcd} \mathbb{A}^n_S \arrow[r] \arrow[d] & \mathbb{A}^n_R \arrow[d] \\ Spec\, S \arrow[r] & Spec\, R \end{tikzcd}$$

Rmk $R \to S$ ring hom $\Rightarrow$ hom on polys (so: $R[x_1, .., x_n] \to S[x_1, .., x_n]$) $\overset{Spec}{\Longrightarrow}$ $\mathbb{A}^n_S \longrightarrow \mathbb{A}^n_R$

Example $R \to R_f \Rightarrow \mathbb{A}^n_{R_f} \longrightarrow \mathbb{A}^n_R$ is the basic open set of $\mathbb{A}^n_R$ for $f \in R \subseteq R[x_1, .., x_n]$

If $U \subseteq Spec\, R$ open $\Longrightarrow U = \cup D_{f_i} \Rightarrow \mathbb{A}^n_U := \cup \mathbb{A}^n_{R_{f_i}} \subseteq \mathbb{A}^n_R$ $\leftarrow$(glued along $Spec\, R_{f_i f_j} = D_{f_i} \cap D_{f_j}$
(some $f_i \in R$) $\quad$ ↖open subsc.

X scheme, affine n-space over X: $\mathbb{A}^n_X := \bigcup \mathbb{A}^n_{X_i}$ where $X = \cup X_i$ affine open cover

$$\left(\begin{tikzcd} \mathbb{A}^n_X \arrow[d] & = & \cup \mathbb{A}^n_{X_i} \arrow[d] \\ X & = & \cup X_i \end{tikzcd}\right)$$

(notice $\mathbb{A}^n_{X_i} = \bigcup_j \mathbb{A}^n_{X_i \cap X_j}$, then identify these copies) ↖glued along $\mathbb{A}^n_{X_i \cap X_j}$ $\leftarrow$ open in affine X_i

Claim $\mathbb{A}^n = Spec\, \mathbb{Z}[x_1, .., x_n]$ represents functor $F : Sch^{op} \to Sets$, $X \mapsto \left\{ \begin{array}{l} \text{Morphs } \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{O}_X \text{ s.t. } \forall U, \\ \mathcal{O}_X(U)^{\oplus n} \to \mathcal{O}_X(U) \text{ is hom of } \mathcal{O}_X(U)\text{-mod} \end{array} \right\}$

Pf $F|_{Top(X)^{op}}$ is a sheaf of sets (easy to check: can glue morphs since $\mathcal{O}_X$ sheaf)

$h_{\mathbb{A}^n}|_{Top(X)^{op}}$ " by Consequence above. Thus if the two functors agree (natural iso) on affines then by sheaf property they agree everywhere. For affine $X = Spec\, R$ just need compare global sections

$F(Spec\, R) = Hom_R(R^n, R)$ $\leftarrow$(here: R-mod homs!)

$h_{\mathbb{A}^n}(Spec\, R) = Mor(Spec\, R, \mathbb{A}^n) \cong Hom(\mathbb{Z}[x_1, .., x_n], R)$

in both cases just need specify where generators go $\left\{ \begin{array}{l} e_i = (0, .., 1, 0, .., 0) \mapsto r_i \\ x_i \longmapsto r_i \end{array} \right.$ □

5. PRODUCTS

5.0 Products in category theory

category theory: $\mathcal{C}$ cat., $C_i \in \mathcal{C}$

product $C_1 \times \dots \times C_n$ (if exists) is an object with morphs π_i to C_i, s.t.

$$\boxed{\begin{tikzcd} \forall Z \arrow[d, dashed, "\exists!"'] \arrow[dr, "\forall p_i"] & \\ C_1 \times \dots \times C_n \arrow[r, "\pi_i"'] & C_i \end{tikzcd}}$$

$\leftarrow$ Yoneda/functor of points interpretation: $\quad$ ↙product of sets

$F : \mathcal{C}^{op} \longrightarrow Sets$, $F(Z) = \prod Mor_{\mathcal{C}^{op}}(C_i, Z) = \prod h_{C_i}(Z)$

Is it representable? if so, call the object $\prod C_i$, $h_{\prod C_i} \cong F = \prod h_{C_i}$

Explicitly: $(p_i) \in \prod h_{C_i}(Z)$ gives unique $\in h_{\prod C_i}(Z) = Mor(Z, \prod C_i)$

Why $\exists$ maps π_j? $\exists$ projections of sets $h_{\prod C_i}(Z) \cong \prod h_{C_i}(Z) \to h_{C_j}(Z)$

but $Mor(h_{\prod C_i}, h_{C_j}) \cong Mor(\prod C_i, C_j) \ni \pi_j$.

coproduct $C_1 \sqcup \dots \sqcup C_n$:

$$\boxed{\begin{tikzcd} \forall Z & \\ C_1 \sqcup \dots \sqcup C_n \arrow[u, dashed, "\exists!"] & C_i \arrow[l, "\pi_i"] \arrow[ul, "\forall p_i"'] \end{tikzcd}}$$

Examples Sets / Top.spaces: $\times$ = product, π_i = projections, $\sqcup$ = disjoint union, π_i are inclusions

Vectorspaces/abeliangps/modules: " , $\sqcup$ = direct sum, π_i are inclusions.

Rings: " , $\sqcup$ = tensor product, $\pi_i(r) = 1 \otimes .. \otimes r \otimes .. \otimes 1$

Fix $B \in \mathcal{C}$ ("base")

Category of B-objects: $\mathcal{C}/B$

obj: morphs $C \to B$ in $\mathcal{C}$, **morphs**:
$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & D \arrow[dl] \\ & B & \end{tikzcd}$$

(think of B as a parameter space and C as a family parametrised by B)

IMPORTANT EXAMPLES:
All schemes X have canonical $X \to Spec\, \mathbb{Z}$ by gluing canonical maps on affines: $Spec\, R \longrightarrow Spec\, \mathbb{Z}$ from $\mathbb{Z} \longrightarrow R$, $1 \mapsto 1$

Schemes over field k means have $X \longrightarrow Spec\, k$, same as saying all $\mathcal{O}_X(U)$ are k-algebras and restrictions are k-alg. homs

fiber product $C \times_B D$ is the product in $\mathcal{C}/B$ of $C \xrightarrow{f} B$, $D \xrightarrow{g} B$ (if exists)

(or pullback, or Cartesian square)

Similarly get $C_1 \times_B \dots \times_B C_n$

so:
$$\begin{tikzcd} \forall Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\forall p_D"] \arrow[ddr, bend right, "\forall p_C"'] & & \\ & C \times_B D \arrow[r, "\pi_D"] \arrow[d, "\pi_C"'] & D \arrow[d, "g"] \\ & C \arrow[r, "f"] & B \end{tikzcd}$$

Functor of points interpretation:
$Hom(Z, C \times_B D) \cong Hom(Z, C) \times_{Hom(Z,B)} Hom(Z, D)$
So we are asking whether $h_C \times_{h_B} h_D$ is representable

Example for Sets or Top. spaces: $C \times_B D = \{ (c, d) \in C \times D : f(c) = g(d) \in B \}$

Pushout The opposite diagram (reverse arrows)

Example: for Rings the pushout of $B \to C$, $B \to D$ is the tensor product $C \otimes_B D$

Example: $B \xrightarrow{f} C$, $B \xrightarrow{g} D$ inclusions of open subschemes, then pushout $C \sqcup_B D$ is the gluing! ↙Sec. 4.2

Exercise: (co)product, fiber product, pushout are unique up to unique iso if they exist.

(Hint: compose unique maps between them (s.t. diagram commutes) then composites = id by uniqueness of self-map)

Examples of fiber products in cat. of Sets or TopSpaces: $C \times_B D = \{(c, d) : f(c) = g(d)\} \subseteq C \times D$

$B = point \Longrightarrow C \times_B D = C \times D$

$C \xrightarrow{\subseteq} B$, $D \xrightarrow{\subseteq} B \Longrightarrow C \times_B D \cong C \cap D$ (via $(c, c) \leftrightarrow c$)

$D \xrightarrow{\subseteq} B \Longrightarrow C \times_B D \cong f^{-1}(D) \subseteq C$ for example $D = point = b \in B$ get fiber $f^{-1}(b)$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d, "\Delta = diagonal"] \\ C \arrow[r, "(f,g)"] & B \times B \end{tikzcd} \Longrightarrow E := C \times_{B \times B} B = \{(c, b) : f(c) = g(c) = b\} \cong \{c \in C : f(c) = g(c)\}$$ "equalizer"

5.1 Fiber products exist in Schemes/B

<u>RmK</u> $B = \operatorname{Spec}\mathbb{Z}$ gives $X\times_B Y = X\times Y$

Fix scheme B, consider category Schemes$/B$

<u>Theorem</u> fiber products $X_1 \times_B \cdots \times_B X_n$ exist

Inductively suffices to do case $n=2$. First need some algebraic preliminaries

An <u>A-algebra</u> R is a ring R together with a ring hom $A \xrightarrow[\psi]{} R$
(A ring) ($\Rightarrow R$ is A-mod via $a\cdot r = \psi(a)r$)

R,S A-algebras $\Rightarrow$ $(R\otimes_A S) = \dfrac{\text{free } R\text{-alg. on } R\times S}{\text{relations}}$ (so general element is $\sum r_i \otimes s_i$; so "generators" are $r\otimes s$)

relations: i) $\otimes$ is bilinear
ii) $a\cdot(r\otimes s) = (\psi_R(a)\cdot r)\otimes s = r\otimes(\psi_S(a)\cdot s)$. (often drop ψ_R, ψ_S from notation.)

In particular $A \longrightarrow R\otimes_A S$ is $a \longmapsto a\cdot(1\otimes 1) = \psi_R(a)\otimes 1 = 1\otimes\psi_S(a)$

The product on generators: $(r\otimes s)\cdot(r'\otimes s') = rr'\otimes ss'$.

<u>RmK</u> R,S rings $\Rightarrow$ $R\otimes S = R\otimes_{\mathbb{Z}} S$

<u>Facts</u>

1) $R\otimes_R S \cong S$ (via $\sum r_i\otimes s_i \longmapsto \sum r_i s_i$)
2) $R[x_1,\dots,x_n]\otimes_R R[y_1,\dots,y_m] \cong R[x_1,\dots,x_n,y_1,\dots,y_m]$
3) $(S/I)\otimes_R T \cong (S\otimes_R T)/(I\otimes 1)\cdot(S\otimes_R T)$ where S,T are R-algebras
4) k field, A k-alg, for A-algs R,S get: $R\otimes_A S \cong (R\otimes_k S)/\langle \psi_R(a)\otimes 1 - 1\otimes\psi_S(a) : a\in A\rangle$

<u>Affine case</u>: $\operatorname{Spec} R \times_{\operatorname{Spec} A} \operatorname{Spec} S = \operatorname{Spec}(R\otimes_A S)$ exists in Aff$/\operatorname{Spec} A$:

have pushout:
(in category of A-algs)

$$\begin{tikzcd} R\otimes_A S & S \arrow[l] \\ R \arrow[u] & A \arrow[l, "\psi_R"] \arrow[u, "\psi_S"'] \end{tikzcd}$$

$s \longmapsto 1\otimes s$

$r \longmapsto r\otimes 1$

Now apply Spec. □

(convince yourself that universal property of pushout for A-algs gives you the univ. prop. for fiber prod. for Aff$_{\operatorname{Spec} A}$)

<u>Claim</u>: this is fiber product also in Sch$/\operatorname{Spec} A$: let $X = \operatorname{Spec} R$
$Y = \operatorname{Spec} S$
$B = \operatorname{Spec} A$
$F = \operatorname{Spec}(R\otimes_A S)$

affine cover of scheme Z:

$$\begin{tikzcd} U_i \arrow[dr, "\text{incl}"] \arrow[drrr, bend left] \arrow[ddrr, bend right] & & & \\ & \text{scheme } Z \arrow[dr, dotted, "\text{want } \exists!"] \arrow[drr, bend left, "\forall p_2"] \arrow[dr, bend right, "\forall p_1"'] & & \\ & & F \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & & X \arrow[r, "f"] & B \end{tikzcd}$$

Recall fiber products are unique up to unique iso if they exist.
By construction (as U_i affine) $\exists!$ $U_i \longrightarrow F$ making diagram commute (used universal property in Aff$/_B$)

If can show these agree on overlaps $U_{ij} = U_i \cap U_j$, then glue to unique $Z \to F$.

If U_{ij} were affine, this would have been immediate.

$U_{ij} \subseteq$ affine U_i, so running same argument with Z replaced by U_{ij}, we can cover U_{ij} by basic open affines $D_{f_k} \subseteq U_i$ and now $D_{f_k}\cap D_{f_\ell} = D_{f_k f_\ell}$ affine!
$\Rightarrow$ glue uniquely to give $U_{ij} \to F$

Recall trick that can pick open cover of U_{ij} that are basic opens simultaneously for U_i, U_j ("USEFUL TRICK" in 3.1)
$\Rightarrow U_{ij} \to F$ and $U_{ji} \to F$ agree.

<u>General case</u> build schemes / morphs by 3 gluing procedures (tedious!)

1) case $U_i \times_B Y$ with B, Y affine, $X = \cup U_i$ affine open cover $\Rightarrow \exists\, X \underset{\text{affine}}{\times} \text{affine}$
2) case $X\times_B V_j$ with B affine, $Y = \cup V_j$ " " " $\Rightarrow \exists\, X \times \underset{\text{affine}}{Y}$
3) case $X\times_{W_k} Y$ with $B = \cup W_k$ " " " $\Rightarrow \exists\, X \underset{\mathbb{Z}}{\times} Y$

Gluings work because agreement on overlaps is ensured by uniqueness up to iso of fiber products. Sketch:

① if know $U_i\times_B Y$ exist, then $\pi_1^{-1}(U_{ij})$ is fiber product $U_{ij}\times_B Y$ (preimage of open set viewed as open subscheme of U_i) (easy check by category theory) so by uniqueness $\exists$ iso $\pi_1^{-1}(U_{ij}) \longrightarrow \pi_1^{-1}(U_{ji})$, so glue & get $X\times_B Y$
(indeed a natural identification since $U_{ij} = U_{ji}$ with sheaf $\mathcal{O}_X|_{U_{ij}}$)

② as in ①, swapping roles X, Y.

③ let $X_k = f^{-1}(W_k)$, $Y_k = g^{-1}(W_k)$ (again: open subschemes since preimages of opens) $\Rightarrow$ $X_k \times_{W_k} Y_k$ exists by ② (W_k affine, X_k, Y_k general)

<u>Key trick</u>: notice $X_k \times_{W_k} Y_k = X_k \times_B Y$
"because images are trapped in W_k, Y_k anyway"
Then use argument in ① to glue the $X_k \times_B Y$. □

(exists as map to B lands in W_k)

$$\begin{tikzcd} \exists! & X_k\times_B Y \arrow[dl, dashed, bend left] \arrow[rr] \arrow[ddr] & & Y_k \arrow[r] \arrow[dd, "g"] & Y \arrow[dd, "g"] \\ X_k\times_{W_k} Y_k \arrow[ur, dashed, bend left, "\exists!"] \arrow[urrr] \arrow[drr] & & & & \\ & & X_k \arrow[r] & W_k \arrow[r] & B \end{tikzcd}$$

<u>RmK</u> Proof shows that $X\times_B Y$ has affine open cover by $\cup(U_i \times_{W_k} V_j)$
where $X = \cup U_i$, $Y = \cup V_j$, $B = \cup W_k$ are " " " with $U_i \to W_k \subseteq B$
$V_j \to W_k \subseteq B$

<u>Examples</u>

1) $\mathbb{A}^n_R \times_{\operatorname{Spec} R} \mathbb{A}^m_R = \operatorname{Spec} R[x_1,\dots,x_n,y_1,\dots,y_m] = \mathbb{A}^{n+m}_R$ (more points than fiber product of sets e.g. $(x-y) \in \operatorname{Spec}\mathbb{Z}[x,y]$)
2) $\operatorname{Spec}\mathbb{Z}/_2 \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/_3 = \operatorname{Spec}(\mathbb{Z}/_2 \otimes_{\mathbb{Z}} \mathbb{Z}/_3) = \operatorname{Spec}(0) = \emptyset$ ($(0) \longmapsto (2) \neq (3)$, $(0) \to (3)$)

<u>Exercise</u> $X\times_Y Y \cong X$, $X\times_B Y \cong Y\times_B X$, $(X\times_B Y)\times_B Z \cong X\times_B (Y\times_B Z)$, $X\times_A B\times_B Y \cong X\times_A Y$.

5.2 Fibers and preimages

$f : X \longrightarrow B$ morph of schemes

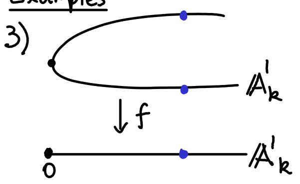
<u>fiber</u> over point $b\in B$: $f^{-1}(b) = \operatorname{Spec} k(b)\times_B X$

<u>preimage of</u> closed subscheme $Y\subseteq B$: $f^{-1}(Y) = Y\times_B X$

$$\begin{tikzcd} f^{-1}(b) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k(b) \arrow[r] & B \\ 0 \arrow[r, mapsto] & b \end{tikzcd}$$

Examples

3)



k = algebraically closed field ← (so classical alg. geometry)

$f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ induced by $f^\# : k[x] \to k[y]$, $x \mapsto y^2$

fiber over 0: (view point 0 as $\operatorname{Spec} k \to \mathbb{A}^1_k$ so $k \cong k[x]/(x)$)

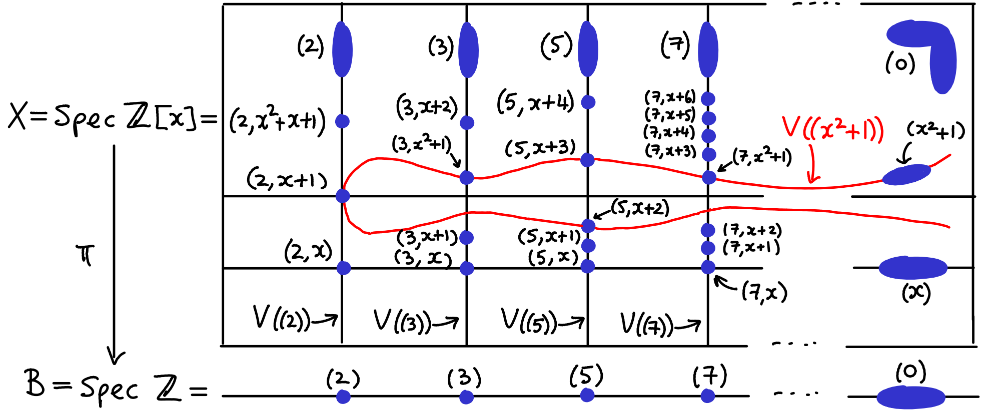
$$\text{fiber} = \operatorname{Spec} k \times_{\operatorname{Spec} k[x]} \mathbb{A}^1_k = \operatorname{Spec}(k \otimes_{k[x]} k[y])$$ ← where $f^\#(x) = y^2$

$$= \operatorname{Spec}\left(\frac{k[y^2]}{(y^2)} \otimes_{k[y^2]} k[y]\right) \cong \operatorname{Spec}(k[y]/(y^2))$$

↖(e.g. use Facts about $\otimes$ from 5.1)

Rmk Notice how a product of affine varieties gave a scheme that was not an affine variety.

4) Mumford's picture of $\operatorname{Spec} \mathbb{Z}[x]$:



π is induced by inclusion $\mathbb{Z} \longrightarrow \mathbb{Z}[x]$

$$\Rightarrow \pi^{-1}((p)) = V((p)) = \{(p), (p, f(x)) : f(x) \bmod p \text{ is irreducible in } \mathbb{F}_p[x]\}$$

(So (p) is a dense point in $\pi^{-1}((p))$.)

(if $p \in I$ then $\mathbb{Z}[x]/I \cong \underbrace{\mathbb{F}_p[x]}_{\text{PID, so } (f) \text{ prime} \Leftrightarrow f \text{ irred or } 0}/I$ where $\mathbb{F}_p = \mathbb{Z}/p$

Rmk curve $V(x^2+1)$ passes through $(p, x+j)$ iff x^2+1 vanishes at that point, so iff $x^2+1 = 0$ in $\mathbb{F}_p[x]/(x+j) \cong \mathbb{F}_p$, $x \mapsto -j$, so iff $j^2 = -1$.
Classical number theory says a square root of -1 exists in $\mathbb{F}_p \Leftrightarrow \left(\begin{matrix} p = 1 \bmod 4 \\ \text{or } p = 2 \end{matrix}\right)$

fiber over (p) : $k(p) = \mathbb{Z}_{(p)}/p\cdot\mathbb{Z}_{(p)} = (\mathbb{Z}/p)_{(p)} = \mathbb{F}_p = \mathbb{Z}/p$

$$\Rightarrow \pi^{-1}(p) = \operatorname{Spec}(k(p) \otimes_{\mathbb{Z}} \mathbb{Z}[x]) = \operatorname{Spec} \mathbb{F}_p[x] = \{(0), (\bar{f}(x))\}$$ ← irred in $\mathbb{F}_p[x]$ nonconstant

fiber over (0) : $k(0) = \mathbb{Z}_{(0)} = \mathbb{Q}$

$$\Rightarrow \pi^{-1}(0) = \operatorname{Spec}(k(0) \otimes_{\mathbb{Z}} \mathbb{Z}[x]) = \operatorname{Spec} \mathbb{Q}[x] = \{(0), (f(x))\}$$

[Gauss's Lemma: For $f \in \mathbb{Z}[x]$ primitive ($\gcd(\text{coeffs}) = 1$) f irred. $\in \mathbb{Z}[x] \Leftrightarrow f$ irred. $\in \mathbb{Q}[x]$] irred in $\mathbb{Q}[x]$ nonconstant } so WLOG irred in $\mathbb{Z}[x]$, nonconstant

Consequence $\operatorname{Spec}\mathbb{Z}[x] = \{(0), (p), (f), (p, \bar{f})\}$ ← $f \in \mathbb{Z}[x]$ irred. mod p nonconstant

↖ $p \in \mathbb{Z}$ prime ↖ $f \in \mathbb{Z}[x]$ irred, nonconstant

Forgetful functor $|\cdot| : \text{Sch} \longrightarrow \text{TopSpaces}$, $X \longmapsto |X|$ = underlying topological space.
morph ⟼ underlying continuous map

Claim $f : X \to B$ morph schemes $\Longrightarrow |f^{-1}(b)| = |f|^{-1}(b)$ ← (fiber is homeomorphic to topological fiber)

Pf WLOG B affine $= \operatorname{Spec} S$ and b is prime ideal $p \subseteq S$

$f^{-1}(B) = \bigcup \operatorname{Spec} R_i$ given by $\varphi_i : S \to R_i$

WLOG just consider one affine, so $R = R_i$, so WLOG $X = \operatorname{Spec} R$

$$\Longrightarrow \operatorname{Spec} k(b) \times_B X = \operatorname{Spec}(k(b) \otimes_S R)$$

($R_p = S_p \otimes_S R$ = (R localised at mult. set $\varphi(S \setminus R)$))

$$k(b) = (S/p)_p \Rightarrow k(b) \otimes_S R = (S/p)_p \otimes_S R = S_p \otimes_S S/p \otimes R = S_p \otimes_S R/\varphi(p)R = R_p/p\cdot R_p$$

$$\Rightarrow \operatorname{Spec}(k(b) \otimes_S R) \overset{1:1}{\longleftrightarrow} \{q \subseteq R \text{ prime ideal containing } \varphi(p) \text{ but not intersecting } \varphi(S \setminus p)\}$$

$q \cdot R_p \longleftrightarrow q$ (= preimage of qR_p via localisation $R \to R_p = S_p \otimes_S R$) $fq = p$

$q \subseteq R \setminus \varphi(S \setminus p) \Rightarrow \varphi^{-1}q \subseteq S \setminus (S \setminus p) = p$
$q \supseteq \varphi(p) \Rightarrow \varphi^{-1}q \supseteq p$ } so get $\{q \in \operatorname{Spec} R : \varphi^{-1}q = p\}$ and can check that closed sets agree via the 1:1 correspondence. □

Cor Given $f : X \to B$, $g : Y \to B$,

(apply $|\cdot|$ to diagram defining $X \times_B Y$ then by universal property in category of topological spaces get unique map ⊛)

fiber of $|X \times_B Y| \xrightarrow{⊛} |X| \times_{|B|} |Y|$ over (x, y) is $\left|\operatorname{Spec}\left(k(x) \otimes_{k(b)} k(y)\right)\right|$ ↙ where $f(x) = g(y) = b$

Pf fiber of $X \times_B Y \longrightarrow X$ over x : $\operatorname{Spec} k(x) \times_X (X \times_B Y) = \operatorname{Spec} k(x) \times_B Y$

fiber of $\operatorname{Spec} k(x) \times_B Y \to Y$ over y : $\operatorname{Spec} k(x) \times_B Y \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} k(x) \times_B \operatorname{Spec} k(y)$

fiber of $\operatorname{Spec} k(x) \times_B \operatorname{Spec} k(y) \to B$ over b : $\operatorname{Spec} k(x) \times_{\operatorname{Spec} k(b)} \operatorname{Spec} k(y) = \operatorname{Spec} k(x) \otimes_{k(b)} k(y)$. □

↑ (lands in $\{b\} \subseteq B$)

(by Claim can work with fiber in Sch before applying $|\cdot|$)

at algebra level: if A_1, A_2 are modules over $S = R_p/pR_p$ then

$$\underbrace{S \otimes_R}_{\substack{\| \\ R_p \otimes_R (R/p) \otimes_R}} (A_1 \otimes_R A_2) \underset{\uparrow}{\cong} A_1 \otimes_S A_2$$

namely: $\frac{r}{t} \otimes a_1 \otimes a_2 \mapsto \frac{r}{t} \cdot (a_1 \otimes a_2)$

or at category level, with abuse of notation:

hence isos

$$\begin{tikzcd}
 & x \times_b y \arrow[l, dotted, shift left, "\exists!"] \arrow[rr, red] \arrow[dd, red] & & y \arrow[d] \arrow[ld] \\
x \times_B y \arrow[ru, dotted, shift left, "\exists!"] \arrow[rrru, bend left] \arrow[rd, bend right] & & B \arrow[rd] & \arrow[d] \\
 & x \arrow[ru] \arrow[rr] & & b
\end{tikzcd}$$

Examples • $|\operatorname{Spec}\mathbb{Z}/2 \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/3| = |\operatorname{Spec}\mathbb{Z}/2| \times_{|\operatorname{Spec}\mathbb{Z}|} |\operatorname{Spec}\mathbb{Z}/3| = \emptyset$ since 1st factor ⟼ (2) and 2nd factor ⟼ (3) $\in \operatorname{Spec}\mathbb{Z}$

• $\mathbb{A}^2_k = \underbrace{\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k}$ $= \operatorname{Spec} k[x, y]$ then $(x+y) \longmapsto (0)$ via both projections to $\mathbb{A}^1_k$ but $(x+y) \neq (0)$

so $|\mathbb{A}^2_k| \neq |\mathbb{A}^1_k| \times |\mathbb{A}^1_k|$: the fiber over $((0), (0))$ is complicated.

(field k) ↖note $\operatorname{Spec} k = \text{point} = \{(0)\}$ so often omit "Spec k" from notation.

Rmk If x, y closed points of schemes X, Y finite type over k, k algebraically closed, then fiber over (x, y) of $X \times_{\operatorname{Spec} k} Y$ is $\operatorname{Spec}(k(x) \otimes_k k(y)) = \operatorname{Spec}(k \otimes_k k) = \operatorname{Spec} k = (0)$ so over closed points you get the product of sets. ← (so classical alg. geom.)

Warning $\mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k$ does not have the product topology, e.g. consider $\mathbb{V}(x-y)$

Non-examinable Rmk Working over an algebraically closed field k, the stalk of $X \times_{\operatorname{Spec} k} Y$ at (x, y) is $\mathcal{O}_{X,x} \otimes_k \mathcal{O}_{Y,y}$ localised at max ideal $m_{X,x} \otimes \mathcal{O}_{Y,y} + \mathcal{O}_{X,x} \otimes m_{Y,y}$

5.3 Base change

$$\begin{tikzcd} X_A := X\times_B A \arrow[r] \arrow[d] & X \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is base-change of $X\to B$ to A via $A\to B$

all schemes →

Example $\mathbb{A}^n_Y = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} Y$ is base change of $\mathbb{A}^n_{\mathbb{Z}}\to \mathrm{Spec}\,\mathbb{Z}$ to Y via $Y\longrightarrow \mathrm{Spec}\,\mathbb{Z}$

Motivation This generalises the idea of changing the "base coefficients"

example : $X = \mathrm{Spec}\,\mathbb{R}[x_1,\dots,x_n]/(f_1,\dots,f_n)$ real affine variety $\subseteq \mathbb{R}^n$

$\left.\begin{array}{l} B = \mathrm{Spec}\,\mathbb{R} \\ A = \mathrm{Spec}\,\mathbb{C}\end{array}\right\}$ and $A\longrightarrow B$ via $\varphi:\mathbb{R}\longrightarrow\mathbb{C}$ inclusion

$X\times_B A$ is Spec of : $\dfrac{\mathbb{R}[x_1,\dots,x_n]}{(f_1,\dots,f_n)}\otimes_{\mathbb{R}}\mathbb{C} \cong \dfrac{\mathbb{C}[x_1,\dots,x_n]}{(\varphi(f_1),\dots,\varphi(f_n))}$ ← (same polys but viewed over $\mathbb{C}$) so affine var $\subseteq\mathbb{C}^n$

Same works if replace $\mathbb{R}\to\mathbb{C}$ by any ring hom $S\to R$.

FACT Many properties of $A\to B$ are inherited by the base change $X_A\to X$:
① affine, ② quasi-compact, ③ locally finite type, ④ finite type, ⑤/⑥ closed/open immersion, ⑦ flat as well as properties from 5.4 : ⑧ separated, ⑨ universally closed, ⑩ proper

5.4 More properties of schemes

(all properties we list are preserved when compose such morphs)

Motivation Topological space X is Hausdorff $\Leftrightarrow$ diagonal $\Delta = \{(x,x): x\in X\}\subseteq X\times X$ closed for product topology

⑧ • $f: X\to B$ morph of schemes is separated if

$\Updownarrow$ $\Delta = \Delta_{X/B}: X\longrightarrow X\times_B X$ is a closed immersion ← (Hwk 3 : enough to check image is a closed set)

• $\forall/\exists$ open cover U_i of B, $f^{-1}(U_i)\longrightarrow U_i$ separated

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & X\times_B X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & B \end{tikzcd}$$

Rmk Often write Δ to mean image $\subseteq X\times_B X$ of morphism Δ.

Rmk Any subscheme $S\subseteq X$ over B is also separated since $\Delta_{S/B} = \Delta_{X/B}\cap(S\times_B S)$

Rmk X separated means separated over spec $\mathbb{Z}$ so $\Delta\subseteq X\times X$ closed

Example for affine varieties (similar for projective varieties) work over $B = \mathrm{Spec}\,k$:

$\mathrm{Spec}\,k[X]\times_k \mathrm{Spec}\,k[X] = \mathrm{Spec}\,k[X]\otimes_k k[X] \supseteq \Delta$ has ideal $\langle f\otimes 1 - 1\otimes f : f\in k[X]\rangle$ ← see next claim

Why good? It disallows pathologies like "affine line with two origins" (Hwk 1 ex. 5) arising by gluing $\begin{array}{l}\mathrm{Spec}\,R[s,s^{-1}]\\ \mathrm{Spec}\,R[t,t^{-1}]\end{array}\hookrightarrow \mathrm{Spec}\,R[x]$ by $\begin{array}{l}x\to s\\ x\to t\end{array}$ (if do $\begin{array}{l}x\to s\\ x\to t^{-1}\end{array}$ then get $\mathbb{P}^1_R$: Hwk.2, ex1)

Claim Affine opens are separated (same proof for Spec $R\to$ Spec S)

Pf $\Delta:\mathrm{Spec}\,R\longrightarrow \underset{=\mathrm{Spec}(R\otimes R)}{\mathrm{Spec}\,R\times\mathrm{Spec}\,R}$ comes from $R\otimes R\xrightarrow[\text{multiply}]{m} R$,

$$\begin{tikzcd} R & R\otimes R \arrow[l, dashed, "\Delta^{\#}"'] & R \arrow[l] \arrow[ll, bend right, "id"'] \\ & R \arrow[u] \arrow[ul, bend left, "id"] & \mathbb{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} r & r\otimes 1 \arrow[l] & s & 1\otimes s \arrow[l] & s \arrow[l] \arrow[ll, bend right] \\ & r \arrow[u] \arrow[ul, bend left] & & & \end{tikzcd}$$

surjective : $m(r,1) = r$ (and $\ker = \langle r\otimes 1 - 1\otimes r : r\in R\rangle$). □

Claim X separated $\Leftrightarrow$ $\forall$ affine opens U_1,U_2 $\left\{\begin{array}{ll} i) & U_1\cap U_2 \text{ affine} \\ ii) & \Gamma(U_1,\mathcal{O}_X)\otimes\Gamma(U_2,\mathcal{O}_X)\overset{surj}{\twoheadrightarrow}\Gamma(U_1\cap U_2,\mathcal{O}_X)\end{array}\right.$ ← multiply restrictions

(enough if holds for cover $\cup U_i$)

Pf (⇒) $\overset{\Delta^{-1}(U_1\times U_2)}{``U_1\cap U_2} \cong (U_1\times U_2)\cap\Delta$, so $U_1\cap U_2\subseteq U_1\times U_2$ closed inside affine $U_1\times U_2$ so affine

U_i affine ⇒ $\Gamma(U_1)\otimes\Gamma(U_2)\cong\Gamma(U_1\times U_2)$. Say $U_1\times U_2 = \mathrm{Spec}\,A$, then :

$U_1\cap U_2 \cong \Delta\cap\mathrm{Spec}\,A = \mathrm{Spec}\,A/I$ some $I\subseteq A$, so $\Gamma(U_1\times U_2)\longrightarrow\Gamma(U_1\cap U_2)$

$\hphantom{U_1\cap U_2 \cong \Delta\cap\mathrm{Spec}\,A = \mathrm{Spec}\,A/I \text{ some } I\subseteq A, \text{ so }} \overset{\cong}{A} \twoheadrightarrow \overset{\cong}{A/I}$

(⇐) Cover $X\times X = \bigcup U_i\times U_j$ by products of affine opens.

$\Gamma(U_1\times U_2)\cong\Gamma(U_1)\otimes\Gamma(U_2)\overset{(ii)}{\twoheadrightarrow}\Gamma(U_1\cap U_2)$ so $U_i\cap U_j\cong\Delta\cap(U_i\times U_j)\subseteq\underbrace{U_i\times U_j}_{\text{affine}}$ closed ← its ideal is ker of hom (ii)

So Δ closed immersion (use 3rd definition in ⑤ Sec.3.6). □

Hwk 3 Claim holds also in case $\Delta_{X/B}$, after tweaking conditions slightly.

Claim X separated $\Leftrightarrow$ $\forall\varphi_1,\varphi_2: Y\longrightarrow X$ if $\varphi_1=\varphi_2$ on dense subset then $\varphi_1=\varphi_2$ as topological maps (so if Y reduced then $\varphi_1=\varphi_2$ as morphisms) ← "equalizers are closed"

↖ see 3.3

Pf (⇒) $\varphi_1\times\varphi_2: Y\longrightarrow X\times X$, $(\varphi_1\times\varphi_2)^{-1}(\Delta)\subseteq Y$ is closed & dense so $= Y$.

(⇐) Let $Y = \overline{\Delta\cap\underbrace{(U_i\times U_j)}_{\text{aff. cover of } X\times X}}$ $\subseteq U_i\times U_j$ (affine) and $\varphi_1,\varphi_2: Y\to X$ projections ⇒ $\varphi_1=\varphi_2$ is precisely the set $\Delta\cap(U_i\times U_j)$. □

Claim $X\xrightarrow{f} Y$, Y separated ⇒ graph $\Gamma_f: X\longrightarrow X\times Y$ closed imm.

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\Gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "id"'] & & \\ & X\times Y \arrow[r] \arrow[d] & Y \\ & X & \end{tikzcd}$$

Pf $f\times id: X\times Y\longrightarrow Y\times Y$, $\Gamma_f\cong(f\times id)^{-1}\Delta$ closed □ ← Non-examinable Rmk: Can also view this as a base change

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X\times Y \arrow[d, "{(f,id)}"] \\ Y \arrow[r, "\Delta"'] & Y\times Y \end{tikzcd}$$

⑨ **Motivation** For top. spaces, X compact $\Leftrightarrow$ $\left(\forall Y,\ \begin{array}{c} X\times Y \\ \downarrow\pi_2 \\ Y\end{array}\text{ is closed map i.e. sends closed sets to closed sets}\right)$

$f: X\to B$ universally closed : every base change is closed map →

$$\begin{tikzcd} X_Y = X\times_B Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "\forall"] & B \end{tikzcd}$$

f is closed map

← Non-example : $\mathbb{A}^1\longrightarrow\mathrm{Spec}\,k$
$\mathbb{A}^1\times_k\mathbb{A}^1\longrightarrow\mathbb{A}^1$
$(a,b)\longmapsto b$
not closed since image of $\mathbb{V}(xy-1)$ is $\mathbb{A}^1\backslash 0$

Fact f univ. closed ⇒ f quasi-compact.

⑩ $f: X\longrightarrow B$ proper $\Leftrightarrow$ ④, ⑧, ⑨ (finite type, separated and universally closed)

Motivation Analogue in smooth world is "preimages of compact sets are compact"

Example Projective n-space $\mathbb{P}^n_B = \mathbb{P}^n_{\mathbb{Z}}\times B$ (build $\mathbb{P}^n_{\mathbb{Z}}$ by gluing in Hwk 2)

$f: X\to Y$ is a projective morphism if factors

$$X\xrightarrow{\text{closed immersion}}\mathbb{P}^n_Y\xrightarrow{\text{projection}} Y$$

Fact if X,Y Noetherian this is proper.

← Non-examinable Rmk
Quasi-projective morph $X\to Y$ if $X\xrightarrow{\text{open imm.}} Z\xrightarrow{\text{Proj. morph}} Y$
If X,Y Noeth this is ④ & ⑧ (finite type & separated)

5.5 Varieties

Def A variety (↙ or abstract variety) is a scheme over k (↙ algebraically closed field) s.t.

(i)	integral
(ii)	$X\longrightarrow\mathrm{Spec}\,k$ finite type ④
(iii)	$X\longrightarrow\mathrm{Spec}\,k$ separated ⑧

← means we're given a morph $X\longrightarrow\mathrm{Spec}\,k$ ⇒ $\mathcal{O}_X(U)$ is k-algebra and restrictions are k-algebra homs. By 2.3 same as giving a hom $k\longrightarrow\Gamma(X,\mathcal{O}_X)$ i.e. a k-algebra structure on $\Gamma(X,\mathcal{O}_X)$

(i) $\Leftrightarrow$ X irreducible, $\mathcal{O}_X(U)$ reduced ← Sometimes don't require irreducibility, just require reduced. But can study one irreducible component at a time.

(ii) $\Leftrightarrow$ X quasi-compact, $\mathcal{O}_X(U)$ are f.g. k-algebras

The definition includes all quasi-projective varieties from classical algebraic geom.

but ∃ more: Nagata (1956) ∃ variety can't embed into any $\mathbb{P}^n_k$ (Rmk finite union of quasi-compacts is quasi-compact)

> You get varieties by gluing together finitely many affine varieties along common open sets (the separated assumption prevents pathologies, see ⑧)

A variety is complete if $X\longrightarrow\mathrm{spec}\,k$ proper ⑩, so extra condition : (iv) universally closed ⑨

Motivation Over $\mathbb{C}$ for "holomorphic spaces" you ask whether a holomorphic map $D^*\to X$ on the punctured disc, meromorphic at 0, can be extended to a holomorphic map $D\to X$ i.e. there are no "missing points in X". (Made rigorous by "valuative criterion for properness")

Hwk3: ■ integral closed subsch. of variety is variety ← exclude e.g. irred. closed subsch. $\mathrm{Spec}\left(\frac{k[x]}{(x^2)}\right)\subseteq\mathbb{A}^1_k$
□ irreducible open subsch. of variety is variety

Examples Complete varieties : $\mathbb{P}^n_k$, projective varieties (■ $\subseteq\mathbb{P}^n_k$), Nagata's 1956 example

Varieties : $\mathbb{A}^n_k$, affine varieties (■ $\subseteq\mathbb{A}^n_k$), quasi-projective varieties (□ $\subseteq$ proj. variety)

↖ not complete (except point, ∅)

Rmk A point $x\in X$ of a variety is closed $\Leftrightarrow$ $K(x)\cong k$ (uses that k is alg. closed). E.g. $\mathbb{A}^1_k = \mathrm{Spec}\,k[x]$, $\begin{array}{l}K((x-a))\cong k\\ K((0)) = k(x)\end{array}$

5.6 Scheme structure on subsets

Motivation: classically, a projective variety is a closed subset of $\mathbb{P}^n_k$. A quasi-proj. var. is an open $\subseteq$ proj. var, so $\Leftrightarrow$ locally closed subset of $\mathbb{P}^n_k$

Claim Any closed subset $C \subseteq X$ of a scheme $\Rightarrow \exists!$ closed reduced subscheme $(C, \mathcal{O}_C) \to X$

Pf $\mathcal{J}(U) := \{ s \in \mathcal{O}_X(U) : s(p) = 0 \in k(p) \; \forall p \in C \cap U \}$ is sheaf of ideals

Locally: $U = \operatorname{Spec} R$, $C \cap U = \mathbb{V}(I)$ for unique radical ideal I

then $s(p) = 0 \in k(p) = (R/p)_p \; \forall p \in \mathbb{V}(I) \Leftrightarrow s \in \bigcap_{p \in \mathbb{V}(I)} P = \sqrt{I} = I$ $\Big\} \Rightarrow \mathcal{J}(\operatorname{Spec} R) = I$

Same trick shows $\mathcal{J}(D_f) = I_f$, so $\mathcal{J}$ is the quasi-coherent ideal sheaf corresponding to I.

Note: $C = \operatorname{supp}(\mathcal{O}_X/\mathcal{J})$ and $C \cap U = \operatorname{Spec} R/I$, and we define $\mathcal{O}_C = \mathcal{O}_X/\mathcal{J}$. □

see 1.15

(so sheafify $C \cap U \longmapsto \mathcal{O}_X(U)/\mathcal{J}(U)$)

Def call this the induced reduced scheme structure on C.

Example When we consider an irreducible component $Z \subseteq X$, we use this scheme structure

Exercise For $C = X \subseteq X$ get the reduced scheme X_{red} (see ⑤ in Sec. 3.6)

Def $Z \subseteq X$ locally closed means $\forall z \in Z, \exists$ open $z \in U$ s.t. $Z \cap U$ is closed in U.
(i.e. $\exists$ closed $C \subseteq X$ with $Z \cap U = C \cap U$)

Non-examinable Pf

Lemma Z locally closed $\Leftrightarrow$ Z open in $\overline{Z}$ ← (i.e. $Z = \overline{Z} \cap U$ some open $U \subseteq X$) ↖by Lemma, $C = \overline{Z}$ works

Pf $\Leftarrow$: $Z = \overline{Z} \cap U$ for open $U \subseteq X \Rightarrow Z \cap U = Z = \overline{Z} \cap U$

$\Rightarrow$: $Z \cap U$ closed in U so equals its closure in U which is: $Cl_U(Z \cap U) = \overline{Z} \cap U$.

$\Rightarrow z \in Z \cap U = \overline{Z} \cap U \subseteq Z$ so Z contains an open neighbourhood of z in $\overline{Z}$ □

$x \in Cl_U(Z \cap U)$
$\Leftrightarrow$ ($\forall$ open $x \in V \subseteq U$, $V \cap Z \neq \emptyset$)
so $x \in \overline{Z}$ but also $x \in U$

Rmk $\overline{Z} \subseteq X$ closed, so $\exists!$ induced reduced scheme structure $\mathcal{O}_{\overline{Z}}$ on $\overline{Z}$

$Z \subseteq \overline{Z}$ is open so get " " " $\boxed{\mathcal{O}_Z = \mathcal{O}_{\overline{Z}}|_Z}$ (so $\mathcal{O}_Z(V) = \mathcal{O}_{\overline{Z}}(V)$)

The local description is the same as above: $Z \cap U = \overline{Z} \cap U = \operatorname{Spec}(R/I)$, $\mathcal{O}_Z|_U \cong \mathcal{O}_{\operatorname{Spec}(R/I)}$

Rmk If Z irreducible ($\Rightarrow \overline{Z}$ irreducible) then $I = p \in \operatorname{Spec} R$ where p is a generic point for both $Z, \overline{Z}$

Z has unique generic point p (see 3.4)
so $Z \subseteq \overline{p} \subseteq \overline{Z}$
so $\overline{p} = \overline{Z}$
$= V(p)$

Hwk 3 Z irred. locally closed $\subseteq$ variety $(X, \mathcal{O}_X) \Rightarrow (Z, \mathcal{O}_Z)$ variety

Hwk 3 $(X, \mathcal{O}_X)$ variety, $Z \subseteq X$ irreducible subspace ← (the irreducibility is not so important if allow varieties to be reducible)

Define sheaf $\mathcal{O}_Z$ on Z: for open $V \subseteq Z$,

$$\mathcal{O}_Z(V) = \Big\{ s : V \to \coprod_{x \in V} k(x) : \forall x \in V \; \exists \text{ open } x \in U \subseteq X, \; t \in \Gamma(U, \mathcal{O}_X) \text{ such that } s(x) = t(x) \in k(x), \forall x \in V \cap U \Big\}$$

Prove that:
$(Z, \mathcal{O}_Z)$ variety $\Rightarrow Z$ locally closed and $\mathcal{O}_Z$ is the induced reduced scheme structure

Idea: we ensure functions on Z are locally restrictions of local functions of X, in classical sense of k-valued functions, rather than germs (recall $k(x) \cong k$ if x is closed point, k alg. closed)

(universal property for the above sheaf)

Lemma With that definition, if Y reduced scheme, $f : Y \to X$ morph of sch. if $f(Y) \subseteq Z$ (as topological spaces) then f factorizes $f : Y \to Z \to X$

Pf Need check sheaves: $s \in \mathcal{O}_Z(U \cap Z)$ for $U \subseteq X$ open then $\exists$ open cover $U \cap Z = \bigcup U_i \cap Z$ and $s_i \in \mathcal{O}_X(U_i)$, $s(x) = s_i(x) \in k(x) \; \forall x \in U_i \cap Z$

$\Rightarrow f^\#(s_i) \in \mathcal{O}_Y(f^{-1}U_i)$, $f^\#(s_i)(y) = f^\#(s_j)(y) \in k(y)$, $\forall y \in f^{-1}(U_i \cap U_j)$ ← (since both are equal to $f^\#_y(s(fy))$ where by 1.10: $f^\#_y : k(fy) \to k(y)$)

$\Rightarrow$ by Sec. 3.3 since Y reduced: $f^\#(s_i)_y = f^\#(s_j)_y \in \mathcal{O}_{Y,y} \; \forall y \in f^{-1}(U_i \cap U_j)$

$\Rightarrow f^\#(s_i)$ glue to a unique section $r \in \mathcal{O}_Y(f^{-1}U)$. Define $\mathcal{O}_Z(U) \to \mathcal{O}_Y(f^{-1}U)$, $s \mapsto r$

and note $\mathcal{O}_X(U_i) \to \mathcal{O}_Z(U_i \cap Z) \to \mathcal{O}_Y(f^{-1}U_i)$, $s_i \mapsto s|_{U_i \cap Z} \mapsto r|_{f^{-1}U_i}$ □

Rmk Applying the Lemma to the case $Y =$ locally closed $Z \subseteq X$ with induced reduced sheaf, implies $\mathcal{O}_Y \cong \mathcal{O}_Z$.

6. SHEAVES OF MODULES

6.1 $\mathcal{O}_X$-modules

Def $\mathcal{O}_X$-module is:
- sheaf $F \in Ab(X)$ ← $(X, \mathcal{O}_X)$ ringed space
- $F(U)$ is an $\mathcal{O}_X(U)$-module ← (often abbreviate $\mathcal{O}_U := \mathcal{O}_X|_U$)
- restrictions are compatible with module structure

(or sheaf of/in $\mathcal{O}_X$-mods)

EXAMPLE: $F = \bigoplus_{i \in I} \mathcal{O}_X$ free $\mathcal{O}_X$-mod

Morphism $F \to G$ of $\mathcal{O}_X$-module is:
- morph $F \xrightarrow{\varphi} G$ of sheaves
- $F(U) \xrightarrow{\varphi_U} G(U)$ is hom of $\mathcal{O}_X(U)$-mods

(if monomorph, i.e. φ_U injective, F is $\mathcal{O}_X$-submod of G)

Rmk stalk F_x is $\mathcal{O}_{X,x}$-mod, and for morphs $F \to G$ get $F_x \to G_x$ is $\mathcal{O}_{X,x}$-mod hom.

Example A sheaf of ideals is an $\mathcal{O}_X$-submod of $\mathcal{O}_X$ ← (just like R-submods of R are ideals

Fact $\boxed{\mathcal{O}_X\text{-Mods} = (\text{category of } \mathcal{O}_X\text{-mods on } X) \text{ is an abelian cat}}$ ← (proof similar to $Ab(X)$ abelian

or: $Mod_{\mathcal{O}_X}(X)$

Indeed notions of submod, quotient mod, Ker, Coker, Im agree with what get in $Ab(X)$
e.g. $F \to G \to H$ exact $\Leftrightarrow$ exact in $Ab(X) \Leftrightarrow$ exact on stalks

Will write $\operatorname{Hom}_{\mathcal{O}_X}$ for morphisms in this category.

6.2 Modules generated by sections

$\boxed{\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F) \xleftrightarrow{1:1} F(X)}$ $\forall F \in \mathcal{O}_X$-Mods ← analogue of $\operatorname{Hom}_R(R, M) \cong M$, $\varphi \mapsto \varphi(1)$

$(\varphi : \mathcal{O}_X \to F) \longleftrightarrow s = \varphi(1)$ since $\varphi_U(r) = \varphi_U(r \cdot 1) = r \cdot s|_U \;\; \forall r \in \mathcal{O}_X(U)$

Similarly $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, F) \xleftrightarrow{1:1} F(X)^{\oplus n}$ defined by n global sections $s_1, \dots, s_n \in F(X)$

Def F is generated by global sections if

$\exists$ surjection $\bigoplus_{i \in I} \mathcal{O}_X \to F$ of $\mathcal{O}_X$-mods ($\Leftrightarrow s_i|_x$ generate $\mathcal{O}_{X,x}$-mod F_x $\forall x \in X$)

same as picking sections $s_i \in F(X)$

Def F is locally generated by sections if $\forall x \in X \; \exists$ open $x \in U$ s.t. $F|_U$ generated by global sections (as $\mathcal{O}_U$-module, $\bigoplus \mathcal{O}_U \twoheadrightarrow F|_U$)

Rmk Can produce $\mathcal{O}_X$-submods from given local sections $s_i \in F(U_i)$ ← sheafify $U \mapsto$ {possible $\mathcal{O}_X(U)$-linear combos of $\{s_i|_U : U \subseteq U_i\}$

Def A sheaf has finite type if locally generated by finitely many sections. ← so $\mathcal{O}_U^{\oplus n} \twoheadrightarrow F|_U$ some open $x \in U$ some $n \in \mathbb{N}$ (not fixed)

6.3 Vector bundles and coherent modules

Def $\mathcal{O}_X$-mod F is locally free $\mathcal{O}_X$-mod of finite rank ("or" vector bundle) if (equivalent definitions)

$\forall x \in X \; \exists$ open $x \in U$: $\boxed{F|_U \cong \mathcal{O}_U^{\oplus n}}$ ← (rank n can depend on U unless we say "of rank n")

as $\mathcal{O}_U$-mods

i.e. locally generated by finite # of "independent sections".

Def X invertible sheaf ("or" line bundle) if $n = 1$ (fixed) ← locally $\mathcal{O}_U \xrightarrow{\cong} \mathcal{O}_U \cdot s = F(U)$ generated by one section $s \in F(U)$

$(X, \mathcal{O}_X)$ locally ringed space

Question Is it enough to ask $F_x \cong \mathcal{O}_{X,x}^{\oplus n}$ (as $\mathcal{O}_{X,x}$-mods) $\forall x$ some $n \in \mathbb{N}$ depending on x? ($\Rightarrow$: clear, $\Leftarrow$: can fail)

Lemma (see Hwk 4) F finite type, $\mathcal{O}_{X,x}^{\oplus n} \xrightarrow{\varphi_x} F_x$ surj $\Rightarrow \exists x \in U \subseteq X$ with surj $\mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} F|_U$, $\varphi|_x = \varphi_x$.

Pf finite type $\Rightarrow \exists$ surj $\mathcal{O}_U^{\oplus m} \xrightarrow{\psi} F|_U$. Let $s_i = \varphi_x(e_i) \in F_x$ ($e_i = 1$ in ith copy of $\mathcal{O}_{X,x}$) so $F_x = \sum \mathcal{O}_{X,x} s_i$. Now $s_i \in F(U_i)$ some $x \in U_i$. Replace U by $U \cap U_1 \cap \dots \cap U_n$ so WLOG $s_i \in F(U)$. Let $f_j = 1 \in$ (j-th copy of $\mathcal{O}_U$) $\Rightarrow \psi(f_j)|_x = \sum r_{ji} \cdot s_i$ some $r_{ji} \in \mathcal{O}_{X,x}$. So $\psi(f_j)|_{V_j} \in \sum \mathcal{O}_{V_j} \cdot s_i|_{V_j}$ some $V_j \subseteq U$, again WLOG $V_j = U$ (replace U by $U \cap V_1 \cap \dots \cap V_m$) $\Rightarrow \psi(f_j) \in \operatorname{Im} \varphi$ for $\varphi : \mathcal{O}_U^{\oplus n} \to F|_U$ with $\varphi(e_i) = s_i$ on U. So φ hits $\mathcal{O}_U$-mod generators $\psi(f_j)$ □

Continuing above Question: We know φ_x is inj at x, but we don't know if the same φ works also for y close to x, so we do not know whether φ_y inj (recall φ inj $\Leftrightarrow \varphi_y$ inj at all stalks at $y \in U$).

Lemma In previous Lemma, if $\mathrm{Ker}\,\varphi$ finite type, φ_x iso $\Rightarrow$ $\varphi : \mathcal{O}_U^{\oplus n} \to F|_U$ iso, some $\underset{x\in}{U}$.

Pf Shrinking U, $\exists$ surj $\mathcal{O}_U^{\oplus m} \xrightarrow{\psi} \ker\varphi$, hence $\mathcal{O}_U^{\oplus m} \xrightarrow{\psi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} F|_U \to 0$ exact. Apply Lemma to $\mathrm{Ker}\,\varphi$: using $(\mathrm{Ker}\,\varphi)_x = 0$ deduce $(\mathrm{Ker}\,\varphi)|_U = 0$ possibly after shrinking U further. So φ is also injective. □

such F are called locally finitely presented

This motivates the definition:

Def $F \in \mathcal{O}_X$-Mods is coherent if $\begin{cases} F \text{ finite type} \\ \mathrm{Ker}\,(\mathcal{O}_U^{\oplus n} \to F|_U) \text{ finite type} \end{cases}$ $\forall\, \mathcal{O}_U$-mod homs $\forall$ open U, $\forall n\in\mathbb{N}$

Rmk $F \in \mathrm{Coh}(X) \Rightarrow F$ locally finitely presented

Pf F finite type $\Rightarrow \exists$ surj $\mathcal{O}_U^{\oplus n} \longrightarrow F|_U$, then consider Ker. □

$\mathrm{Vect}(X) = \{\text{vector bundles on } X\} \subseteq \mathcal{O}_X$-Mods, but not an abelian cat (ker, coker need not be free

$\mathrm{Coh}(X) = \{\text{coherent } \mathcal{O}_X\text{-mods}\} \leftarrow$ Fact abelian category! (explains partly its importance)

Claim $F \in \mathrm{Coh}(X)$ and $F_x \cong \mathcal{O}_{X,x}^{\oplus n}$ $\forall x \Longrightarrow F \in \mathrm{Vect}(X)$ ($\forall x\in X$, some $n \in \mathbb{N}$ depending on x unless we fix the rank)

Claim follows by Lemmas. Converse of Claim?

Cor X locally Noetherian scheme $\Rightarrow \mathrm{Vect}(X) = \{F \in \mathrm{Coh}\,X : \forall x,\ F_{X,x} \cong \mathcal{O}_{X,x}^{\oplus n} \text{ some } n\} \subseteq \mathrm{Coh}(X)$

Pf $F \in \mathrm{Vect}(X) \Rightarrow F$ finite type, in general

$\mathrm{Ker}\,(\mathcal{O}_U^{\oplus n} \underset{\text{given}}{\xrightarrow{\varphi}} F|_U)$ (need show finite type for small U) shrinking U WLOG U affine $= \mathrm{Spec}\,R$ ←Noetherian

In sections below we will prove that because $\mathcal{O}_U^{\oplus n}, F|_U$ are "quasi-coherent" the problem reduces to taking global sections: $\mathrm{Ker}\,(R^n \xrightarrow{\varphi} F(U))$ and this is finitely generated since R Noeth (so get exact sequence $R^m \to R^n \xrightarrow{\varphi} F(U) \to 0$ and this will imply $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} F \to 0$ exact). □

R^n is f.g. R-mod, so its R-submods are f.g. since R Noeth.

6.4 $\mathcal{O}_X$-module $\widetilde{M}$ on $X = \mathrm{Spec}\,R$, for R-mod M

sheaf $\widetilde{M}$ on $X = \mathrm{Spec}\,R$ by Sec. 1.12 method:

- $\widetilde{M}(D_f) = M_f$ (so $\widetilde{M}(X) = \widetilde{M}(D_1) = M$)
- $D_g \subseteq D_f \Rightarrow M_f \longrightarrow M_g$ induced by $R_f \longrightarrow R_g$
- stalk $\widetilde{M}_p = \varinjlim_{D_f \ni p} \widetilde{M}(D_f) = \varinjlim_{D_f\ni p} M_f \cong M_p$ ← ($\varinjlim M\otimes R_f \cong M \otimes \varinjlim R_f \cong M\otimes R_p$
- $\widetilde{M}(U) = \{ s : U \to \bigsqcup_{p\in \mathrm{Spec}R} M_p : s(p) \in M_p$ which are locally compatible: $\forall p \in U$, $\exists$ open nbhd $p \in D_f \subseteq U$ ←some $f\in R$, $\exists\, t \in \widetilde{M}(D_f)$ with $s(x) = t_x$ $\forall x\in D_f$ $\}$

$\widetilde{M}(D_f) = M_f \ni t = \frac{m}{f^n}$ ← $m\in M$; $t_x = \frac{m}{f^n} \in \widetilde{M}_x$ is image via natural $M_f \to M_x$

M_f = localisation of M at f $\cong M \otimes_R R_f$

$M_p = S^{-1}M$ localisation of M at $S = R\backslash p$ $\cong M \otimes_R R_p$

with the obvious restriction maps.

Rmk · could assume $t = \frac{m}{f}$ since can replace D_f with D_{fm} $(= D_f)$.

- could just ask $s(x) = t_x$ on a smaller open $p \in V \subseteq D_f$.
- $\boxed{\widetilde{M} = \text{sheafification of } U \longmapsto M \otimes_R \mathcal{O}_X(U)}$

Call $\widetilde{M}$ the sheaf associated to M

EXAMPLES. $\widetilde{R} = \mathcal{O}_X$ $(X = \mathrm{Spec}\,R)$

- $\widetilde{\bigoplus_{i\in I} M_i} \cong \bigoplus_{i\in I} \widetilde{M}_i$, so $\widetilde{\bigoplus_{i\in I} R} \cong \bigoplus_{i\in I} \mathcal{O}_X$

UPSHOT $\widetilde{M}$ is $\mathcal{O}_X$-module on $X = \mathrm{Spec}\,R$

$\varphi: M \to N$ R-mod hom $\Longrightarrow \widetilde{M} \to \widetilde{N}$ $\mathcal{O}_X$-mod morph by gluing

$$\begin{tikzcd} \widetilde{M}(D_f) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \widetilde{N}(D_f) \arrow[d, phantom, "\cong" rotate=90] \\ M_f \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & N_f \arrow[d, phantom, "\cong" rotate=90] \\ M\otimes R_f \arrow[r, "\varphi\otimes id"'] & N\otimes R_f \end{tikzcd}$$

(just need check stalks, then use Sec. 3.0) ↖ for converse take global sections

$\Rightarrow$ fully faithful exact functor $\boxed{R\text{-Mods} \longrightarrow \mathcal{O}_{\mathrm{Spec}(R)}\text{-Mods}}$

6.5 Direct image and inverse image

$$\begin{tikzcd} F \arrow[d, dash] & f_*F \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\mathcal{O}_X$-mod → F; ringed sp. → X; Y ← top. sp.

f_*F is $f_*\mathcal{O}_X$-mod

$(f_*F)(U) = F(f^{-1}(U))$ is $\mathcal{O}_X(f^{-1}(U))$-mod, $= f_*\mathcal{O}_X(U)$

Example $\alpha : \mathrm{Spec}\,S \longrightarrow \mathrm{Spec}\,R$, $\varphi = \alpha^{\#} : R \to S$

N S-mod $\Rightarrow \boxed{\alpha_*\widetilde{N} = \widetilde{{}_RN}}$ ← ${}_RN = N$ viewed as R-mod via φ

Pf $(\alpha_*\widetilde{N})(D_f) = \widetilde{N}(D_{\varphi f}) = N_{\varphi f} = ({}_RN)_f$ compatible with restrictions □

Algebra: Recall $R \xrightarrow{\varphi} S$ hom of rings, then S is R-mod via $r\cdot s = \varphi(r)s$.

$f: X \to Y$ morph of ringed spaces, then: ← (recall $\mathrm{Mor}(\mathcal{O}_Y, f_*\mathcal{O}_X) \cong \mathrm{Mor}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$)

$f^{-1}\mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X$ an $f^{-1}\mathcal{O}_Y$-mod on ringed space $(X, f^{-1}\mathcal{O}_Y)$

$$\begin{tikzcd} f^{-1}F \arrow[d, dash] & F \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

top. sp. → X; F ← $\mathcal{O}_Y$-mod; Y ← ringed sp.

$f^{-1}(F)$ is $f^{-1}(\mathcal{O}_Y)$-mod

$(f^{-1}F)(U) = \varinjlim_{V \supseteq fU} F(V)$ (presheaf) ↻ act by $\mathcal{O}_Y(V)$ so can act by $(f^{-1}\mathcal{O}_Y)(U) = \varinjlim_{V\supseteq fU} \mathcal{O}_Y(V)$

6.6 Operations on $\mathcal{O}_X$-mods

$\boxed{\mathcal{H}om_{\mathcal{O}_X}(F, G) : U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(F|_U, G|_U)}$ is a sheaf of $\mathcal{O}_X$-mods.

left exact in F and in G

Warning: $\mathrm{Hom}_{\mathcal{O}_X(U)}(F(U), G(U))$ would not work since do not get restriction maps.

coproduct in $\mathcal{O}_X$-Mod: F_i $\mathcal{O}_X$-mods, $\boxed{\oplus F_i = \text{sheafify}\,(U \longrightarrow \oplus F_i(U))}$

(Need sheafify: could get ∞ sums when globalise, e.g. $X = \mathbb{N}$, $F_i = \begin{cases}\mathbb{Z} \text{ on } \{i\} \\ 0 \text{ else}\end{cases}$, $s_n = (\underbrace{1,\ldots,1}_{n}, 0, \ldots)$ at $\{n\}$, try globalise)

Fact $\exists$ canonical iso $\mathrm{Mor}\,(\oplus F_i, G) \cong \prod \mathrm{Mor}_{\mathcal{O}_X}(F_i, G)$ natural in F_i, G.

right exact in F, G

product in $\mathcal{O}_X$-Mod: $\boxed{F \otimes_{\mathcal{O}_X} G = \text{sheafify}\,(U \longrightarrow F(U) \otimes_{\mathcal{O}_X(U)} G(U))}$

Fact $\exists!$ $\mathcal{O}_X$-mod structure s.t. $F(U) \otimes_{\mathcal{O}_X(U)} G(U) \longrightarrow (F \underset{\mathcal{O}_X}{\otimes} G)(U)$ hom of $\mathcal{O}_X(U)$-mods

Universal property: $\mathrm{Hom}_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G, H) = \mathrm{Bilinear}_{\mathcal{O}_X}(F \times G, H)$

Rmk Stalks are $\mathrm{Hom}_{\mathcal{O}_{X,x}}(F_x, G_x)$, $\oplus(F_i)_x$, $F_x \otimes_{\mathcal{O}_{X,x}} G_x$.

for this require M finitely presented: $\exists$ exact $\underset{\text{finite}}{\oplus} R \to \underset{\text{finite}}{\oplus} R \to M \to 0$

Examples on $X = \mathrm{Spec}\,R$: $\widetilde{\oplus M_i} \cong \oplus \widetilde{M}_i$, $\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, $\widetilde{\mathrm{Hom}_R(M,N)} \cong \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$

Algebra $\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ canonically, for R-mods M, N, P (so ⊗ & Hom are adjoint)

Fact $\mathcal{H}om_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G, H) \cong \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{H}om_{\mathcal{O}_X}(G, H))$ canonically & functorial in F, G, H.

Cor $F \otimes_{\mathcal{O}_X} \cdot$, $\mathcal{H}om_{\mathcal{O}_X}(G, \cdot)$ adjoint, $F \otimes_{\mathcal{O}_X} \cdot$ right exact, $\mathcal{H}om_{\mathcal{O}_X}(G, \cdot)$ left exact.

Fact $f: X \to Y \Longrightarrow f^{-1}(F \underset{\mathcal{O}_Y}{\otimes} G) \cong f^{-1}F \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}G$ canonically (F, G $\mathcal{O}_Y$-mod)

can prove this using universal property, or by hand thinking about ⊗ of presheaves.

6.7 Pullback

Rmk $R \to S$ rings, M R-mod, N S-mod

$\Rightarrow M \otimes_R N$ is $\begin{cases} \text{R-mod since N R-mod via } R \to S & (r\cdot(m\otimes n) = (rm)\otimes n = m \otimes rn) \\ \text{S-mod by } s\cdot(m\otimes n) = m \otimes sn \end{cases}$

similarly:
$$\begin{tikzcd} & F \arrow[d, dash] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

F ← $\mathcal{O}_Y$-mod; ringed

$\Rightarrow$ $\boxed{f^*F = f^{-1}(F) \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X}$ is an $f^{-1}\mathcal{O}_Y$-mod but also an $\mathcal{O}_X$-mod!

Fact $\exists!$ $\mathcal{O}_X$-mod structure s.t. : presheaf tensor product $= \underbrace{f^{-1}(F)(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U)}_{\mathcal{O}_X(U)\text{-mod as by Rmk.}} \longrightarrow f^*F(U)$ is $\mathcal{O}_X(U)$-mod hom

Example $f^*\mathcal{O}_Y = \mathcal{O}_X$ (since $f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$ canonically)

Exercise
- $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow f^* \circ g^* = (g\circ f)^*$ ⟸ (use last Fact in 6.6, using Sec.1.9)
- $f^*(F \otimes_{\mathcal{O}_Y} G) = f^*F \otimes_{\mathcal{O}_X} f^*G$ canonically & functorial

Upshot $f: X \longrightarrow Y$ morph of ringed spaces $\Longrightarrow$ $\mathrm{Mod}_{\mathcal{O}_X}(X) \xrightarrow{f_*} \mathrm{Mod}_{\mathcal{O}_Y}(Y)$ and $\xleftarrow{f^*}$

Theorem (exercise) $\boxed{f^*, f_* \text{ are adjoint functors: } \mathrm{Mor}_{\mathcal{O}_X}(f^*F, G) \cong \mathrm{Mor}_{\mathcal{O}_Y}(F, f_*G)}$

hence f_* left exact, f^* right exact

HWK 3 f_* commutes with limits $\underleftarrow{\lim}$ for example $\prod$, f^* commutes with colimits $\underrightarrow{\lim}$ for example $\oplus$

(product in category $\mathcal{O}_X$-Mods) (coproduct in cat. of $\mathcal{O}_X$-Mods)

Example $f^*(\oplus \mathcal{O}_Y) = \oplus f^*\mathcal{O}_Y = \oplus \mathcal{O}_X$.

Exercise Deduce from that $f^*(\mathrm{Vect}(Y)) \subseteq \mathrm{Vect}(X)$.

6.8 $\widetilde{M}$ on any scheme

ASSUME given a ring hom $R \to \Gamma(X)$

M R-mod,

$$\begin{tikzcd} X \arrow[r, "\text{canonical}"] \arrow[rr, bend right, "\alpha"'] & \mathrm{Spec}\,\Gamma(X,\mathcal{O}_X) \arrow[r] & \mathrm{Spec}\,R \end{tikzcd}$$

then get $F_M := \alpha^*\widetilde{M}$

Easier: $(X, \mathcal{O}_X) \xrightarrow{\pi}$ ringed space (point, R) (on sheaves $\pi_*\mathcal{O}_X = \Gamma(X) \longleftarrow R$) ← GIVEN

$F_M := \pi^*M$

$= \text{sheafify}\,(U \longmapsto M \otimes_R \mathcal{O}_X(U))$ ⟵ (since $\pi^{-1}M \underset{\pi^{-1}R}{\otimes} \mathcal{O}_X$ and $(\pi^{-1}R)(U) = R$, $(\pi^{-1}M)(U) = M$

(get same answer since $X \xrightarrow{\alpha} \mathrm{Spec}\,R \xrightarrow{\pi_1} (\text{point}, R)$, $\widetilde{M} = \pi_1^*M$ by construction, $\pi^* = \alpha^*\pi_1^*$)

Claim $f: Y \longrightarrow X$ (morph of ringed spaces) $\Longrightarrow f^*F_M = F_N$ where $N = M \otimes_{\Gamma(X)} \Gamma(Y)$ is $\Gamma(Y)$-module

M $\Gamma(X)$-module (case $R \xrightarrow{id} \Gamma(X)$)

Pf

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\pi_X"] \\ (\text{point}, \Gamma(Y)) \arrow[r, "\psi"] & (\text{point}, \Gamma(X)) \end{tikzcd}$$

using $f^\#: \Gamma(X) \to \Gamma(Y)$

$f^*\pi_X^*M = \pi_Y^*\psi^*M$

$\psi^*M = \psi^{-1}M \otimes_{\psi^{-1}\Gamma(X)} \Gamma(Y) = M \otimes_{\Gamma(X)} \Gamma(Y)$ □

Cor $\boxed{\begin{array}{l} \alpha: \mathrm{Spec}\,S \longrightarrow \mathrm{Spec}\,R \\ M \text{ R-mod} \end{array} \Longrightarrow \alpha^*\widetilde{M} = \widetilde{M \otimes_R S}}$ ⟵ (S is R-mod via the ring hom $R \to S$ $\alpha^\#$)

Example $D_f = \mathrm{Spec}\,R_f \hookrightarrow \mathrm{Spec}\,R \Longrightarrow \widetilde{M}|_{D_f} = \widetilde{M \otimes_R R_f} = \widetilde{M_f}$ ⟵ stronger statement than saying $\widetilde{M}(D_f) = M_f$

6.9 Classification of $\mathcal{O}_X$-homs $\widetilde{M} \to F$

Lemma (compare Sec. 2.3) $X = \mathrm{Spec}\,R \Longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, F) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_R(M, \underbrace{\Gamma(X,F)}_{= F(X)})$ $\forall \mathcal{O}_X$-mod F

$\varphi \longmapsto \varphi_X$

Pf $\pi: (X, \mathcal{O}_X) \longrightarrow (\text{point}, R)$ morph of ringed spaces ($\pi^\#: R \xrightarrow{id} \pi_*\mathcal{O}_X = \mathcal{O}_X(X) = R$)

$\widetilde{M} = \pi^*M$, $\Gamma(X,F) = \pi_*F$

$\Rightarrow \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, F) = \mathrm{Hom}_{\mathcal{O}_X}(\pi^*M, F) \cong \mathrm{Hom}_R(M, \pi_*F) = \mathrm{Hom}_R(M, \Gamma(X,F))$. □ ← π^*, π_* adjoint

Exercise Using 6.8: $\mathrm{Hom}_{\mathcal{O}_X}(F_M, F) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_R(M, F(X))$ using $R \xrightarrow{\text{given}} \Gamma(X, \mathcal{O}_X)$ to make $F(X)$ an R-mod.

6.10 Flatness

Def F is flat $\mathcal{O}_X$-mod if $F \otimes_{\mathcal{O}_X} \cdot$ is exact

so $\Longleftrightarrow F_x$ flat $\mathcal{O}_{X,x}$-mod $\forall x$. ⟵ since exactness can be checked on stalks

Example $U \xrightarrow{i} X$ open subsch. $\Rightarrow i_*\mathcal{O}_U$ is flat $\mathcal{O}_X$-mod ⟵ stalk is either 0 or $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} \cdot = id$

Rmk Morph of schemes $f: X \to Y$ is flat $\Longleftrightarrow$ (see ⑦ in Sec.3.6) $\mathcal{O}_X$ flat $f^{-1}\mathcal{O}_Y$-module ⟵ since recall $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,fx}$

Claim $f: X \to Y$ flat $\Longrightarrow f^*: \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ is exact (not just right exact)

Pf f^{-1} is exact (Sec. 1.9) $\Longrightarrow \mathcal{O}_Y\text{-Mod} \xrightarrow{f^{-1}} f^{-1}\mathcal{O}_Y\text{-Mod}$ exact,

$F \longmapsto f^{-1}F$

- $\otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ exact by Rmk $\Longrightarrow f^*F = f^{-1}F \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is composite of two exact functors □

Facts ← $(X, \mathcal{O}_X)$ ringed space
- free $\Rightarrow$ flat
- Can take $\oplus$ of flat mods
- $0 \to F_1 \to F_2 \to F_3 \to 0$ exact : outer two (F_1, F_3) or last two (F_2, F_3) flat $\Rightarrow$ all flat ← so kernels are flat
- " , F_3 flat $\Rightarrow$ sequence $\otimes_{\mathcal{O}_X}$ any $\mathcal{O}_X$-mod G is exact
- $\cdots \to F_2 \to F_1 \to F_0 \to F \to 0$ exact, all flat $\Rightarrow$ "

(so "flat resolution of flat $\mathcal{O}_X$-mod F")

Combine (break into SES's, show images $(F_n \to F_{n-1})$ flat)

⟵ Taking stalks, all follow from analogous statements for R-mods

7. (QUASI-)COHERENT SHEAVES

7.1 QCoh(X)

Recall (Sec. 6.3) F coherent $\Longrightarrow F$ locally finitely presented
and "$\Longleftarrow$" holds if X locally Noetherian scheme.

now weaken this condition by dropping finiteness

Fact "$\Longleftarrow$" holds also if just assume $\mathcal{O}_X$ is coherent

Def F quasi-coherent $\Longleftrightarrow$ $\boxed{\begin{array}{l} F \text{ is locally presented, i.e. } \forall x, \exists \text{ open } x \in U \subseteq X \\ \exists \bigoplus_{i\in I} \mathcal{O}_U \longrightarrow \bigoplus_{j\in J} \mathcal{O}_U \longrightarrow F|_U \longrightarrow 0 \text{ exact.} \end{array}}$

$\in$ $\mathcal{O}_X$-Mods (any ringed space $(X, \mathcal{O}_X)$)

where the maps are morphs of $\mathcal{O}_U$-mods

← where $\mathcal{O}_U = \mathcal{O}_X|_U$

SUMMARY: coherent $\Longrightarrow$ locally finitely presented $\Longrightarrow$ quasi-coherent (= locally presented) $\Longrightarrow$ locally generated by sections

locally finitely presented $\Longrightarrow$ locally generated by finitely many sections

vector bundle $\Longrightarrow$ locally generated by finitely many sections $\Longrightarrow$ locally generated by sections

Lemma For $X = \mathrm{Spec}\,R$: $\left(\begin{array}{l} \exists \text{ exact sequence of } \mathcal{O}_X\text{-mods} \\ \bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \bigoplus_{i\in J} \mathcal{O}_X \to F \to 0 \end{array}\right) \Longleftrightarrow (F \cong \widetilde{M} \text{ some R-module } M)$

Pf $(\Rightarrow)$ Let $M = \bigoplus_J R / \mathrm{Im}(\bigoplus_I R \to \bigoplus_J R)$ (taking global sections)

by exact functor from 6.4:

$$\begin{tikzcd} \bigoplus_I \mathcal{O}_X \arrow[r] \arrow[d, phantom, "\cong"] & \bigoplus_J \mathcal{O}_X \arrow[r] \arrow[d, phantom, "\cong"] & F \arrow[r] \arrow[d, phantom, "="] & 0 & \text{exact} \\ \bigoplus_I \widetilde{R} \arrow[r] & \bigoplus_J \widetilde{R} \arrow[r] & \widetilde{M} \arrow[r] & 0 & \text{exact} \end{tikzcd}$$

by uniqueness of cokernels up to iso: $F \cong \widetilde{M}$

$(\Leftarrow)$ $F = \widetilde{M}$: pick J = set of generators m_j for R-mod M (e.g. $J = M$)

pick I = " " " k_i " " $\mathrm{Ker}(\bigoplus_J R \longrightarrow M)$ ← send 1 in j-th copy of R to m_j

apply $\sim$ to $\bigoplus_I R \to \bigoplus_J R \longrightarrow M \to 0$. □ ← send 1 in i-th copy of R to k_i

Cor $\forall$ scheme X

$$\boxed{\begin{array}{l} F\in QCoh(X) \iff \forall x\in X\ \exists \text{ affine open } x\in U\cong \operatorname{Spec}R,\ F|_U\cong\widetilde{M} \text{ some } R\text{-mod } M \\ F\in Coh(X) \iff \text{in addition require } M \text{ is } \underline{\text{coherent}}\ R-\text{mod}\end{array}}$$

(WLOG $M=F(U)$, $R=\mathcal{O}_X(U)$ as $F|_U(U)=\widetilde{M}(U)=M$)

(Pf $\forall x$ pick U so that Lemma applies.)

$\begin{cases} \cdot\ M \text{ finitely generated} \\ \cdot\ \ker(R^n \xrightarrow{\varphi} M) \text{ is f.g., any } n\in\mathbb{N} \end{cases}$ ← any hom of R-mods

Idea: want $\forall$ f.g. submod of M to have finite presentation, indeed get exact sequence $R^m \to R^n \xrightarrow{\varphi} \operatorname{Im}\varphi \to 0$ (map to gens. of $\ker\varphi$)

Rmk If R Noeth., coherent = f.g. (since R^n f.g., so its submods are f.g. as R Noeth)

Example X loc. Noeth. scheme $\Rightarrow$ $\mathcal{O}_X$ is coherent
$\Rightarrow$ ideal sheaf of any closed subsch. is coherent.

Rmk $\forall$ scheme: $F\in QCoh(X) \iff \exists$ affine open cover $X=\bigcup U_i$ s.t. $F|_{U_i}\cong\widetilde{M_i}$ for R_i-mods M_i, $U_i=\operatorname{Spec}R_i$
$F\in Coh(X) \iff$ " and M_i coherent.
(WLOG: $R_i=\mathcal{O}_X(U_i)$, $M_i=F(U_i)$)

(immediate from Cor)

Rmk restriction to open $V\subseteq X$: $QCoh(X)\longrightarrow QCoh(V)$, $Coh(X)\longrightarrow Coh(V)$

Pf $x\in V\cap U=\bigcup D_{f_i}$ for $f_i\in R$ then $F|_U|_{D_{f_i}}\cong \widetilde{M}|_{D_{f_i}} \cong \widetilde{M_{f_i}}$ (and use fact that localization preserves "coherent" property) ← Example in 6.8

so again locally $\widetilde{\text{module}}$. □

Why is quasi-coherence a good notion?

$\text{Rings}^{op}\longrightarrow \text{Aff}$, $R\longmapsto(\operatorname{Spec}(R),\mathcal{O}_{\operatorname{Spec}R})$ equivalence of cats

$R-\text{Mods}\longrightarrow \mathcal{O}_{\operatorname{Spec}(R)}-\text{Mods}$, $M\longmapsto\widetilde{M}$ not equivalence of cats

Example $X=\operatorname{Spec}k[x]=\mathbb{A}^1_k$, skyscraper sheaf at 0: $F(U)=\begin{cases}k[x] & \text{if } 0\in U\\ 0 & \text{else}\end{cases}$ ← notice $F\in\mathcal{O}_X$-Mods

$\Rightarrow$ if the above were an equivalence of cats, then $F\cong\widetilde{M}$ some $k[x]$-mod M

so $k[x]=F(X)\cong\widetilde{M}(X)=M$. But $\widetilde{k[x]}=\mathcal{O}_X$ is not isomorphic to F!

Solution restrict which $\mathcal{O}_X$-mods you allow: want them locally to look like $\widetilde{M}$, just like when we studied sheaves of ideals that locally look like $\widetilde{I}$

Will show later: $\boxed{\text{For } X=\operatorname{Spec}R: R-\text{Mods}\longrightarrow QCoh(X) \text{ equivalence of categories}}$ $\begin{array}{l} M\longmapsto\widetilde{M}\\ F(X)\longleftarrow F\end{array}$

7.2 Overview of general properties of QCoh(X) and Coh(X) for X scheme

(for Coh X properties enough if X ringed)

1) $Coh(X)$ abelian category, and $\quad Coh(X)\xrightarrow{incl}\mathcal{O}_X-\text{Mod}$
$QCoh(X)$ " " " $\quad Coh(X)\subseteq QCoh(X)\xrightarrow{incl}$ are exact functors

In particular can take Ker, Coker, Image in both (not in Vect(X)) ← Easy for QCoh since locally hom of mods $M_1\to M_2$ so take $\sim$ of Ker, Coker, Im

2) $0\to F_1\to F_2\to F_3\to 0$ exact in $\mathcal{O}_X$-Mods.

Two of the $F_i\in QCoh(X)\Rightarrow$ all three are. Same holds for $Coh(X)$ (not for Vect(X))

Trick $0\to F_1\to F_2\to F_3$ exact, and F_2,F_3 are, then F_1 is. (Pf. $F_1\cong\operatorname{Ker}(F_2\to F_3)$, use (1).□)

3) Can take finite $\oplus$, $\cdot\otimes_{\mathcal{O}_X}\cdot$, $\operatorname{Hom}_{\mathcal{O}_X}(\cdot,\cdot)$ in $QCoh(X)$, $Coh(X)$ and $Vect(X)$

← for QCoh, $\operatorname{Hom}_{\mathcal{O}_X}(F,G)$ need assume F loc. finitely presented

4) Gabriel–Rosenberg thm

X quasi-compact & separated (e.g. variety) $\Rightarrow$ X is determined up to iso by $QCoh\,X$!

5) $\cdot$ X loc. Noeth. scheme, $Z\xrightarrow{i}X$ closed subsc. $\Rightarrow 0\to\mathcal{J}_{Z/X}\to\mathcal{O}_X\to i_*\mathcal{O}_Z\to 0$ exact in $\underline{\underline{Coh X}}$

$\cdot$ finite type subsheaf $F\subseteq G$, $G\in Coh(X)\Rightarrow F\in Coh(X)$ } ← combine to prove Kernels exist in Coh X

Hwk 4 { $\cdot$ $\varphi: F\to G$, $G\in Coh X$, F finite type $\Rightarrow$ $\ker\varphi$ finite type

$\cdot$ $\varphi: F\to G$, $G\in Coh X$, F finite type, $\varphi_x: F_x\to G_x$ injective $\Rightarrow$ $\varphi|_U: F|_U\to G|_U$ inj. some $U\ni x$

← we proved it in case $F=0$ in Pf. claim in Sec. 6.3

Hwk 4: Picard group $\operatorname{Pic}(X)=\{\text{isomorphism classes of invertible sheaves}\}$
group operation is $\cdot\otimes_{\mathcal{O}_X}\cdot$. (abelian group as $F\otimes_{\mathcal{O}_X}G\cong G\otimes_{\mathcal{O}_X}F$)

7.3 Pullback preserves quasi-coherence

$f: X\to Y$ morph ringed spaces

Claim $f^*: QCoh(Y)\longrightarrow QCoh(X)$. If X loc. Noeth. scheme $\Rightarrow f^*: Coh\,Y\to Coh\,X$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Vect}\,Y\xrightarrow{f^*}\text{Vect}\,X$ (with $\text{Vect}\,Y\subseteq Coh\,Y$, $\text{Vect}\,X\subseteq Coh\,X$)

← without this can fail e.g. $f^*\mathcal{O}_Y=\mathcal{O}_X$ so if $\mathcal{O}_Y$ coh, $\mathcal{O}_X$ not coh, then fails

Pf If $\bigoplus_I\mathcal{O}_Y|_U\to\bigoplus_J\mathcal{O}_Y|_U\to G|_U\to 0$ exact ($fx\in U\subseteq Y$ open)

apply g^* where $g=f|_{f^{-1}U}: f^{-1}U\to U$, using g^* right exact & commutes with $\oplus$: ← Sec. 6.7

$\bigoplus_I\mathcal{O}_X|_{f^{-1}U}\to\bigoplus_J\mathcal{O}_X|_{f^{-1}U}\to f^*G|_{f^{-1}U}\to 0$ exact, and $x\in f^{-1}U$ open.

$F\in Coh(Y)\Rightarrow F$ locally finitely presented $\Rightarrow f^*F$ loc. finitely presented $\Rightarrow f^*F\in Coh(X)$ □

(above proof for I,J finite) ← using X loc. Noeth

7.4 Push-forwards for X Noetherian

Claim $f:X\to Y$ morph of schemes, X Noetherian $\Rightarrow f_*: QCoh\,X\to QCoh\,Y$

← issue is f^{-1}(affine) need not be affine. For affine morphs you get result by Sec. 6.5

Pf $0\to F\xrightarrow{restr.}\prod F|_{U_i}\longrightarrow\prod F|_{U_{ijk}}$ exact by sheaf property, where $X=\bigcup U_i$ affine open cover, $U_i\cap U_j=\bigcup U_{ijk}$ " " "

← take differences of sections on overlaps (Sec. 1.4)

Recall f_* left-exact & commutes with limits e.g. with $\prod\Rightarrow 0\to f_*F\to\prod f_*(F|_{U_i})\to\prod f_*(F|_{U_{ijk}})$ exact ← Sec. 6.7

WLOG Y open affine $=\operatorname{Spec}R$ (replace X by $f^{-1}(\operatorname{Spec}R)$), WLOG $F|_{U_i}=\widetilde{F(U_i)}$, so $f_*(F|_{U_i})={}_R\widetilde{F(U_i)}$ ← Sec. 6.5

similarly for U_{ijk}. If show $\prod f_*(F|_{U_i}),\prod f_*(F|_{U_{ijk}})\in QCoh(Y)$ then $f_*F\in QCoh(Y)$ ← Trick (2) in 7.2

X Noeth $\Rightarrow U_{ij}$ quasi-compact $\Rightarrow$ finite covers $\Rightarrow\prod$ is $\oplus$, but $\sim$ commutes with $\oplus$ so finally done! □ ← Sec. 3.1

Rmk X quasi-compact, separated $\Rightarrow f_*: QCoh\,X\to QCoh\,Y$ ← proof above but easier $U_{ijk}=U_i\cap U_j$ affine!

Non-examinable fact f proper, X,Y loc. Noeth. $\Rightarrow f_*: Coh\,X\to Coh\,Y$ ← e.g. $\mathbb{A}^1_k\xrightarrow{f}\operatorname{Spec}k$, $f_*\mathcal{O}_{\mathbb{A}^1_k}=k[x]$ not finite k-mod

Otherwise in general f_* can ruin (quasi)-coherence

(e.g. $\bigsqcup_{n\in\mathbb{N}}\mathbb{A}^1\xrightarrow{f}\mathbb{A}^1$ obvious morph, $F=\widetilde{\prod k[t]}$, $f_*F=\widetilde{\prod k[t]}$ if assume $\in QCoh$ ← by 7.6
but notice $(\frac{1}{t^n})\in F(\bigsqcup D_t)=(f_*F)(D_t)$ but $\notin(\widetilde{\prod k[t]})(D_t)=(\prod k[t])_t\neq\prod(k[t]_t)$.)

7.5 Gluing modules

Similar to Sec. 4.1: R ring $\ni f_1,\dots,f_n$ s.t. $1\in\langle\text{all } f_i\rangle$ ⊛

data: $\cdot\ M_i$ R_{f_i}-mod ← (so have $\widetilde{M_i}$ on $D_{f_i}\subseteq\operatorname{Spec}R$)
$\cdot\ \psi_{ij}: (M_i)_{f_j}\to(M_j)_{f_i}$ iso of $R_{f_if_j}$-mods ← (so $\widetilde{M_i}\cong\widetilde{M_j}$ on $D_{f_if_j}\subseteq\operatorname{Spec}R$)
$\cdot\ \psi_{ii}=id$

} cocycle condition

$$\begin{tikzcd} (M_i)_{f_jf_k} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"'] & & (M_k)_{f_if_j} \\ & (M_j)_{f_if_k} \arrow[ur, "\psi_{jk}"'] & \end{tikzcd}$$

← case $k=i$ get $\psi_{ji}=\psi_{ij}^{-1}$. Take $\sim$ get cond^{ns} of Sec. 4.1

Define $M:=\operatorname{Ker}\left(\begin{array}{ccc}\bigoplus_i M_i & \xrightarrow{\varphi} & \bigoplus_{i,j}(M_i)_{f_j}\\ (m_i) & \longmapsto & \left(\frac{m_i}{1}-\psi_{ji}\left(\frac{m_j}{1}\right)\right)\end{array}\right)$ ← Idea: local data which agrees on overlaps

Call $\pi_i: M\to M_i$ the projections.

Gluing lemma π_i induces isos $M_{f_i}\to M_i$ and $\psi_{ij}\circ\frac{\pi_i(m)}{1}=\frac{\pi_j(m)}{1}$ $\forall m\in M$ (with $\frac{\pi_i(m)}{1}\in(M_i)_{f_j}$, $\frac{\pi_j(m)}{1}\in(M_j)_{f_i}$)

Pf Enough to show π_ℓ iso after localising at every prime $q\in\operatorname{Spec}R_{f_\ell}$ ← see Sec. 3.0

$\Rightarrow q=pR_{f_\ell}$ with $f_\ell\notin p\in\operatorname{Spec}R$. By exactness of localisation

$(M_{f_\ell})_q=M_p=\operatorname{Ker}(\bigoplus(M_i)_p\xrightarrow{\varphi_p}\bigoplus((M_i)_p)_{f_j})$ ← R_p-mods

$f_\ell\in R_p$ is unit so WLOG replace: $R\rightsquigarrow R_p$, $M\rightsquigarrow M_p$, $M_i\rightsquigarrow(M_i)_p$, $f_\ell\rightsquigarrow 1$. ← "WLOG" in sense that localising at f_ℓ is like localising at 1 since f_ℓ is a unit in R_p

Abbreviate $N=M_\ell$ so: $\pi_\ell: M=\ker\varphi_p\subseteq(N\oplus\bigoplus_{i\neq\ell}M_i)\longrightarrow N$
$\psi_{\ell i}: N_{f_i}\xrightarrow{\cong}(M_i)_1=M_i$

WLOG $M_i = N_{f_i}$ (identify via $\psi_{\ell i}$), so cocycle cond. becomes:

$$\begin{tikzcd} N_{f_j f_k} \arrow[r, "="] \arrow[d, "="'] & (M_k)_{f_j} \\ (M_j)_{f_k} \arrow[ru, "\psi_{jk}"'] & \end{tikzcd}$$

$\psi_{\ell k}$ is now id; hence $\psi_{jk}=$ id

$$\Rightarrow \quad 0 \to N \xrightarrow{\text{natural}} \bigoplus_i N_{f_i} \xrightarrow{\varphi_\ell} \bigoplus_{i,j} N_{f_i f_j}$$

$(N \to N \oplus \bigoplus_{i\neq \ell} N_{f_i},\ n \mapsto n \oplus \bigoplus_{i \neq \ell} \frac{n}{1})$ $\quad (x_i) \mapsto \left(\frac{x_i}{1} - \frac{x_j}{1}\right)$

Sub-claim This is exact ($\Rightarrow N = \operatorname{Ker}\varphi_p = M$, π_ℓ iso, $\psi_{jk} = \text{id}$ under identifications via π maps)

Pf Enough to prove after localising at each max ideal m ← See 3.0

By ⊛ not all $f_i \in m$ otherwise $1 \in \langle \text{all } f_i \rangle \subseteq m$ ↯

Say $f_k \notin m$, so WLOG replace $N \rightsquigarrow N_m$, $R \rightsquigarrow R_m$, $f_k \rightsquigarrow 1$: ← $f_k \notin M$ so unit in local ring R_m

$$\Rightarrow \quad 0 \to N \longrightarrow \underbrace{N}_{= N_{f_k}} \oplus \bigoplus_{i \neq k} N_{f_i} \longrightarrow \bigoplus_{i,j} N_{f_i f_j}$$

clearly injective

$\underbrace{n \oplus \bigoplus_{i\neq k} n_i}_{\text{hence } = n \oplus \bigoplus \frac{n}{1} \text{ so image of } n \text{ via previous map}} \in \operatorname{Ker}$ then $\frac{n}{1} = \frac{n_i}{1} \in N_{f_i f_k} = N_{f_i}$ $\forall i$ ☐

7.6 QCoh(X), Coh(X), Vect(X) for X = Spec R

Theorem For $X = \operatorname{Spec} R$, $\exists$ equivalence of categories

$$\boxed{\begin{array}{ccc} R\text{-Mods} & \longrightarrow & \operatorname{QCoh}(X) \\ M & \longmapsto & \widetilde{M} \\ F(X) = \Gamma(X,F) & \longleftarrow\!\!| & F \end{array}}$$

← means: the two given functors compose to functors which are naturally iso to identity functors

Pf. Easy direction: $M \mapsto F = \widetilde{M} \mapsto F(X) = \widetilde{M}(X) = M$. Converse: given F want $F \cong \widetilde{F(X)}$.

$\Rightarrow$ locally $\forall p \in X, \exists p \in D_f$ s.t. $F|_{D_f} \underset{\cong}{\xrightarrow{\varphi_f}} \widetilde{N}$ some R_f-mod N

cover X by finitely many such, say N_i on D_{f_i}, $i = 1, \dots, n$, so $1 \in \langle \text{all } f_i \rangle$

} By Cor in 7.1 using that D_f are basis of topology and Spec R quasi-compact

$\Rightarrow$ On overlaps: $\psi_{ij} : \widetilde{(N_i)_{f_j}} \underset{\cong}{\xrightarrow{\varphi_{f_i}^{-1}}} F|_{D_{f_i f_j}} \underset{\cong}{\xrightarrow{\varphi_{f_j}}} \widetilde{(N_j)_{f_i}}$ satisfy cocycle condition ← since $\widetilde{(N_i)}_{f_j f_k}$ and other two are identified with $F|_{D_{f_i f_j f_k}}$

$\Rightarrow$ by gluing thm $\exists M$ with $M_{f_i} = N_i$ compatibly with the ψ_{ij}

But then $\widetilde{M}, F$ have isomorphic local gluing data for cover $X = D_{f_1} \cup \dots \cup D_{f_n}$ so $\widetilde{M} \cong F$. ☐

(Explicitly: $m \in M \mapsto m_i = \frac{m}{1} \in M_{f_i} = N_i \xrightarrow{\varphi_{f_i}^{-1}} s_i \in F(D_{f_i})$ and $s_i|_{D_{f_i f_j}} = s_j|_{D_{f_i f_j}}$ so globalises to unique $s \in F(X)$. Recall $M \to F(X)$ determines $\widetilde{M} \to F$ by Sec. 6.9)

Cor $X = \operatorname{Spec} R$: $F \in \operatorname{Coh} X \Leftrightarrow F = \widetilde{M}$ for coherent module M $(\cong F(X))$ ← and if R Noeth. get: $\Leftrightarrow F(X)$ f.g. R-mod

Pf $F = \widetilde{F(X)}$ by Theorem. In definition of coherent take global sections $\Rightarrow F(X)$ coherent R-mod, and conversely if M coherent get $\widetilde{M}$ coherent since $\sim$ is exact & fully faithful. ☐

Fact $X = \operatorname{Spec} R$: $F \in \operatorname{Vect} X \Leftrightarrow \left(F = \widetilde{M} \text{ for finitely presented } \underline{\text{flat}} \text{ R-mod } M \right)$ ($\Leftrightarrow$ f.g. projective R-mod)

(see Hwk 4)

projective means in R-mods Hom$(M, \cdot)$ exact. ($\Leftrightarrow$ M is a direct summand of some free R-mod)

8. Čech Cohomology

← Motivation for cohomology: assign group or rings of "invariants" to a space i.e. iso. spaces give isos of) e.g. if $H^*(X) \not\cong H^*(Y)$ then $X \not\cong Y$ are not iso. spaces

8.1 Čech complex

X top. space, $X = \bigcup U_i$ open cover

notation: $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$ ...

$U_I = U_{i_0} \cap \dots \cap U_{i_n}$ for $I = (i_0, \dots, i_n)$ multi-index, abbreviate $|I| = n$

ordered, allow repetitions; size is actually n+1

$$\boxed{C^n = \check{C}^n_{\{U_i\}} = \prod_{|I|=n} \Gamma(U_I, F)}$$

$F \in \operatorname{Ab}(X)$

← so $s \in C^n$ is a collection $s_I \in F(U_I)$ called cochain

$d = d^n : C^n \longrightarrow C^{n+1}$

$$\boxed{(ds)_I = \sum_{j=0}^{n+1} (-1)^j \underbrace{s_{I_j}|_{U_I}}_{\in F(U_I)}}$$

← where $I_j = (i_0, \dots, \hat{i_j}, \dots, i_{n+1})$ (omit)

later also use notation $I_{jk\dots}$ if omit $i_j, i_k, \dots$

$\in F(U_I)$ so sum makes sense.

Example $C^0 = \prod_i \Gamma(U_i) \xrightarrow{d} \prod_{i,j} \Gamma(U_{ij}) = C^1$

$(s_i) \longmapsto \left(s_j|_{U_{ij}} - s_i|_{U_{ij}} \right)$ ← $i_0 = i$, $i_1 = j$; $I = (i_0, i_1)$; $I_0 = (i_1) = j$

$C^1 = \prod_{i,j} \Gamma(U_{ij}) \xrightarrow{d} \prod_{i,j,k} \Gamma(U_{ijk}) = C^2$

$(s_{ij}) \longmapsto \left(s_{jk}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{ij}|_{U_{ijk}} \right)$ ← if you took C3.1 Algebraic Top. notice similar to simplicial differential

Claim $d^2 = 0$, so (C^*, d) is a complex

Pf

$$(dds)_J = \sum_{k=0}^{n+2} (-1)^k (ds)_{J_k}\Big|_{U_J} = \sum_{k=0}^{n+2} \left(\sum_{j<k} (-1)^{k+j} s_{J_{kj}} \Big|_{U_J} + \sum_{j>k} \underline{\underline{(-1)^{k+j-1}}} s_{J_{kj}}\Big|_{U_J} \right)$$

$J_k = (j_0, \dots, \hat{j_k}, \dots, j_{n+2})$; $\cdot|_{U_{J_k}}|_{U_J} = \cdot|_{U_J}$; since j_k missing in J_k

$= 0$. ☐

anti-symmetry if swap j, k (notice full sum is over all $j \neq k$)

Def $\boxed{H^n(X,F) = \check{H}^n_{\{U_i\}}(X,F) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}}$ ← ($H^n(X,F)$ depend on choice of U_i)

Rmk $d^n \circ d^{n-1} = 0$ so $\operatorname{Im} d^{n-1} \subseteq \operatorname{Ker} d^n$

$\operatorname{Im} d^{n-1}$ called coboundaries, $\operatorname{Ker} d^n$ called cocycles; "co" sometimes omitted. Emphasizes doing cohomology

Lemma $H^0(X,F) = \Gamma(X,F)$

Pf $s_j|_{U_{ij}} = s_i|_{U_{ij}}$ says s glues to global section. ☐

Terminology 1) hom of complexes $f : C^n \to C^n$ is chain map if $f \circ d = d \circ f$

2) $h : C^n \to C^{n-1}$ is chain homotopy between chain maps f, g if $f - g = d \circ h + h \circ d$

Consequences: 1) $f : H^n \to H^n$ via $f[c] = [fc]$ well-defined ← $[c] = [c + db]$ but $[fdb] = [dfb] = 0$

2) $f = g : H^n \to H^n$ ← $(dc = 0 \Rightarrow [fc - gc] = [dhc] = 0)$

Key trick To show $H^* = 0$ can find chain homotopy between id, 0.

i.e. C^* is exact, also called acyclic

Rmk If a homomorphism $d_n : C_n \to C_{n-1}$ decreases the degree by 1, and $d_{n-1} \circ d_n = 0$ then $H_n = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ is called the homology of (C_*, d_*). In this case a chain homotopy is degree increasing: $h : C_n \to C_{n+1}$ with $f_n - g_n = d_{n+1} \circ h_n - h_{n-1} \circ d_n$.

8.2 $\check{C}$ech complex with ordering

Repetitions of indices are annoying since $C^n \neq 0$ all $n \geqslant 0$ even if finite # U_i ← e.g. if X quasi-compact

Trick pick total ordering on indices

C^n_+ : as C^n but only allow $I=(i_0,\dots,i_n)$ if $i_0<i_1<\dots<i_n$, d as before ← so if finite cover with N sets, $C^n_+=0$ for $n \geqslant N$; $H^n_+=0$ "

$\Longrightarrow C^n_+ \subseteq C^n$ subcomplex

Claim $H^n_+ \cong H^n$

Non-examinable Proof ("Serre's Trick") ← I'm doing a hands-on proof based on
Serre "FAC" 1955 Sec. 20, p. 214
Godement "Théorie des Faisceaux" 1958 p. 60
Eilenberg & Skenrod "Foundations of Alg. Top." 1952, VI.6

Let $S_* =$ free abelian group generated by all index sets I, so : $S_n = \langle I : |I|=n \rangle$ ← (I is really a function $\{0,1,\dots,n\} \to \{\text{indices}\}$)

Differential: $\partial I = \sum (-1)^j I_j$ so $\partial : S_n \to S_{n-1}$.

$S^+_* =$ subgroup generated by strictly ordered index sets I ← (so strictly increasing function for chosen total order on set)

Step 1 S_*, S^+_* are acyclic

Pf $h : S^+_* \to S^+_{*+1}$, $h(I) = \begin{cases} (\ell, I) & \text{if } \ell \neq i_0 \\ 0 & \text{if } \ell = i_0 \end{cases}$ ← $\ell :=$ minimal index $\Rightarrow$ if $\ell \neq i_0$: $\partial h I = \partial(\ell,I) = I + \sum(-1)^{j+1}(\ell, I_j)$
$h \partial I = h \sum (-1)^j I_j = \sum (-1)^j (\ell, I_j)$

$\Rightarrow I = (\partial h + h \partial) I$. Exercise: check same holds if $\ell = i_0$.

$\Rightarrow id - 0 = \partial h + h \partial$ ✓ For S_* it is even easier: $h(I) = (\ell, I)$ works. □

Step 2 $f(I) := \begin{cases} 0 & \text{if } \exists \text{ repeated indices in } I \\ \text{sign}(\sigma)\cdot \sigma(I) & \text{otherwise, where } \sigma \text{ unique permutation s.t. } \sigma I \text{ ordered} \end{cases}$ ← $\sigma j_0 < \sigma j_1 < \dots$

$\Rightarrow f$ chain map, $f=id$ on S_0, $f(S_*) \subseteq S^+_*$, $f \circ f = f$ (i.e. f is id on S^+_*, f is a projection to S^+_*)

Pf $\sigma(I) \in S^+_*$ and if I is ordered then $\sigma = id$. On S_0 : $f((i_0)) = (i_0)$.

$\partial f I = \sum (-1)^j \text{sign}(\sigma)\, \sigma(I)_j$ ← for $k = \sigma^{-1}(j)$ get same set, $\text{sign}(\sigma) = \text{sign}(\tau)\cdot(-1)^{k-j}$ since
$f \partial I = \sum (-1)^k \text{sign}(\tau)\, \tau(I_k)$ — σ does an extra $k-j$ transpositions to move i_j to position k □

Step 3 General trick: C_* free acyclic complex, a chain map $f : C_* \to C_*$ has $f_0 = id : C_0 \to C_0$ then f, id are chain homotopic: $\exists k : C_* \to C_{*+1}$ with $f - id = \partial k + k \partial$

Pf Build k inductively by equation $\partial_{n+1} \circ k_n = f_n - id - k_{n-1} \circ \partial_n$

$$\begin{tikzcd} 0 & C_0 \arrow[l, "\partial_0"'] \arrow[d, "f_0=id"'] \arrow[dr, "k_0"] & C_1 \arrow[l, "\partial_1"'] \arrow[d, "f_1"] \\ 0 & C_0 \arrow[l, "\partial_0"] & C_1 \arrow[l, "\partial_1"] \end{tikzcd}$$

Warning the k_i do not make diagram commute

$n=0$: $\partial_1 \circ k_0 = \underbrace{f_0 - id}_{=0} - \underbrace{k_{-1}}_{=0} \circ \partial_0$ so just define $k_0 = 0$.

Assume true for $n-1$: $\partial_n k_{n-1} = f_{n-1} - id - k_{n-2}\partial_{n-1}$ ⊛

$$\begin{tikzcd} C_{n-2} \arrow[d, "f_{n-2}"'] \arrow[dr, "k_{n-2}"'] & C_{n-1} \arrow[l, "\partial_{n-1}"'] \arrow[d, "f_{n-1}"] \arrow[dr, "k_{n-1}"] & C_n \arrow[l, "\partial_n"'] \arrow[d, "f_n"] \\ C_{n-2} & C_{n-1} \arrow[l, "\partial_{n-1}"] & C_n \arrow[l, "\partial_n"] \end{tikzcd}$$

$\partial_n (f_n - id - k_{n-1} \circ \partial_n) = f_{n-1}\partial_n - \partial_n - (\partial_n \circ k_{n-1})\partial_n$

⊛ $\rightarrow = f_{n-1}\partial_n - \partial_n - (f_{n-1} - id - k_{n-2}\partial_{n-1})\partial_n$

$= 0$ since $\partial \circ \partial = 0$ and f is chain map

C_* exact $\Rightarrow \forall c_n \in C_n, (f_n - id - k_{n-1}\partial_n) c_n = \partial_{n+1} c_{n+1}$ some $c_{n+1} \in C_{n+1}$. As C_n is free we can pick basis elts c_n of C_n and pick such c_{n+1}, then define $k_n(c_n) := c_{n+1}$ and extend k_n linearly to get $k_n : C_n \to C_{n+1}$ $\Rightarrow$ get required equation for n. ✓

Step 4 chain maps/homotopies on S_*, S^+_* induce corresponding chain maps/hpies on C^*, C^*_+

Pf If $\varphi(I) = \sum n_{II'} \cdot I'$, $n_{II'} \in \mathbb{Z}$ then define $(\check{\varphi}(s))_I = \sum n_{II'} \cdot s_{I'}|_{U_I}$
(φ hom on S_* or S^+_*) ($\check{\varphi}$ hom on C^* or C^*_+ respectively)

Example $d = \check{\partial}$, and for f of Step 2 : $(\check{f}(s))_I = \begin{cases} 0 & \text{if } \exists \text{ repeated indices in } I \\ \text{sign}(\sigma)\cdot s_{\sigma(I)}|_{U_I} & \text{else} \end{cases}$

Conclusion: $\check{f} : C^* \to C^*$ chain hpic to id and surjects onto $C^*_+ \Rightarrow [\check{f}] = id : H^* \to H^*_+ \cap H^*$ ← hence equal. □

Cor • H^*_+ is independent of choice of total ordering on set of indices (since $H^*_+ \cong H^*$)

• $\check{H}^m_{\{U_i\}}(X,F) = 0$ for $m \geqslant N$ if $X = \cup U_i$ if finite cover with N sets (since $U_I = \emptyset$ in $\check{C}^*_+$ if $|I| \geqslant N$)

Example $X = \mathbb{P}^n_k$ with cover by $N = n+1$ affine sets $U_i \cong \mathbb{A}^n_k$ (Hwk 2)

8.3 Affines have no cohomology except H^0

Theorem $X = \text{Spec } R$
$F \in QCoh(X)$ $\Longrightarrow \check{H}^n(X,F) = 0$ for $n \geqslant 1$ ← (compare $H^*(\mathbb{C}^n)=0$ for $* \geqslant 1$ in algebraic topology)
$X = \cup U_i$ finite affine open cover

Pf X separated (since affine) $\Longrightarrow U_I$ all affine (Sec. 5.3, ⑧)

use ordered Čech cohomology. $s \in C^n$, $hs \in C^{n-1}$, $I = (i_0,\dots,i_{n-1})$, $i_0 < i_1 < \dots < i_{n-1}$

Easy case: minimal index ℓ satisfies $U_\ell = X$

notice this is Step 1 & 4 of Sec. 8.2

chain homotopy: $(hs)_I = \begin{cases} 0 & \text{if } i_0 = \ell \\ s_{\ell,I} & \text{if } i_0 \neq \ell \end{cases}$ (so $\ell < i_0$) ← $s \in \check{C}^{n-1}$, $n \geqslant 1$ ← $U_{\ell,I} = U_\ell \cap U_I = X \cap U_I = U_I$

for I with $i_0 \neq \ell$:

$(d(hs))_I = \sum (-1)^j (hs)_{I_j} = \sum (-1)^j s_{\ell, I_j}$
$(h(ds))_I = (ds)_{\ell,I} = s_I + \sum (-1)^{j+1} s_{\ell, I_j}$

$\Rightarrow id = dh + hd$ ← Exercise check case $I = (\ell, i_1, \dots)$ also works.

$\Rightarrow$ Key Trick ✓ (Sec. 8.1)

General case $X = \text{Spec } R = \cup U_i$, $U_i = \text{Spec } R_i$

By easy case, know result for space U_ℓ with covering $\cup(U_\ell \cap U_i)$, for minimal ℓ. Ordering of indices does not affect H^*, so know result for any ℓ by Cor of 8.2

$\Rightarrow$ Reduce to claim: if C^* exact when restrict to U_i $\forall i$, then C^* exact

$F \in QCoh(X)$, U_I affine say Spec $R_I \overset{7.6}{\Longrightarrow} F|_{U_I} \cong \widetilde{M_I}$ some R_I-module M_I

$C^n = \prod\limits_{|I|=n} F(U_I, F) = \prod\limits_{|I|=n} M_I$ finite product so $= \oplus$ (since finite cover U_i) (in particular, an R-mod since $R \to R_I$ from $U_I \to X$)

$\Longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \to \dots$ is a complex of R-mods

and by assumption of exactness on U_i have: ← using $F|_{U_I}|_{U_i} = \widetilde{M_I}|_{U_i} \cong \widetilde{M_I \otimes R_i}$ by 6.8 and $\oplus \widetilde{N_i} = \widetilde{\oplus N_i}$

$C^0 \otimes_R R_i \to C^1 \otimes_R R_i \to \dots$ exact $\forall i$

$\Longrightarrow$ localising further by $\cdot \otimes_{R_i} (R_i)_p$ get exactness of localisation of C^* at each $p \in \text{Spec } R$. ← U_i cover X so $p \in U_i$ some i

$\Longrightarrow$ by Sec. 3.0 deduce exactness of C^*. □

RmK Chain homotopy trick above can be used to show $\check{H}^*(X, \underline{A}) = 0$ for $* \geqslant 1$ if X irreducible scheme and $\underline{A}$ is constant sheaf with values in abelian group A. (e.g. $\check{H}^1 = 0$: given cocycle g_{ij} fix index i_0 define $h \in \check{C}^0$ by $h_i = g_{i_0 i} \in \Gamma(U_{i_0 i}) = A = \Gamma(U_i)$ Cocycle $\Rightarrow g_{i_0 j} = g_{i_0 i} + g_{ij}$ so $(dh)_{ij} = h_j - h_i = g_{ij}$)

8.4 Independence of cover

Theorem X separated, quasi-compact $\Longrightarrow \check{H}^*(X,F)$ independent of choice of finite affine open cover ← $F \in Coh(X)$

Pf Will use ordered Čech cohomology.

X separated $\Longrightarrow \bigcap\limits_{\text{finite}}$ affines is affine (Sec. 5.3, ⑧)

$X = \cup U_i$, $X = \cup V_j$ ← finite covers, take mixed intersections: $C^{n,m} = \prod\limits_{\substack{|I|=n \\ |J|=m}} \Gamma(U_I \cap V_J, F)$ ← "bi-complex"

$C^{n,\bullet} \cong \prod\limits_{|I|=n} \check{C}_{\{V_j \cap U_I\}}(F|_{U_I})$ ← finite affine cover of the affine U_I so by 8.3 $H^*=0$

$C^{\bullet,m} \cong \prod\limits_{|J|=m} \check{C}_{\{U_i \cap V_J\}}(F|_{V_J})$ ← similar

$$\begin{tikzcd} \dots & \dots & \dots \\ C^{0,2} \arrow[u] \arrow[r] & C^{1,2} \arrow[u] \arrow[r] & C^{2,2} \arrow[u] \arrow[r] & \dots \\ C^{0,1} \arrow[u] \arrow[r] & C^{1,1} \arrow[u] \arrow[r] & C^{2,1} \arrow[u] \arrow[r] & \dots \\ C^{0,0} \arrow[u] \arrow[r] & C^{1,0} \arrow[u] \arrow[r] & C^{2,0} \arrow[u] \arrow[r] & \dots \end{tikzcd}$$

$\Longrightarrow$ rows & columns are exact except for degree 0:

$H^0(C^{n,\bullet}) = \prod\limits_{|I|=n} \Gamma(U_I, F) = \check{C}_{\{U_i\}}(F)$

$H^0(C^{\bullet,m}) = \prod\limits_{|J|=m} \Gamma(V_J, F) = \check{C}_{\{V_i\}}(F)$

General fact from homological algebra

$C^{i,j}$ bi-complex, $H^i(C^{n,\bullet})=0 \ \forall i>0, \forall n$ ⟸ $i,j\geq 0$
$H^i(C^{\bullet,m})=0 \ \forall i>0, \forall m$

$\Rightarrow$ $H^0(C^{n,\bullet}) =: A^n$ complex in n; $H^0(C^{\bullet,m}) =: B^m$ " " m } with iso cohomology $H^*(A^\bullet)\cong H^*(B^\bullet)$

Sketch Pf

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & B^1 \arrow[r] \arrow[u] & C^{0,1} \arrow[r] \arrow[u] & C^{1,1} \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & B^0 \arrow[r] \arrow[u] & C^{0,0} \arrow[r] \arrow[u] & C^{1,0} \arrow[r] \arrow[u] & {} \\
 & & A^0 \arrow[r] \arrow[u] & A^1 \arrow[r] \arrow[u] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

(Note $A^i = \ker(C^{i,0}\to C^{i,1})$, $B^i = \ker(C^{0,i}\to C^{1,i})$)

Now rows & cols are exact, so can diagram chase, and get a "zig-zag":

$$\begin{tikzcd}
\exists c_3 \arrow[r] & c_2 \arrow[r] & 0 & \\
H^1(B^\bullet) \arrow[u, phantom, "\ni" description] & \exists c_1 \arrow[r] \arrow[u] & c \arrow[r] \arrow[u] & 0 \\
 & {} \arrow[r] & c \arrow[r] \arrow[u] & 0 \\
 & & H^1(A^\bullet) \arrow[u, phantom, "\ni" description] &
\end{tikzcd}$$

so $H^1(A^\bullet)\to H^1(B^\bullet)$, $c\longmapsto c_3$ via the iso □

8.5 Induced Long Exact Sequence on $\check{H}^\bullet$

Recall $\Gamma(X,\cdot): Ab(X)\to Ab$ is always left exact (Sec. 1.9)

Lemma U open affine $\subseteq$ scheme $X \Rightarrow \Gamma(U,\cdot): QCoh\, X \longrightarrow Ab$ is exact

Pf Given $F_1\to F_2\to F_3$ exact. Exactness is local condition (indeed stalks)
$\Rightarrow$ WLOG $F_i|_U = \widetilde{M}_i$. $\widetilde{M}_1\to\widetilde{M}_2\to\widetilde{M}_3$ exact $\Leftrightarrow$ $M_1\to M_2\to M_3$ exact □

Recall Sec. 6.4
$R\text{-mod}\to QCoh(\operatorname{Spec} R)$
$M\longmapsto \widetilde{M}$
is exact and fully faithful

Claim X separated, $0\to F_1\to F_2\to F_3\to 0$ SES in $QCoh(X)$

SES = short exact sequence
LES = long "

$\Longrightarrow$ get LES (using affine cover) $0\to H^0(X,F_1)\to H^0(X,F_2)\longrightarrow H^0(X,F_3)\to H^1(X,F_1)\to H^1(X,F_2)\to\cdots$

with $H^0(X,F_1) = \Gamma(X,F_1)$, $H^0(X,F_2) = \Gamma(X,F_2)$, $H^0(X,F_3) = \Gamma(X,F_3)$

(e.g. Ker measures failure of $\Gamma(X,\cdot)$ being right-exact

Pf $0\to F_1(U_I)\to F_2(U_I)\to F_3(U_I)\to 0$ exact by Lemma. (U_I affine since X separated)
$\Longrightarrow 0\to \check{C}^\bullet(F_1)\to\check{C}^\bullet(F_2)\to\check{C}^\bullet(F_3)\to 0$ exact, claim follows □

homological algebra: SES of chain complexes induces LES on cohomology (e.g. see my C3.1 notes)

8.6 Dealing with infinite covers

A refinement of an open cover $X=\bigcup U_i$ (top. space) is an open cover $X=\bigcup V_j$ s.t. $\forall j$, $V_j\subseteq U_i$ some i (∃ choice!)

Make choices ($i\mapsto i(j)$) $\Rightarrow$ restrictions $F(U_{i(j)})\to F(V_j)$ (sheaf) $\Rightarrow \check{C}_{\{U_i\}}(X,F)\longrightarrow\check{C}_{\{V_j\}}(X,F)$ chain map.
(on V_J get restriction from $F(U_{i(j_0)\dots i(j_n)})$)

Fact $\check{H}_{\{U_i\}}(X,F)\longrightarrow\check{H}_{\{V_j\}}(X,F)$ does not depend on choices made (Serre, "FAC", Sec.21)

Def ✪ $\boxed{\check{H}(X,F) = \varinjlim \check{H}_{\{U_i\}}(X,F)}$ ← (so each class is represented by a Čech cocycle for some cover, and identify cocycles if they differ by a boundary after passing to some common refinement

Non-examinable Rmk For any topological space homotopy equivalent to a CW complex (e.g. any manifold)
$\check{H}^*(X,\underline{A}) \cong H^*(X;A)$ = singular cohomology of X with coefficients in A ← so for smooth manifolds, and $A=\mathbb{R}$, get de Rham cohomology
($\underline{A}$ is "constant sheaf" with values in A: actually means sheafify, so $\underline{A}(U)=\{$locally constant $U\to A\}$)

Rmk X quasi-compact scheme (e.g. affine) $\Longrightarrow$ can use finite covers by affine opens, and can refine any cover by such a cover
$\Longrightarrow$ can calculate ✪ by only using finite affine covers

$U_i = \bigcup_j A_{ij}$ (affine)
$X = \bigcup_{i,j} A_{ij}$
pick finite subcover

Cor Theorem in 8.3 holds $\forall$ cover (using definition ✪)

Cor X separated quasi-compact sch. $\Rightarrow$ can calculate ✪ with one cover!
(by Theorem 8.4 $\Rightarrow$ maps in $\varinjlim$ for such covers are isos so $\check{H}_{\{U_i\}}(X,F)\to\varinjlim\dots$ is iso.)

8.7 Application: line bundles and $\check{H}^1(X,\mathcal{O}_X^*)$

X scheme, $F\in Vect(X)$
$\Longrightarrow \exists$ open cover $X=\bigcup U_i$ with $F|_{U_i}\xrightarrow[\varphi_i]{\cong}\mathcal{O}_{U_i}^{\oplus n_i}$ some $n_i\in\mathbb{N}$ (called a trivialization over U_i)
and can compare trivializations on overlaps:

$$\begin{tikzcd}
F|_{U_{ij}} \arrow[r, "\cong", "\varphi_i"'] \arrow[d, equal] & \mathcal{O}_{U_{ij}}^{\oplus n_i} \arrow[d, dashed, "\alpha_{ij}", "\cong"'] \\
F|_{U_{ji}} \arrow[r, "\cong", "\varphi_j"'] & {\mathcal{O}_{U_{ji}}^{\oplus n_j} = \mathcal{O}_{U_{ij}}^{\oplus n_j}}
\end{tikzcd}$$

α_{ij} called transition maps

$\mathcal{O}_{U_{ij}}$-module iso described by an invertible $n_j\times n_i$ matrix with entries in $\mathcal{O}_{U_{ij}}(U_{ij})$

(see Sec. 6.2: $Hom_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n},\mathcal{O}_X)\cong\Gamma(X,\mathcal{O}_X)^{\oplus n}$)

$\Longrightarrow n_i=n_j$ if $U_{ij}\neq\emptyset$, so the rank of F is locally constant.

Conversely, given such data φ_i,α_{ij} satisfying the cocycle condition $\alpha_{jk}\circ\alpha_{ij}=\alpha_{ik}$ on U_{ijk} determines by gluing a vector bundle. ← This is the actual definition of vector bundle in terms of compatible local trivializations.

Rmk $\alpha_{ji}=\alpha_{ij}^{-1}$

Def $\mathcal{O}_X^*\subseteq\mathcal{O}_X$ sheaf of invertible functions. So $\mathcal{O}_X^*(U)=\{f\in\mathcal{O}_X(U): \exists g\in\mathcal{O}_X(U) \text{ s.t. } f\cdot g=1\}$
Note that $\mathcal{O}_X^*(U)$ is an abelian group under multiplication.

Theorem $\left\{\begin{array}{l}\text{isomorphism classes of line bundles}\\ \text{that admit a trivialization over } U_i\end{array}\right\} \overset{1:1}{\longleftrightarrow} \check{H}^1_{\{U_i\}}(X,\mathcal{O}_X^*)$

and $Pic(X)\cong\check{H}^1(X,\mathcal{O}_X^*)$ as groups. ← (Pic X defined in 7.2)

Pf $\alpha_{ij}:\mathcal{O}_{U_{ij}}\longrightarrow\mathcal{O}_{U_{ij}}$ given by multiplication by element $\in\mathcal{O}^*_{U_{ij}}$

- tensoring line bundles that admit a trivialization on U_{ij}:

$$\begin{tikzcd}
\mathcal{O}_{U_{ij}} \cong \mathcal{O}_{U_{ij}}\otimes_{\mathcal{O}_{U_{ij}}}\mathcal{O}_{U_{ij}} \arrow[r, "\alpha_{ij}\otimes\tilde{\alpha}_{ij}"] \arrow[rr, bend right, "\text{multiplication by } \alpha_{ij}\cdot\tilde{\alpha}_{ij}\in\mathcal{O}^*_{U_{ij}}"'] & \mathcal{O}_{U_{ij}}\otimes_{\mathcal{O}_{U_{ij}}}\mathcal{O}_{U_{ij}} \cong & \mathcal{O}_{U_{ij}}
\end{tikzcd}$$

- Cocycle condition can be rewritten: $\alpha_{jk}\cdot\alpha_{ik}^{-1}\cdot\alpha_{ij}=1$
(which is the statement $s_{jk}-s_{ik}+s_{ij}=0$ in multiplicative notation)

$\Longrightarrow(\alpha_{ij})\in\check{H}^1_{\{U_i\}}(X,\mathcal{O}_X^*)$

In $\check{H}^1$ we identify $[(\tilde{\alpha}_{ij})]=[(\alpha_{ij})] \Longleftrightarrow \alpha_{ij}=\tilde{\alpha}_{ij}\beta_j\beta_i^{-1}$ some $\beta_i\in\mathcal{O}^*_{U_i}$ ← ($(s_i)\in\check{C}^0$, $d(s_i)=s_j-s_i$ on U_{ij} in additive notation)

This corresponds precisely to how the $\check{C}^1$ class changes under an iso of line bundles $\mathcal{L}\cong\tilde{\mathcal{L}}$ as in claim:

$\beta_i :=$ composite $\left(\mathcal{O}_{U_i}\xleftarrow[\tilde{\varphi}_i]{\cong}\tilde{\mathcal{L}}|_{U_i}\cong\mathcal{L}|_{U_i}\xrightarrow[\varphi_i]{\cong}\mathcal{O}_{U_i}\right)\in\mathcal{O}^*_{U_i}$

$$\begin{tikzcd}
\mathcal{O}_{U_{ij}} \arrow[d, "\tilde{\alpha}_{ij}"'] \arrow[rrr, bend left, "\beta_i"] & \tilde{\mathcal{L}}|_{U_{ij}} \arrow[l, "\cong", "\tilde{\varphi}_i"'] \arrow[r, phantom, "\cong"] \arrow[d, equal] & \mathcal{L}|_{U_{ij}} \arrow[r, "\cong", "\varphi_i"'] \arrow[d, equal] & \mathcal{O}_{U_{ij}} \arrow[d, "\alpha_{ij}"] \\
\mathcal{O}_{U_{ij}} \arrow[rrr, bend right, "\beta_j"'] & \tilde{\mathcal{L}}|_{U_{ji}} \arrow[l, "\cong", "\tilde{\varphi}_j"'] \arrow[r, phantom, "\cong"] & \mathcal{L}|_{U_{ji}} \arrow[r, "\cong", "\varphi_j"'] & \mathcal{O}_{U_{ji}}
\end{tikzcd}$$

□

In the case $\mathcal{L}=\tilde{\mathcal{L}}$ the diagram shows that the $\check{C}^1$ class changes by a boundary chain if we change the choice of trivialization on each U_i →

$$\begin{tikzcd}
F|_{U_i} \arrow[r, "\cong", "\varphi_i"'] \arrow[d, equal] & \mathcal{O}_{U_i} \arrow[d, dashed, "\beta_i"] \\
F|_{U_i} \arrow[r, "\cong", "\tilde{\varphi}_i"'] & \mathcal{O}_{U_i}
\end{tikzcd}$$

Hence the $\check{H}^1$ class does not depend on the choices of the φ_i.

Rmk $\mathcal{L}$ line bundle with transition maps α_{ij}

$\Rightarrow \mathcal{L}^{-1}$ 〃 〃 〃 〃 α_{ij}^{-1} $\Big\}$ and $\mathcal{L}\underset{\mathcal{O}_X}{\otimes}\mathcal{L}^{-1} \cong \mathcal{O}_X$ = trivial line bundle

FACT line bundles on $\mathbb{A}^n$ are always trivial

indeed vector bundles on $\mathbb{A}^n$ are always trivial ← (Serre's Conjecture 1955 / Quillen–Suslin Theorem 1976)

EXAMPLE Pic($\mathbb{P}^1$) $\mathbb{P}^1_k = A_0 \cup A_1$, $A_0 = \operatorname{Spec} k[t]$, $A_1 = \operatorname{Spec} k[t^{-1}]$ ← In C3.4 course: view $\mathbb{P}^1_k = k^2\setminus 0 \,/\, k^*\text{-rescaling}$. Have homogeneous coordinates $[x_0:x_1]$ and A_0 corresponds to $\{[1:t] : t\in\mathbb{A}^1\}$ where $t = x_1/x_0$

$\mathcal{L}$ line bundle on $\mathbb{P}^1_k \Rightarrow \mathcal{L}|_{A_i}$ trivial since $A_i \cong \mathbb{A}^1$.

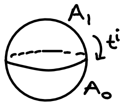
$(\alpha_{10} : \mathcal{L}|_{A^1} \longrightarrow \mathcal{L}|_{A^0}) \in k[t,t^{-1}]^* = \{a t^i : a\in k^*, i\in\mathbb{Z}\}$ ← note: $A_0\cap A_1 = \operatorname{Spec} k[t,t^{-1}]$

(exercise)

$\beta_0 \in k[t]^* = k^*$, $\beta_1 \in k[t^{-1}]^* = k^*$

$$\Rightarrow \boxed{\begin{matrix} \operatorname{Pic}(\mathbb{P}^1_k) \cong & \check{H}^1(\mathbb{P}^1, \mathcal{O}^*_{\mathbb{P}^1_k}) & \cong \mathbb{Z} \\ \mathcal{O}(i) \longleftrightarrow & (\alpha_{10} = t^i) & \longleftrightarrow i \end{matrix}}$$

so define $\mathcal{O}(i)$ by using $\alpha_{10} = t^i$, $\alpha_{01} = t^{-i}$



Rmk $\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^1}$ trivial line bdle.

HWK 4 Ideal sheaf of a closed point in $\mathbb{P}^1_k$ is $\cong \mathcal{O}(-1)$, for disjoint union of n closed pts get $\cong \mathcal{O}(-n)$ for order n point $(t^n) \subseteq k[t]$ (i.e. closed subscheme $\operatorname{Spec} k[t]/(t^n) \subseteq A_0 \subseteq \mathbb{P}^1_k$) get $\mathcal{O}(-n)$.

Non-examinable Rmk (for differential geometers): i determines the Chern class $c_1(\mathcal{L})$: $i = \int_{\mathbb{P}^1_{\mathbb{C}}} c_1(\mathcal{L})$

$T\mathbb{P}^1_{\mathbb{C}}$ is $\mathcal{O}(2)$ since $2 = \chi(\mathbb{P}^1_{\mathbb{C}}) = \chi(S^2)$ and $c_1(T\mathbb{P}^1_{\mathbb{C}})$ = Euler class of $\mathbb{P}^1_{\mathbb{C}}$, and $T^*\mathbb{P}^1_{\mathbb{C}} = \mathcal{O}(-2)$.

$\mathcal{O}(-1) \to \mathbb{P}^1_{\mathbb{C}}$ is blow-up of $\mathbb{C}^2$ at 0 : the lines through 0 in $\mathbb{C}^2$ are the fibres.

Theorem

Cultural Rmk → 1), 2)

Symmetry is "Serre duality". For $\mathbb{P}^1$: $\check{H}^1(\mathcal{O}(i)) \cong \check{H}^0(\underbrace{\mathcal{O}(i)^{-1}\otimes\mathcal{O}(-2)}_{=\mathcal{O}(-i-2)})^*$ ← dual v.s.

1) $\check{H}^0(\mathbb{P}^1_k, \mathcal{O}(i)) = \begin{cases} 0 & i<0 \\ \{f\in k[t] : \deg f \le i\} \cong k[x_0,x_1]_i & i\geq 0\end{cases}$

$t = x_1/x_0$; i-th graded part, so homogeneous polys in x_0, x_1 of degree i

2) $\check{H}^1(\mathbb{P}^1_k, \mathcal{O}(i)) = \begin{cases} 0 & i \ge -1 \\ k[t^{-1}]/k + t^i k[t^{-1}] \cong k[x_0,x_1]_{-i-2} & i<-1\end{cases}$ (exercise)

3) $\check{H}^n(\mathbb{P}^1_k, \mathcal{O}(i)) = 0$ for $n \geq 2$

Pf By 8.6, since $\mathbb{P}^1_k$ separated & quasi-compact, enough to calculate $\check{H}^*_{\{A_0,A_1\}}(\mathbb{P}^1_k, \mathcal{O}(i))$.

3) no triple ordered overlaps or higher

$g \in \mathcal{O}_{A_1} \cong \mathcal{O}(i)|_{A_1} \xrightarrow{\alpha_{10}} \mathcal{O}_{A_0} \cong \mathcal{O}(i)|_{A_0} \ni f$ where α_{10} is defined on $A_0\cap A_1$

1) $\check{H}^0 = \Gamma$: $g(t^{-1}) \in k[t^{-1}]$ on A_1, $f(t)\in k[t]$ on A_0, on overlap: $t^i g(t^{-1}) = f(t) \in k[t,t^{-1}]$

$\Rightarrow \deg f \le i$ and g is determined by f from equation

2) $\mathcal{L} = \mathcal{O}(i)$

$$\underbrace{\Gamma(A_0,\mathcal{L})}_{\overset{\shortparallel}{k[t]}} \oplus \underbrace{\Gamma(A_1,\mathcal{L})}_{\overset{\shortparallel}{k[t^{-1}]}} \xrightarrow{d} \underbrace{\Gamma(A_0\cap A_1, \mathcal{L})}_{\overset{\shortparallel}{k[t,t^{-1}]}} \xrightarrow{d} 0$$

$$(f,g) \longmapsto \underbrace{t^i \cdot g(t^{-1})} - f(t)$$

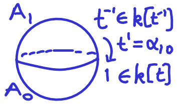
(strictly speaking $\mathcal{L}(A_0) \cong \mathcal{O}_{A_0}(A_0) \cong k[t]$); $\mathcal{L}(A_1) \cong \mathcal{O}_{A_1}(A_1)$; $\mathcal{L}(A_{01}) \cong \mathcal{O}_{A_0}(A_{01})$

↖ need to transition from $g(t^{-1}) \in \mathcal{O}_{A_1}(A_{10})$ to $\mathcal{O}_{A_0}(A_{01})$ via α_{10}: $\mathcal{O}_{A_{10}} \cong \mathcal{L}|_{A_{10}} = \mathcal{L}|_{A_{01}} \cong \mathcal{O}_{A_{01}}$

$\check{H}^1 = k[t,t^{-1}] / \underbrace{k[t] + t^i k[t^{-1}]}$

- is all of $k[t,t^{-1}]$ if $i \geq -1$
- does not contain $t^{-1}, t^{-2}, \dots, t^{i+1}$ if $i<-1$

example $\mathcal{O}(1)$
$s = 1$ on A_0
$s = t^{-1}$ on A_1
is global section



□

EXAMPLE: $\mathbb{P}^n$

$X = \mathbb{P}^n_k = A_0 \cup A_1 \cup \dots \cup A_n$ $\qquad A_i = \operatorname{Spec} k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ ← omit $\frac{x_i}{x_i}$

(called hyperplane bundle or Serre's twisting sheaf) → $\mathcal{O}(1)$ = line bundle with $\alpha_{ij} = \left(\frac{x_i}{x_j}\right) : k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}] \to k[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}]$ (both equal to $\Gamma(A_i\cap A_j, \mathcal{O}_X)$) ← $\mathbb{P}^1_k$ case: $t = x_1/x_0$, $\alpha_{01} : k[t] \to k[t^{-1}]$ is multiplication by $\frac{x_0}{x_1} = t^{-1}$ ✓

$\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ so $\alpha_{ij} = \left(\frac{x_i}{x_j}\right)^m$ ($m\in\mathbb{Z}$; tensor m times)

Rmk $\mathcal{O}(-1)$ called <u>tautological line bundle</u> because in C3.4 course each (closed) point of $\mathbb{P}^n_k$ is a 1-dim vector subspace $V \subseteq k^{n+1}$ ($\mathbb{P}^n_k = k^{n+1}\setminus 0 \,/\, k^*\text{-rescaling}$) so get obvious line bundle: over the point $[V]\in\mathbb{P}^n$ have the line V.

Hwk 4 $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ generated by the $\mathcal{O}(m)$

$\Gamma(\mathbb{P}^n_k, \mathcal{O}(m)) = \begin{cases} k[x_0,\dots,x_n]_m & \text{if } m\geq 0 \\ 0 & \text{if } m<0\end{cases}$ ← So homogeneous polys of deg = m. So on A_i get polys of deg $\le m$ in the variables $\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}$

8.8 Divisors

Let $(X, \mathcal{O}_X = \mathcal{O})$ be an <u>integral scheme</u> (i.e. reduced & irreducible) ← see Sec. 3.5

Recall from Sec. 3.5 that $\forall$ open $\emptyset \neq U \subseteq X$ can view $\mathcal{O}(U) \overset{\text{injective}}{\subseteq} K(X)$ = function field. (stalk of $\mathcal{O}_X$ at unique generic pt; rational functions on X)

Abbreviate: $K = K(X)$, $K^* = K\setminus\{0\}$ (non-zero rational functions are invertible)

$\mathcal{O}^* \subseteq \mathcal{O}$ subsheaf of sections of $\mathcal{O}$ admitting inverses in $\mathcal{O}$ (so can view $\mathcal{O}^* \subseteq K^*$)

$X = \bigcup U_i$ open cover

$f_i \in K^*$ s.t. $\frac{f_i}{f_j}\Big|_{U_i\cap U_j} \in \mathcal{O}^*(U_i\cap U_j)$ $\Big\}$ this data is called a <u>Cartier divisor</u> (see ⊛ below)

(so $\mathcal{O}^*(U) \subseteq K^*$ or can also view K, K^* as constant sheaves on X with values in K, K^*)

$\Rightarrow$ get line bundle $\mathcal{L} \subseteq K$ via $\boxed{\mathcal{L}(U) := \mathcal{O}(U)\cdot \frac{1}{f_i} \subseteq K}$ ← (so free $\mathcal{O}(U)$-mod with basis $1/f_i$)

Exercise

① Obvious trivialisations $\varphi_i : \mathcal{L}(U_i) \longrightarrow \mathcal{O}(U_i)$, $g\cdot\frac{1}{f_i} \mapsto g$

Yields transition maps $\alpha_{ij} = \varphi_j \circ \varphi_i^{-1}|_{U_i\cap U_j} = \frac{f_j}{f_i}$ (from U_i to U_j)

② If $D_1 = (U_i, f_i)$, $D_2 = (V_k, g_k)$ are two Cartier divisors on X yielding line bundles $\mathcal{L}_1, \mathcal{L}_2$ then

$D_1 + D_2 = (U_i\cap V_k, f_i g_k)$ is a Cartier divisor yielding the line bundle $\mathcal{L}_1 \underset{\mathcal{O}_X}{\otimes} \mathcal{L}_2$

$D_1 - D_2 = (U_i\cap V_k, \frac{f_i}{g_k})$ 〃 〃 〃 〃 $\mathcal{L}_1\otimes\mathcal{L}_2^{-1}$

[in particular $-D_1$ is $(U_i, \frac{1}{f_i})$ yields $\mathcal{L}_1^{-1}$]

Key Example Recall $\mathbb{P}^n = \bigcup U_i$ for $U_i = \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}] \cong \mathbb{A}^n$

Let $m\in\mathbb{Z}$, $f_0 = 1$, $f_i = \left(\frac{x_0}{x_i}\right)^m \in K(\mathbb{P}^n) = \{\frac{p}{q} \in \mathbb{Q}(x_0,..,x_n) : p,q\in\mathbb{Z}[x_0,..,x_n]$ homogeneous of same degree$\}$

(side remark: $K(\mathbb{P}^n) \cong k(U_i) \cong k(\mathbb{A}^n) \cong \mathbb{Q}(z_1, \dots, z_n)$)

$\mathcal{L}(U_0) = \mathcal{O}_{\mathbb{P}^n}(U_0)\cdot 1 \subseteq K(\mathbb{P}^n)$

$\mathcal{L}(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i)\cdot\left(\frac{x_i}{x_0}\right)^m \subseteq$ $\qquad$ transition $\alpha_{ij} = \left(\frac{x_0}{x_j}\cdot\frac{x_i}{x_0}\right)^m = \left(\frac{x_i}{x_j}\right)^m$ (from U_i to U_j) so $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(m)$.

RmK This does not look very "symmetric" in the x_i. One can define an $\mathcal{O}_{\mathbb{P}^n}$-module F by $F(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i)\cdot x_i^m$ which is a line bundle with the same transitions $\alpha_{ij} = \left(\frac{x_i}{x_j}\right)^m$.

So $F \cong \mathcal{L}$ above, but we cannot pick $f_i = x_i^m$ for the Cartier divisor since $x_i^m \notin K(\mathbb{P}^n)$.

⊛ Actually want to identify Cartier divisors related by refining the cover, so if $X = \bigcup U_i = \bigcup V_j$ and $V_j \subseteq U_{i(j)}$ ← compare Sec. 8.6 then identify (U_i, f_i) and $(V_j, f_{i(j)})$.

Also identify (U_i, f_i) with $(U_i, f_i g_i)$ if $g_i \in \mathcal{O}^*(U_i)$ ← (i.e. rescaling f_i by invertible regular fns)

Viewing K, K^* as constant sheaves, have an exact sequence

$$0 \to \mathcal{O}^* \longrightarrow K^* \longrightarrow K^*/\mathcal{O}^* \longrightarrow 0$$

Because of ⊛, a Cartier divisor is just a global section of $K^*/\mathcal{O}^*$ so $\in \check{H}^0(X, K^*/\mathcal{O}^*)$

Take LES: $0 \to \check{H}^0(X,\mathcal{O}^*) \to \check{H}^0(X,K^*) \to \check{H}^0(X, K^*/\mathcal{O}^*) \to \check{H}^1(X,\mathcal{O}^*) = \mathrm{Pic}(X) \to \underbrace{\check{H}^1(X,K^*)}_{\overset{\shortparallel}{0}}$

A Cartier divisor in image of ↗ is called principal ← (i.e. use cover X and one $f \in K^*$ or (U_i, f_i) and $f_i \in \mathcal{O}^*(U_i)\cdot f, \forall i$)

Two Cartier divisors D, D' are linearly equivalent if $D - D'$ is principal. Write $D \sim D'$.

Get abelian group $CaCl(X)$ of Cartier divisors modulo linear equivalence. [because K^* is constant sheaf and X is irreducible (end of Sec 8.3)

$\Longrightarrow CaCl(X) \cong \mathrm{Pic}(X)$ by the LES, in particular $\left(\begin{matrix}D, D' \text{ yield iso. line}\\ \text{bundles } \mathcal{L}(D) \cong \mathcal{L}(D')\end{matrix}\right) \Leftrightarrow D \sim D'$.

Cultural Rmks (Non-examinable) There is another notion of divisor: Weil divisor. (dim $\mathcal{O}_{X,z} = 1$ at generic point $z \in Z$)

This means a formal sum $\sum n_i Z_i$ ($\leftarrow n_i \in \mathbb{Z}$) of integral closed subschemes Z_i of codim=1 (think hypersurfaces)

Example rational function $f \in K(X) \Rightarrow \exists$ an "order of vanishing" $\mathrm{ord}_Z(f)$ of f along such subschemes Z.

$\Rightarrow$ Weil divisor $\mathrm{div}(f) := \sum_Z \mathrm{ord}_Z(f)\cdot Z$ called principal Weil divisor ($f = \frac{a}{b} \in K(X)$ then $\mathrm{ord}_Z(f)$ is $\mathrm{length}_R(R/a) - \mathrm{length}_R(R/b)$ for $R = \mathcal{O}_{X,z}$)

Example Cartier divisor (U_i, f_i) yields Weil divisor $W = \sum_Z \sum_i \mathrm{ord}_{Z\cap U_i}(f_i)\cdot Z$ ← (notice compute the order of $f_i|_{U_i} \in K(U_i)$ along $Z \cap U_i \subseteq U_i$)

On $\mathbb{P}^1$: Cartier divisor $(U_0, 1), (U_1, \frac{x_0}{x_1})$ yields $W = +$ point $[0:1]$ ← (since $f_1 = \frac{x_0}{x_1}$ has a simple zero at $x_0 = 0$ but ignore pole $x_1 = 0$ since $[1:0] \notin U_1$)

Cartier divisor $(U_0, 1), (U_1, (\frac{x_0}{x_1})^m)$ yields $W = m \cdot p$ where $m \in \mathbb{Z}$, $p = [0:1]$ (case $m<0$ is when f_1 has a pole at $p = (x_0 = 0)$)

On $\mathbb{P}^n$: $(U_0, 1), (U_i, \frac{x_0}{x_i})$ yields $W = H$ where $H \cong \mathbb{P}^{n-1}$ is the hyperplane $x_0 = 0$ ← so $H \cap U_i$ is $\mathrm{Spec}\left(\mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}] / (\frac{x_0}{x_i})\right)$

The lack of "symmetry" mentioned in Rmk above is because it involves a choice of Weil divisor H. We could have picked any hyperplane to get $\mathcal{L} \cong \mathcal{O}(+1)$. More complicated choices are possible e.g. Cartier divisor D on $\mathbb{P}^1$ with $W = \sum n_i \cdot p_i$ any points p_i, and $n_i \in \mathbb{Z}$, yields $\mathcal{L}(D) \cong \mathcal{O}_{\mathbb{P}^1}(\sum n_i)$ (compare Hwk 4)

Weil divisors $\mathrm{Div}(X)$ modulo principal Weil divisors define the class group $Cl(X)$ (abelian group).

Weil divisor D defines an $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ by $\Gamma(U, \mathcal{O}_X(D)) = \{0\} \cup \{f \in K : \mathrm{div}(f) + D \geq 0 \text{ on } U\}$

But $\mathcal{O}_X(D)$ need not be a line bundle (i.e. invertible sheaf). When it is a line bundle the Weil divisor is Cartier since on some cover $X = \cup U_i$ have trivialisations $\mathcal{O}(U_i) \cong \Gamma(\mathcal{O}_X(D), U_i)$, $1 \longmapsto f_i \in K$

$\Rightarrow$ Cartier divisor (U_i, f_i) and $\mathcal{L}(U_i) = \mathcal{O}(U_i)\cdot \frac{1}{f_i} = \Gamma(U_i, \mathcal{O}_X(D))$.

[means the "n_Z" above are ≥ 0 so if D has $m_Z \cdot Z$ and $m_Z \geq 0$ then f is allowed to have a pole of order $\leq m_Z$ along $Z \cap U$ if $m_Z \leq 0$ then f must vanish with order $\geq -m_Z$ along $Z \cap U$]

Weil divisor is Cartier if locally principal: so locally looks like div(rational fn) (e.g. $D = \mathrm{div}(f)$ gives $\mathcal{O}_X(D) \cong \mathcal{O}$ via $g \mapsto g\cdot f$)

(also need mild condition: X is "normal")

X non-singular variety $\Rightarrow CaCl(X) \cong Cl(X) \cong \mathrm{Pic}(X)$ e.g. get $\mathbb{Z}$ for $\mathbb{P}^n$

← more generally if local rings are UFD.

For X singular it can fail: $X = \mathrm{Spec}\, k[x,y,z]/(xy - z^2) \subseteq \mathbb{A}^3_k$ has $CaCl(X) = 0$ but $Cl(X) = \mathbb{Z}/2$ generated by the hypersurface $Z = (y = z = 0)$. ← (At $0 \in Z$ we really need 2 equations to cut out Z, one is not enough, so not locally principal.

[Rmk in a UFD, height 1 prime ideals are principal, so asking that local rings are UFD ensures Weil divisors are locally cut out by one equation, hence Cartier.

Cultural Remark: Riemann–Roch Theorem (non-examinable)

C projective non-singular algebraic curve / alg. closed field k

$\mathcal{F} = \mathcal{O}_C(D)$ for divisor D of degree d

$$\chi(C, \mathcal{F}) := \sum (-1)^m \dim \check{H}^m(C, \mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) = \deg D + \underbrace{\chi(C, \mathcal{O}_C)}_{= 1 - \mathrm{genus}(C)}$$
$$= d + 1 - g$$

h^0 ← dim(global sections) often written $\ell(D)$.

↖ $h^m = \dim_k \check{H}^k(C, \cdot)$

$\deg D$: $\sum n_i$ if $D = \sum n_i p_i$

usual genus if $k = \mathbb{C}$ so Riemann surface

8.9 Čech cohomology computations on $\mathbb{P}^n$

Recall the Key example in Sec. 8.8:

$\mathbb{P}^n = \bigcup_{i=0}^n U_i$ where $U_i = D_{x_i} = \mathrm{Spec}\, \underbrace{\mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}]}_{\text{written: } (\mathbb{Z}[x_0,\ldots,x_n][\frac{1}{x_i}])_0} \cong \mathbb{A}^n$ (can work also over any ring R, so $\mathbb{P}^n_R$ e.g. $R = k$ field)

Line bundle $\boxed{\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)}$ for $d \in \mathbb{Z}$ has:

$\Gamma(U_i, \mathcal{L}) = (\mathbb{Z}[x_0, \ldots, x_n][\frac{1}{x_i}])_d$ ← so $\frac{\text{Poly. in } x\text{'s of degree } N+d}{x_i^N}$ any $N \geq 0$.

($[\frac{1}{x_i}]$ says localise at x_i; $(\;)_0$ says take degree 0 part)

example: $d = 0$ gives $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}$ and $\Gamma(U_i, \mathcal{L}) = \left\{\frac{p(x)}{x_i^N} : \begin{matrix} p \in \mathbb{Z}[x_0,\ldots,x_n] \\ \deg p = N, N \geq 0\end{matrix}\right\}$ ← the classical functions on U_i, well-defined when rescale homogeneous coords.

Use ordered Čech cohomology using obvious ordering of $i \in \{0,1,\ldots,n\}$.

$\Gamma(U_{i_0 \ldots i_k}, \mathcal{L}) = \left(\mathbb{Z}[x_0,\ldots,x_n]\left[\frac{1}{x_{i_0}\cdot x_{i_1}\cdots x_{i_k}}\right]\right)_d$ ← $\left(\begin{matrix}U_{i_0\ldots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}\\ 0 \leq i_0 < \cdots < i_k \leq n\end{matrix}\right)$

Warm-up example $\check{H}^1(\mathbb{P}^2, \mathcal{L}) = 0$

Proof $c_{ij} \in \check{C}^1$ is $\mathbb{Z}$-combo of terms $\frac{x_0^{m_0} x_1^{m_1} x_2^{m_2}}{(x_i x_j)^N}$ where total degree $\sum m_i - 2N = d$

c cocycle $\Rightarrow (dc)_{012} = 0 \overset{\star}{=} c_{12} - c_{02} + c_{01} \in \Gamma(U_{012}, \mathcal{L})$ ← (e.g. $c_{12} \in \Gamma(U_{12}, \mathcal{L})$ and we restrict to U_{012}

Want to show cocycle c is a coboundary i.e. $\exists b_i \in \Gamma(U_i, \mathcal{L})$, $(db)_{ij} = b_j - b_i = c_{ij}$.

Want $b_i \in \Gamma(U_i, \mathcal{L})$ so only x_i denominators allowed.

Key observation: c_{12} cannot have both $x_1 x_2$ arising as a denominator (after simplify) (e.g. $\frac{x_1^2 x_3}{x_1 x_2} = \frac{x_1 x_3}{x_2}$) because c_{02} has no x_2's at denom, c_{01} has no x_1's at denom.

Expand terms depending on denominators: e.g. c_{12,x_1} = terms of c_{12} which have x_1 denominators

$$\begin{aligned} c_{12} &= c_{12,x_2} + c_{12,x_1} + p_{12} \\ -c_{02} &= -c_{02,x_2} - c_{02,x_0} - p_{02} \\ c_{01} &= c_{01,x_1} + c_{01,x_0} + p_{01} \end{aligned}$$

p_{ij} are leftover terms, so no denoms (so polys of degree d if $d \geq 0$, otherwise zero)

must cancel by ★ they are the only terms with x_2 in denominator

similarly these pairs cancel by ★

$\Longrightarrow$ calling $b_2 = c_{12,x_2}$, $b_1 = -c_{12,x_1}$, $b_0 = -c_{02,x_0}$ get $\begin{cases} c_{12} = b_2 - b_1 + p_{12} \\ -c_{02} = -b_2 + b_0 - p_{02} \\ c_{01} = b_1 - b_0 + p_{01}\end{cases}$

$\Longrightarrow$ replacing c by $c - \underset{\smile}{db}$ (coboundary so does not change coh. class $[c]$) remains to consider the case $c_{ij} = p_{ij}$ (no denominators)

Trick 1 Let $q_2 = \alpha$, $q_0 = q_1 = 0$ then $(dq)_{ij} = q_j - q_i = \begin{cases}\alpha & \text{if } (i,j) = (0,2) \\ \alpha & \text{if } (i,j) = (1,2) \\ 0 & \text{else}\end{cases}$

Taking $\alpha = p_{12}$ we can replace c by $c - dq$ and assume $p_{12} = 0$ (and redefine p_{02} due to ↗)

Whereas for $q_2 = q_1 = 0$, $q_0 = -\beta$ get $(dq)_{ij} = \begin{cases}\beta & \text{if } (0,2) \\ \beta & \text{if } (0,1) \\ 0 & \text{else}\end{cases}$

Taking $\beta = p_{02}$ replacing c by $c - dq$ we can assume $p_{02} = 0$, so also $p_{01} = 0$, so $c = 0$. □ (since $dc = 0$, $p_{12} = 0$)

Lemma $\check{H}^1(\mathbb{P}^n, \mathcal{L}) = 0 \quad \forall n \geq 2$ ⇇ (n=1 fails because don't have triple overlaps We computed the n=1 case in Sec. 8.7

Proof ← TRY ON YOUR OWN FIRST! The first part of proof of $n=2$ case is same: replace $0,1,2$ by indices i_0, i_1, i_2.

So reduce to case of cocycle $c \in \check{C}^1$ with c_{ij} having no denominators (so polys of degree d when $d \geq 0$)

Doing Trick 1 now is messy I think, so I'll use another trick first.

Trick 2 $\frac{c_{ij}}{x_0^d} \in \left(\mathbb{Z}[x_0,\dots,x_n][\frac{1}{x_0}]\right)_0 = \mathbb{Z}[z_1,\dots,z_n] =$ global sections on $U_0 \cong \mathbb{A}^n$

↖ for $0 \neq i < j$ ↖ $z_i = x_i/x_0$

This is a 1-cocycle on $\mathbb{A}^n$ and we know $\check{H}^1(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = 0$ ($\check{H}^* = 0$ for $* \geq 1$ by Sec. 8.3 since $\mathbb{A}^n$ affine)

So $\exists\ \beta_i \in \mathbb{Z}[z_1,\dots,z_n][\frac{1}{z_i}]$ with $(d\beta)_{ij} = \frac{c_{ij}}{x_0^d}$ for $1 \leq i < j$

Since c_{ij} has no denominators, β_i cannot have any z_i denominator. (drop deg $\neq d$ terms from β_i won't affect)

Since c_{ij} is homog. of deg $= d$ in the x's, WLOG β_i is homogeneous of deg $= d$ in z's

$\Rightarrow$ Take $b_i = x_0^d \beta_i =$ homog. deg d poly in x's with $(db)_{ij} = c_{ij}$ for $1 \leq i < j$.

$\Rightarrow$ Replace c by $c - db$, can assume $c_{ij} = 0$ for $i \neq 0$.

Final trick $(dc)_{0ij} = 0 = 0 - c_{0j} + c_{0i}$ so all c_{0i} are the same say $= \beta$, so use Trick 1 with $q_i = 0$ for $i \neq 0$, $q_0 = -\beta$ then $(dq)_{ij} = \begin{cases} \beta & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$ so $c = dq$. □

Theorem For $\mathcal{L} = \mathcal{O}(d)$, $d \in \mathbb{Z}$, $n \geq 2$

$$\check{H}^*(\mathbb{P}^n, \mathcal{L}) = \begin{cases} \mathbb{Z}[x_0,\dots,x_n]_d & \text{for } * = 0 \\ 0 & \text{for } 0 < * < n \\ \mathbb{Z}\left\{\frac{1}{x_0 x_1 \cdots x_n} \cdot \frac{1}{x^{\underline{m}}} \text{ of total degree } d \text{ and all } m_i \geq 0\right\} & \text{for } * = n \\ 0 & \text{for } * > n \end{cases}$$

degree d homog. polys (so $\{0\}$ if $d<0$)

← Hwk 4, global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$

$(x^{\underline{m}} = x_0^{m_0} \cdots x_n^{m_n})$

← ($\mathbb{Z}\{\dots\}$ means free $\mathbb{Z}$-module with that basis)

← no $n+2$ overlaps or higher since $n+1$ sets U_i cover

(same for $\mathbb{P}^n_R$ if replace $\mathbb{Z}$ by a ring R)

Proof $0 < * = k < n$ is same as for $\check{H}^1$: exercise for you.

(Hint. $\pm b_{i_0 \dots \hat{i}_a \dots \hat{i}_b \dots i_k}$ = terms in $c_{i_0 \dots \hat{i}_a \dots i_b \dots i_k}$ with no x_{i_b} at denominator
notice those must cancel with similar terms in $c_{i_0 \dots i_a \dots \hat{i}_b \dots i_k}$
Pick sign it has as a term in $(db)_{i_0 \dots \hat{i}_a \dots i_k}$ ← since want this to give $c_{i_0 \dots \hat{i}_a \dots i_k}$)

Case $* = n$: only one possible overlap: $U_{012\dots n}$, any chain $c \in \check{C}^n$ is cocycle since no higher overlaps. Question becomes what are possible $(db)_{01\dots n}$ for $b_i \in \Gamma(U_{0\dots\hat{i}\dots n}, \mathcal{L})$.

$(db)_{01\dots n} = \underbrace{b_{12\dots n}}_{\text{no } x_0 \text{ at denom}} - \underbrace{b_{02\dots n}}_{\text{no } x_1 \text{ at denom}\dots} + b_{013\dots n} - \cdots$ so can get all $x^{\underline{m}}$ with some $m_i \geq 0$ (i.e. some x_i not in denom.)

$$\begin{aligned} \Rightarrow \check{H}^n &= \mathbb{Z}\{x^{\underline{m}} : \Sigma m_i = d\} / \mathbb{Z}\{x^{\underline{m}} : \Sigma m_i = d, \underline{\text{some}}\ m_i \geq 0\} \\ &\cong \mathbb{Z}\{x^{\underline{m}} : \Sigma m_i = d, \underline{\text{all}}\ m_i < 0\} \\ &= \frac{1}{x_0 \cdots x_n} \cdot \mathbb{Z}\left\{\frac{1}{x^{\underline{m}}} : \Sigma m_i = -d-n-1, \underline{\text{all}}\ m_i \geq 0\right\}. \ \square \end{aligned}$$

Rmk (non-examinable) Serre duality thm for $\mathbb{P}^n_k$: $\check{H}^i(\mathbb{P}^n_k, L) \cong \check{H}^{n-i}(\mathbb{P}^n_k, K \otimes L^{-1})^*$ where $K = \mathcal{O}(-n-1)$, L line bdle. So explains this "symmetry"

Exercise deduce the ranks $h^i = \text{rank}_{\mathbb{Z}} \check{H}^i$ are $h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \binom{n+d}{n} & \text{if } i = 0 \\ \binom{-d-1}{n} & \text{if } i = n \\ 0 & \text{else} \end{cases}$

(means 0 for $d < 0$) (means 0 for $d \geq -n$)

Motivation for chapter 9: Now that we know $\check{H}^*(\mathbb{P}^n, \mathcal{O}(d))$, one might hope to compute $\check{H}^*(\mathbb{P}^n, F)$ for other $F \in Coh(\mathbb{P}^n)$ by first finding a resolution $\dots \to \mathcal{L}_2 \to \mathcal{L}_1 \to F \to 0$ with $\mathcal{L}_i = \bigoplus_j \mathcal{O}(d_{ij})$ and exploiting LES's.

8.10 Product on Čech cohomology (Non-examinable section)

$(X, \mathcal{O}_X)$ any ringed space

$$\begin{aligned} \check{H}^p_{\{U_i\}}(X, F) \times \check{H}^q_{\{U_i\}}(X, G) &\longrightarrow \check{H}^{p+q}_{\{U_i\}}(X, F \otimes_{\mathcal{O}_X} G) \\ ((s_I), (t_I)) &\longmapsto (s_I \otimes t_I) \end{aligned}$$

Rmk In 8.6 where we took constant coefficients $F = G = \underline{\mathbb{Z}}$ (note: $\underline{\mathbb{Z}} \otimes_{\mathcal{O}_X} \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$) we recover the cup product on singular cohomology (respectively on de Rham cohomology) [using $F = G = \underline{\mathbb{R}}$ for $\mathcal{O}_X$ = smooth real functions so $\underline{\mathbb{R}} \otimes_{\mathcal{O}_X} \underline{\mathbb{R}} \cong \underline{\mathbb{R}}$

9. Sheaf cohomology

9.1 Resolutions

←(Reference for more details: Lang, Algebra, Chapter XX §4–6)

Motivation: "represent" an object in an abelian category A by "nicer objects" at the cost of using a chain cx (Sec. 1.8)

right resolution of $M \in A$ means an exact sequence $\underbrace{0 \to M \to I^0 \to I^1 \to I^2 \to \cdots}_{\text{abbreviated as } M \to I^\bullet}$ in A

left resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, or $P_\bullet \to M$

Def I injective if $\text{Hom}(\cdot, I)$ exact
P projective if $\text{Hom}(P, \cdot)$ exact (both always left exact)

Exercise I injective is equivalent to: $\forall$ inj $A \overset{i}{\hookrightarrow} B$, $\forall\ \varphi : A \to I$ can "extend" φ:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & I \\ B \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Fact Injective resolution $M \to I^\bullet$ means I^n are injective
Projective resolution $P_\bullet \to M$ " P_n " projective

$f, g : A \to B$ additive functors of abelian cats (see 1.7)

$$\begin{tikzcd} I^i \arrow[r] \arrow[rr, bend right, "0"'] & I^{i+1} \arrow[r] & I^{i+2} \end{tikzcd}$$

so:

$$\begin{tikzcd} fI^i \arrow[r] \arrow[rr, bend right, "0"'] & fI^{i+1} \arrow[r] & fI^{i+2} \end{tikzcd}$$

so $fI^\bullet$ is complex (see 1.8)

f left exact $\Rightarrow$ right-derived functor $\boxed{R^n f(M) = H^n(f(I^\bullet))}$ ← $M \to I^\bullet$ inj. res.

see 1.8

g right exact $\Rightarrow$ left-derived functor $\boxed{L_n g(M) = H_n(g(P_\bullet))}$ ← $P_\bullet \to M$ proj. res.

Later will see why choice of $I^\bullet$, $P_\bullet$ does not matter.

Warning f left exact only implies $0 \to fM \to fI^0 \to f(\text{Im}(I^0 \to I^1)) \to 0$ exact. Deduce: $R^0 f(M) \cong fM$ ($\text{Ker}(fI^0 \to fI^1) \cong \text{Im}(fM \to fI^0)$)
Similarly $L_0 g \cong g$, so $R^0 f$, $L_0 g$ remember the functors f, g.

Classical Examples $A = S$-Mods (ring), $f = \text{Hom}(M, \cdot)$ (S-mod)

$\Rightarrow$ $\mathbf{Ext}^n_S(M, N) = (R^n f)(N) = H^n(\text{Hom}_S(M, I^\bullet))$ ← $N \to I^\bullet$ inj. res. ($\text{Ext}^0_S(M,N) \cong \text{Hom}_S(M,N)$)

(Similarly: $f = \text{Hom}(\cdot, N) : S\text{-Mods}^{op} \to Ab$ (left exact), $\text{Ext}^n_S(M,N) = (R^n f)(M) = H_n(\text{Hom}(P_\bullet, N))$ ← $P_\bullet \to M$ proj. res.

$g = M \otimes_S \cdot$ right exact $\Rightarrow$ $\mathbf{Tor}^n_S(M,N) = (L_n g)(N) = H_n(M \otimes_S P_\bullet)$ ($\text{Tor}^0_S(M,N) \cong M \otimes_S N$)

(Similarly: $g = \cdot \otimes_S N$, $\text{Tor}^n_S(M,N) = (L_n g)(M) = H_n(P_\bullet \otimes_S N)$ for $P_\bullet \to M$ proj. res.)

For R-mods: I injective $\Leftrightarrow$ if $I \subseteq$ any mod M then $\exists$ mod J : $I \oplus J = M$ ← (compare linear algebra "extending a basis")
P projective $\Leftrightarrow$ P is a direct summand of a free R-mod

Fact $\begin{array}{l} M \to I^\bullet \\ N \to J^\bullet \end{array}$ inj. res., $\begin{array}{c} M \\ \downarrow \\ N \end{array}$ morph $\Rightarrow$ can extend

$$\begin{tikzcd} M \arrow[r] \arrow[d] & I^\bullet \arrow[d, "\exists"] \\ N \arrow[r] & J^\bullet \end{tikzcd}$$

← and any 2 choices are chain homotopic $\Rightarrow$

$$\begin{tikzcd} f(M) \arrow[r] \arrow[d] & H^*(f(I^\bullet)) \arrow[d, "\exists!"] \\ f(N) \arrow[r] & H^*(f(J^\bullet)) \end{tikzcd}$$

Key idea I inj $\Rightarrow$ $\text{Hom}(\cdot, I)$ right exact $\Rightarrow$ if $A \overset{\text{mono}}{\hookrightarrow} B$ then any $A \to I$ can be extended to $B \to I$. E.g.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & I^0 \\ N \arrow[r] \arrow[ur] & J^0 \end{tikzcd} \Rightarrow \begin{tikzcd} M \arrow[r, hook] \arrow[d] & I^0 \arrow[d, "\exists"] \\ N \arrow[r] \arrow[ur] & J^0 \end{tikzcd}$$

then consider $\begin{array}{c} \text{Coker}(M \hookrightarrow I^0) \hookrightarrow I^1 \\ \downarrow \\ \text{Coker}(N \to J^0) \to J^1 \end{array}$ and continue inductively. Try proving the rest.

Cor 1) $R^n f(M) = H^n(fI^\bullet)$ independent of choice of inj. res. $M \to I^\bullet$
2) $M \to N$ induces $R^n f(M) \to R^n f(N)$, indeed $R^n f : A \to A$ is functor.

Pf 1) Apply fact to $M = N$, get $H^*(fI^\bullet) \to H^*(fJ^\bullet) \to H^*(fI^\bullet)$ composite is id by uniqueness.
2) By Fact, $Rf^n(M) = H^n(fI^\bullet) \to H^n(fJ^\bullet) = Rf^n(N)$. Exercise: check functor. □

Lemma f left exact, $0\to M_1\to M_2\to M_3\to 0$ SES $\Rightarrow \exists$ canonical & functorial LES

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & R^0f(M_1) \arrow[r] \arrow[d, phantom, "\shortparallel"] & R^0f(M_2) \arrow[r] \arrow[d, phantom, "\shortparallel"] & R^0f(M_3) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \underbrace{R^1f(M_1) \to R^1f(M_2) \to R^1f(M_3)} \arrow[r] & R^2f(M_1) \arrow[r] & \cdots \\
& fM_1 & fM_2 & fM_3 & & &
\end{tikzcd}$$

where these triples are just R^nf applied to the SES

Sketch Pf

$$\begin{tikzcd}
0 \arrow[r] & I_1^\bullet \arrow[r] & I_2^\bullet = I_1^\bullet \oplus I_3^\bullet \arrow[r] & I_3^\bullet \arrow[r] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & M_3 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

← first pick inj. res. $I_1^\bullet, I_3^\bullet$ then define $I_2^\bullet$ that way so get obvious SES.

use obvious map $M_2\to M_3\to I_3^0$
and $M_1 \hookrightarrow I_1^0$ extends via $M_1\to M_2$ to $M_2\to I_1^0$
Exercise: $M_2 \hookrightarrow I_2^0 = I_1^0\oplus I_3^0$ is injective.
Then take cokernels $M_i' = \mathrm{Coker}(M_i\to I_i^0)$, check that
$0\to M_1'\to M_2'\to M_3'\to 0$ exact, and repeat construction.

(Fact additive functors preserve $\oplus$)

$$\Longrightarrow \begin{tikzcd}
0 \arrow[r] & fI_1^\bullet \arrow[r] & fI_2^\bullet = fI_1^\bullet \oplus fI_3^\bullet \arrow[r] & fI_3^\bullet \arrow[r] & 0 \\
0 \arrow[r] & fM_1 \arrow[r] \arrow[u] & fM_2 \arrow[r] \arrow[u] & fM_3 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

← f may only be left exact, but here clearly $fI_2^\bullet$ surjects onto $fI_3^\bullet$ since have projection onto $fI_3^\bullet$ summand.

Finally take the LES associated to the SES of complexes $0\to fI_1^\bullet\to fI_2^\bullet\to fI_3^\bullet\to 0$. □

Rmk Indeed $R^\bullet f$ satisfies universal property that "$R^0f=f$ and Lemma holds," then it follows that $R^\bullet f(M) = H^\bullet(f(I^\bullet))$ for any inj. res. $M\to I^\bullet$ (see end of next section)

Hwk 4 $Ab(X)$ has enough injectives i.e. can build inj. resolutions of any object $F\in Ab(X)$.

$\Gamma(X,\cdot): Ab(X)\to Ab$ left exact (Sec. 1.9) $\Rightarrow$ can define sheaf cohomology $\boxed{H^n(X,F) = R^n\Gamma(X,F)}$

We now ask how this relates to $\check{H}^n(X,F)$ for $F\in QCoh(X)\subseteq Ab(X)$ and X scheme.

9.2 Acyclic resolutions

(in an abelian cat.)

Rmk If I inj. object $\Rightarrow$ resolution $0\to I\xrightarrow{id} I^0 = I\to 0\to 0\to\cdots \Rightarrow R^nf(I)=0 \;\forall n\geq 1$

So for sheaf cohomology: $H^n(X,I)=0\;\forall n\geq 1$ if I injective sheaf.

Def An acyclic resolution of F is an exact sequence $0\to F\to J^0\to J^1\to\cdots$ with
$$H^n(X,J^k)=0 \quad \forall n\geq 1$$
← (so we weakened the condition of being an inj. resolution)

Claim Any acyclic resolution can be used to compute sheaf cohomology, i.e.

$$\boxed{H^n(X,F) = \text{cohomology of chain complex } \Gamma(X,J^0)\to\Gamma(X,J^1)\to\cdots}$$

Pf Trick "break down into SES and take LES":

let $C_1 = \mathrm{Coker}(F\to J_0) \overset{\text{exactness}}{\cong} \mathrm{Im}(J_0\to J_1)$ so $\exists$ natural monomorph. $C_1\hookrightarrow J_1$

$C_{n+1} = \mathrm{Coker}(C_n\to J_n) \cong \mathrm{Im}(J_n\to J_{n+1})$ " " $C_{n+1}\hookrightarrow J_{n+1}$

$$\left.\begin{array}{l} 0\longrightarrow F\longrightarrow J_0\longrightarrow C_1\longrightarrow 0\\ 0\longrightarrow C_1\longrightarrow J_1\longrightarrow C_2\longrightarrow 0\\ 0\longrightarrow C_n\longrightarrow J_n\longrightarrow C_{n+1}\longrightarrow 0\end{array}\right\}\text{ exact, and }
\begin{tikzcd}[column sep=small]
0 \arrow[r] & F \arrow[r] & J_0 \arrow[r] \arrow[dr] & J_1 \arrow[r] \arrow[dr] & J_2 \arrow[r] & J_3 \arrow[r] & \cdots \\
& & & C_1 \arrow[u] & C_2 \arrow[u] & \cdots &
\end{tikzcd}$$

Technical Lemma (only uses LES in H^*) $0\to F\to I\to G\to 0$ SES with $H^n(I)=0$ $n\geq 1$ $\Longrightarrow$ $H^n(F)\cong H^{n-1}(G)$ $n\geq 2$
$H^1(F)\cong \mathrm{Coker}(H^0I\to H^0G)$

Pf $0\to H^0F\to H^0I \xrightarrow{\circledast} H^0G \to H^1(F)\to \underset{\underset{0}{\shortparallel}}{H^1(I)}\to H^1(G)\xrightarrow[\text{so}\cong]{} H^2(F)\to \underset{\underset{0}{\shortparallel}}{H^2(I)}\to\cdots$ □

so surj. so $H^1F = \mathrm{Coker}\circledast$

Finish proof, abbreviate $H^n(F)=H^n(X,F)$, $\Gamma(F)=\Gamma(X,F)$:

$$H^n(F)\cong H^{n-1}(C_1)\cong H^{n-2}(C_2)\cong\cdots\cong H^1(C_{n-1})\cong \mathrm{Coker}(H^0(J_{n-1})\to H^0(C_n))$$

Γ left exact
$\Downarrow$
exactness of:
$0\to\Gamma(C_n)\xrightarrow{i_n}\Gamma(J_n)\xrightarrow{p_n}\Gamma(C_{n+1})$
hence inj.
$\mathrm{Ker}\,p_n = \mathrm{Im}\,i_n$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \Gamma(J_{n-1}) \arrow[rr, "\alpha_{n-1}"] \arrow[dr, "p_{n-1}"'] & & \Gamma(J_n) \arrow[rr, "\alpha_n"] \arrow[dr, "p_n"'] & & \Gamma(J_{n+1}) \arrow[r] & \cdots \\
& & \Gamma(C_n) \arrow[ur, hook, "i_n"'] & & \Gamma(C_{n+1}) \arrow[ur, hook] & &
\end{tikzcd}$$

$\Gamma(J_{n-1}) = H^0(J_{n-1})$, $\Gamma(C_n) = H^0(C_n)$

$H^n(F) = \mathrm{Coker}\;p_{n-1}$

$$\begin{aligned}
\mathrm{Ker}\,\alpha_n/\mathrm{Im}\,\alpha_{n-1} &= \mathrm{Ker}\,p_n/\mathrm{Im}\,i_n\circ p_{n-1}\\
&= \mathrm{Im}\,i_n/\mathrm{Im}\,i_n\circ p_{n-1}\\
&\cong \Gamma(C_n)/\mathrm{Im}\,p_{n-1}\quad\text{via } i_n\\
&= \mathrm{Coker}\,p_{n-1}\\
&= H^n(F).
\end{aligned}$$
□

Non-examinable:

Rmk For a left-exact functor $f: A\to B$ of abelian cats, a resolution $0\to M\to I^\bullet$ is f-acyclic if $R^n(f(I^k))=0\;\forall n\geq 1$. Similarly for right exact functors g, for $P_\bullet\to M\to 0$ says $L_n(g(P_k))=0\;\forall n\geq 1$.

Fact Injective resolutions are acyclic resolutions for left exact functors
Projective " " " " right " "

9.3 Čech cohomology vs sheaf cohomology

Theorem X separated, quasi-compact scheme. Suppose $\mathbb{H}^n: QCoh(X)\longrightarrow Ab$ are functors s.t.

i) $\mathbb{H}^0(X,F) = \Gamma(X,F)$.

ii) $\underset{\text{affine open}}{\varphi: U\hookrightarrow X} \Longrightarrow \mathbb{H}^n(X,\varphi_*F)=0 \quad \forall n\geq 1, \forall F\in QCoh(U)$.

φ_*F ← $\in QCoh(X)$ by Sec. 7.4 Rmk

iii) SES induces a LES on $\mathbb{H}^*$

holds for Čech cohomology since $\check{H}^n_{\{U_i\}}(X,\varphi_*F) = \check{H}^n_{\{\varphi^{-1}U_i\}}(\varphi^{-1}X,F) = \check{H}^n_{\{U\cap U_i\}}(U,F)\overset{U\text{ affine}}{=}0$, $n\geq 1$

Then $\mathbb{H}^*\cong\check{H}^*$

Pf $X=\bigcup U_i$ finite affine open cover (use X quasi-compact)
U_I affine since X separated (using ordered I)

Notice that the Čech complex

$$\check{C}^n = \prod_{|I|=n} F(U_I) = \prod_{|I|=n}\Gamma(U_I,F) = \prod_{|I|=n}\Gamma(X,\varphi_{I*}(F|_{U_I})) = \Gamma\Big(X,\underbrace{\prod_{|I|=n}\varphi_{I*}(F|_{U_I})}_{\text{call this }J^n}\Big)$$

where $\varphi_I: U_I\hookrightarrow X$ is the inclusion

$\Longrightarrow \check{C}^n = \Gamma(X,J^n)$ and have sequence $0\to F\longrightarrow J^0\longrightarrow J^1\longrightarrow\cdots$

use restriction maps $F\longrightarrow\varphi_{i*}(F|_{U_i})$

other maps are defined on any open $V\subseteq X$ by the Čech differential on V for cover $V\cap U_I$

By Sec. 9.2 it is enough to check this is an acyclic resolution, since then

$$\mathbb{H}^n(X,F)\cong H^n(\Gamma(X,J^\bullet)) = H^n(\check{C}^\bullet_{\{U_i\}}(X,F)) = \check{H}^n(X,F)$$

By (ii): $\mathbb{H}^n(X,\varphi_{I*}(F|_{U_I}))=0\quad\forall n\geq 1$

$\prod_{|I|=n}$ is a finite product so $\cong$ finite $\oplus$.

So $\mathbb{H}^n(X,J^k)=0\;\forall n\geq 1$ follows by induction by following Trick:

Trick If $G_1, G_2 \in QCoh\, X$, $H^n(X,G_i)=0\ \forall n\geq 1 \Rightarrow G_1\oplus G_2$ also, since:

$0\to G_1 \to G_1\oplus G_2 \to G_2 \to 0$ SES $\underset{(iii)}{\Rightarrow}$ take LES get $H^n(X, G_1\oplus G_2)=0$, $n\geq 1$ ✓

$0\to F \to J^\bullet$ exact $\iff$ exact on stalks $\underset{(\not\Rightarrow)}{\Leftarrow}$ $0\to\Gamma(U,F)\to\Gamma(U,J^\bullet)$ exact $\forall$ affine open U

$$\underbrace{0\to\Gamma(U,F)\to\Gamma(U,J_0)\to\overbrace{\Gamma(U,J_1)}}\to\cdots$$

exact since $\Gamma(U,\cdot)$ left exact (Sec.1.9)

exact since $\overset{\vee}{H}{}^n(U,F)=0$ for $n\geq 1$ ← for cover $U\subset U$: since U affine, using sec. 8.3 □

Cor X separated, Noetherian ← stronger than quasi-compact $\Rightarrow$ sheaf cohomology $\boxed{H^n(X,F)\cong \check{H}^n(X,F)}$ $\forall F\in QCoh(X)$

← Non-examinable

Pf Sheaf cohomology $H(X,F)=$ cohomology of $\Gamma(X,I^0)\to\Gamma(X,I^1)\to\cdots$ for $F\to I^\bullet$ any injective resolution.

Check the conditions of Theorem:

i) $\Gamma(X,\cdot)$ left exact $\Rightarrow H^0(X,F)\cong\Gamma(X,F)$ ← general consequence see 9.1, or explicitly: $0\to\Gamma(X,F)\to\Gamma(X,I^0)\to\Gamma(X,I^1)$ exact, so im of ↑ is ker of ↑ which is H^0

iii) Lemma in 9.1 proves $\exists$ LES

ii) by the Theorem below. □

Theorem R Noeth., $F\in QCoh(Spec\, R) \Rightarrow H^n(Spec\, R, F)=0\ \forall n\geq 1$

← **Cultural Rmk** **Serre's Theorem:** X Noeth. scheme then: X affine $\iff H^n(X,F)=0$ $\forall n\geq 1$ $\forall F\in QCoh(X)$

Non-examinable proof ideas The cleanest proof is to build machinery:

1) A sheaf F is flasque if all restrictions $F(U)\longrightarrow F(V)$ are surjective.
2) $\forall$ flasque F on a top. space X, have $H^n(X,F)=0\ \forall n\geq 1$ (Hartshorne III.2.5)
3) $\forall$ injective R-module I, and R Noeth. $\Rightarrow \widetilde{I}$ on Spec R is flasque (Hartshorne III.3.4)

Cor Flasque resolutions are acyclic by (2), so can be used to compute $H^n(X,F)$ by 9.2

Pf Thm $F\cong\widetilde{M}$ for $M=\Gamma(X,F)$ by 7.6. Pick injective resolution of the R-module $M: 0\to M\to I^\bullet$

$\Rightarrow 0\to\widetilde{M}\to\widetilde{I}^\bullet$ exact, each $\widetilde{I}^n$ flasque, so can use this to compute $H^n(X,F)$ by Cor

$\Rightarrow H^n(X,\widetilde{M}) = H^n(\Gamma(X,\widetilde{I}^\bullet)) = H^n(I^\bullet) \underset{n\geq 1}{=} 0$ since $I^\bullet$ exact sequence except in degree 0. □

(in deg=0 get M, and $H^0(X,\widetilde{M})=\widetilde{M}(X)=M$)

Rmk Injective $\mathcal{O}_X$-mods are flasque (Hartshorne III.2.4)

Cultural Rmk For any scheme X and sheaf F of abelian groups have $\check{H}^0(X,F)\cong H^0(X,F)=\Gamma(X,F)$ but also in degree 1: $\exists\ \check{H}^1(X,F)\xrightarrow{\cong}H^1(X,F)$. So for example $Pic(X)\cong\check{H}^1(X,\mathcal{O}^*)\cong H^1(X,\mathcal{O}^*)$ in 8.7.

9.4 Product on sheaf cohomology

(Non-examinable section) $(X,\mathcal{O}_X)$ any ringed space

Fact $\exists$ product $H^p(X,F)\times H^q(X,G)\longrightarrow H^{p+q}(X, F\otimes_{\mathcal{O}_X} G)$

idea $\begin{matrix}0\to F\to I^\bullet\\ 0\to G\to J^\bullet\end{matrix} \Rightarrow 0\to F\otimes G\to I^\bullet\otimes J^\bullet$ unfortunately not a resolution ← rows & cols not exact

← bi-complex (compare 8.4) with maps $d\otimes id$, $id\otimes d$ then take total complex: total degree is sum of degrees

(e.g. degree 2 part is $(I^2\otimes J^0)\oplus(I^1\otimes J^1)\oplus(I^0\otimes J^2)$)

need $I^\bullet, J^\bullet$ to be "pure acyclic resolutions" to ensure this is resolution. Then given any inj. res. $F\otimes G\to K^\bullet$, the identity $F\otimes G\xrightarrow{id}F\otimes G$ extends to $I^\bullet\otimes J^\bullet\longrightarrow K^\bullet$. (see key Idea under the Fact in 9.1)

Taking $\Gamma(X,\cdot)$ yiels the result.

10. QCoh($\mathbb{P}^n$), GRADED MODULES, PROJ(R)

(Non-examinable chapter)

10.1 Graded modules and QCoh($\mathbb{P}^n$)

Def graded ring means a ring R s.t.

$R=R_0\oplus R_1\oplus R_2\oplus\cdots$ as abelian groups (so a graded abelian gp graded by $\mathbb{N}$)

$R_i\cdot R_j\subseteq R_{i+j}$ ← Rmk $R_0\subseteq R$ subring since $R_0\cdot R_0\subseteq R_0$

The elements of R_n are called homogeneous elements of degree n

Graded module means R-mod M s.t.

$M=\cdots\oplus M_{-2}\oplus M_{-1}\oplus M_0\oplus M_1\oplus M_2\oplus\cdots$ as abelian groups (so graded by $\mathbb{Z}$)

$R_i\cdot M_j\subseteq M_{i+j}$ ← (often write $M_\bullet$ to emphasize $\exists$ grading $\bullet$)

A morphism of graded R-mods is R-mod hom $M\xrightarrow{\varphi}N$, with $\varphi(M_n)\subseteq N_n\ \forall n$

From now on: $R=k[x_0,\ldots,x_n]$ $R_m=$ homogeneous polys of deg $=m$ (so $R_0=k$)

$X=\mathbb{P}^n_k = A_0\cup A_1\cup\cdots\cup A_n$ for

$A_i := Spec\ k\left[\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i}\right] = Spec\left(\left(k[x_0,\ldots,x_n]_{x_i}\right)_0\right)$ ← omit $\frac{x_i}{x_i}$

means take 0-th graded part so $\frac{p(x_0,x_1,\ldots,x_n)}{x_i^{\deg(p)}}$ ← poly

Recall in C3.4 the 0-graded part $\left\{\frac{poly}{poly}\right\}$ ← homogeneous in $x_0,\ldots,x_n$ of same degree is the part which gives well-defined functions (invariant under k^*-rescaling) on open subset of $\mathbb{P}^n$ where denominator $\neq 0$.

$A_i\cap A_j = Spec\ k\left[\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i},\frac{x_i}{x_j}\right] = Spec\left(\left(k[x_0,\ldots,x_n]_{x_ix_j}\right)_0\right)$

Claim $\exists$ exact, full & faithful functor

$$\boxed{\begin{matrix}\{\text{graded } R\text{-mods}\} \longrightarrow QCoh(\mathbb{P}^n)\\ M \longmapsto \widetilde{M}\end{matrix}}$$

Pf Let $M_i := (M_{x_i})_0$ ← 0-th graded piece and $M_{ij}=(M_{x_ix_j})_0$

Define $\widetilde{M}|_{A_i} = \widetilde{M_i}$ these glue since $\widetilde{M_i}|_{A_i\cap A_j}\cong\widetilde{M_{ij}}\cong\widetilde{M_j}|_{A_i\cap A_j}$ ← using $((M_{x_i})_0)_{\frac{x_j}{x_i}}\cong(M_{x_ix_j})_0$

Exactness is a local condition, so it holds since it holds in affine case.

Full & faithful: $Hom(\widetilde{M}|_{A_i},\widetilde{N}|_{A_i}) = Hom(\widetilde{M_i},\widetilde{N_i}) = Hom_{(R_{x_i})_0\text{-mods}}((M_{x_i})_0,(N_{x_i})_0)$

this reduces the problem to an exercise in graded R-mods. (omitted here) □

Warning Not an equivalence of categories because: ← unlike case from 7.6: R-Mods $\simeq QCoh(Spec\, R)$, $M\longmapsto\widetilde{M}$, $F(M)\longleftarrow F$

Hwk 4 if $M_n=N_n$ for $n\geq N$ then $\widetilde{M}\cong\widetilde{N}$

Fact If work with graded R-mods "modulo" identifying those which would give rise to "same" $\widetilde{M}$, then get equivalence of categories. So work with $\{R\text{-mods } M\}/\{R\text{-mods } M: \widetilde{M}=0\}$. ⛤

For $X=\mathbb{P}^n$, $\widetilde{M}=0 \iff M$ is locally nilpotent, i.e. $\forall m\in M, \exists\, d$ s.t. $x_i^d\cdot m=0\ \forall i$.

If M is f.g., then $\widetilde{M}=0\iff M$ is finite dim v.s./k)

In reverse direction: $\{\text{graded } R\text{-mods}\}\longleftarrow QCoh(\mathbb{P}^n)$

$\Gamma_\bullet(F) := \bigoplus_{d\geq 0}\Gamma(\mathbb{P}^n, F(d)) \longleftarrow F$ where $F(d)=F\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}(d)$ ← called twisting

($\bullet$ stands for grading $d\geq 0$)

Fact $\boxed{F \cong \widetilde{\Gamma_\bullet(F)}}$

When we mod out by the M with $\widetilde{M}=0$ as in ⊛, this functor together with the functor of claim define an equivalence of cats.

$\mathrm{Coh}(\mathbb{P}^n)$ corresponds to the f.g. graded modules under the equivalence.

Rmk The preferred representative of M in the quotient ⊛ is the saturation $\Gamma_\bullet(\widetilde{M})$ of M. Call M a saturated module if $M \cong \Gamma_\bullet(\widetilde{M})$. ← (think of this like a sheafification)

Def $M[d]$ new graded R-mod with $M[d]_i = M_{d+i}$ ←

so shift the module down by m :

	-1	0	1	2	...	
$M = \cdots$	...	M_0	M_1	M_2	...	...
$M[1] = \cdots$	M_0	M_1	M_2	...	...	...

Example $\mathcal{L} := \widetilde{R[d]}$ on $\mathbb{P}^n$ ← (so $\widetilde{k[x_0,\ldots,x_n][d]}$)

$$\mathcal{L}(A_i) = (R[d]_{x_i})_0 \underset{\nwarrow \text{exercise}}{=} x_i^d\, k\left[\tfrac{x_0}{x_i},\ldots,\tfrac{x_n}{x_i}\right] = x_i^d \cdot (R_{x_i})_0$$ ←

line bdle with $\alpha_{ij} = (x_i/x_j)^d$. Hence $\mathcal{L} = \mathcal{O}(d)$.

$(\mathcal{O}_{\mathbb{P}^n}|_{A_{ij}} \xrightarrow{\cong} \mathcal{L}|_{A_{ij}} = \mathcal{L}|_{A_{ji}} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^n}|_{A_{ji}}$, $f \mapsto x_i^d f \mapsto x_j^{-d} x_i^d f)$

so line bundle, since on each A_i: have $(R_{x_i})_0 \xrightarrow{\cong} \mathcal{L}(A_i)$, $1 \mapsto x_i^d$
Note $\mathcal{O}_{\mathbb{P}^n}(A_i) = (R_{x_i})_0$ (see box at top of page)
and $\mathcal{L}|_{A_i} = \widetilde{\mathcal{L}(A_i)}$, $\mathcal{O}_{\mathbb{P}^n}|_{A_i} = \widetilde{\mathcal{O}_{\mathbb{P}^n}(A_i)}$
$\Rightarrow \mathcal{O}_{\mathbb{P}^n}|_{A_i} \cong \mathcal{L}|_{A_i}$

Exercise $\widetilde{M[d]} \cong \widetilde{M}(d) \quad (= \widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(d))$ ← (e.g. $\widetilde{R[d]} = \widetilde{R}(d) = \mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}(d) = \mathcal{O}(d)$)

Rmk $k[x_0,\ldots,x_n] = \bigoplus_{d\geq 0} \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ (but this does not generalise due to above issue about cats)

The construction of $\widetilde{M}$ is so similar to the Spec R case of $\widetilde{M}$, because ∃ analogue of Spec R: Proj R

10.2 Proj (R) and QCoh (Proj R)

$\mathrm{Proj}(R) = \{$ graded prime ideals $I \subseteq R$ not containing the irrelevant ideal $\}$

R any graded ring

graded: means $I = \bigoplus_{n\geq 0}(I \cap R_n)$ (or "homogeneous") ($\Leftrightarrow$ generated by homogeneous elts)

irrelevant ideal: $R_+ := \bigoplus_{n>0} R_n$

in $\mathbb{P}^n$ we remove the max ideal $(x_0,\ldots,x_n)$ (irredundant ideal) because don't allow the closed point $[0:\ldots:0]$

$\mathbb{V}(I) = \{p \in \mathrm{Proj}\, R : p \supseteq I\}$ define closed sets of Zariski topology
(I: graded ideal)

f homogeneous of degree $>0 \Rightarrow D_f = \mathrm{Proj}\, R \setminus \mathbb{V}(f) = \{p \in \mathrm{Proj}\, R : f \notin p\}$ basis of open sets

Warning $\mathrm{Proj}\, R = \bigcup D_{f_i} \Leftrightarrow R_+ \subseteq \sqrt{\langle \text{all } f_i \rangle}$ ← (example: $\mathbb{P}^n = D_{x_0} \cup \ldots \cup D_{x_n}$ and $(x_0,\ldots,x_n) = k[x_0,\ldots,x_n]_+$)

Fact $D_f \cong \mathrm{Spec}((R_f)_0)$ as topological spaces
$p \longmapsto pR_f \cap (R_f)_0$ (inverse map: $p_0 \longmapsto \bigoplus_{k\geq 0}\{a_k \in R_k : \frac{a_k^{\deg(f)}}{f^k} \in p_0\}$)

Sheaf $\mathcal{O} := \mathcal{O}_{\mathrm{Proj}(R)}$:
$$\mathcal{O}|_{D_f} = \mathcal{O}_{\mathrm{Spec}((R_f)_0)} \quad \text{then glue.}$$ ← (on $D_{fg} = D_f \cap D_g$ get $\mathcal{O}_{\mathrm{Spec}((R_{fg})_0)}$)

Warning Proj is not functorial like Spec
If $\varphi : R \to S$ graded hom of rings, $\varphi(R_+) \supseteq S_+$ then get morph $\varphi^\# : \mathrm{Proj}\, S \to \mathrm{Proj}\, R$, $I \mapsto \varphi^{-1}(I)$
but not all morphs arise in this way.
(more generally, suffices $\sqrt{\varphi(R_+)\cdot S} = S_+$)

Examples

1) $S = R[x_0,\ldots,x_n]$ (R any ring) with usual grading $\Rightarrow \mathrm{Proj}\, R = \mathbb{P}^n_R$ (or $\mathbb{P}^n_{\mathrm{Spec}\,R}$)

2) $R^{(d)} := \bigoplus_{n\geq 0} R_{d\cdot n}$ then the inclusion $R^{(d)} \to R$ induces an iso $\mathrm{Proj}\, R \cong \mathrm{Proj}\, R^{(d)}$

3) S graded ring generated as an S_0-algebra (recall $R_0 \hookrightarrow R$ subring) by n+1 elements $s_0,\ldots,s_n \in S_1$
$$\Rightarrow S_0[x_0,\ldots,x_n] \overset{\varphi}{\twoheadrightarrow} S,\ x_i \longmapsto s_i \Rightarrow S \cong S_0[x_0,\ldots,x_n]/\underbrace{\mathrm{Ker}\,\varphi}_{\text{call this } I} \Rightarrow \mathrm{Proj}\, S \cong \mathbb{V}(I) \subseteq \mathbb{P}^n_{S_0}$$
closed subscheme

Example $k[x,y]^{(2)} = k[x^2, xy, y^2]$
$k[X,Y,Z] \twoheadrightarrow k[x^2,xy,y^2]$, $X \mapsto x^2$, $Y \mapsto xy$, $Z \mapsto y^2$
$\Rightarrow \mathbb{P}^1 = \mathrm{Proj}\, k[x,y] \cong \mathrm{Proj}\, k[x,y]^{(2)} \cong \mathrm{Proj}\; k[X,Y,Z]/(XZ - Y^2)$ closed subscheme of $\mathbb{P}^2$
is the Veronese embedding $\nu_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. Similarly get $\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ ← $N = \#$degree d monomials in $x_0,\ldots,x_n$ so $N = \binom{n+d}{d}$

4) every closed subscheme of Proj R arises as $\mathrm{Proj}(R/I)$ some graded ideal I.

Fact $R = \bigoplus_{n\geq 0} R_n$ graded ring $\Rightarrow$ get line bundles $\mathcal{O}(d) = \widetilde{R_d}$ on Proj R, and
∃ exact, full & faithful functor

Note: this tells us QCoh(·) ∀ proj. variety!

$$\boxed{\begin{array}{c} \{\text{graded } R\text{-mods}\} \longrightarrow \mathrm{QCoh}(\mathrm{Proj}\, R) \\ M \longmapsto \widetilde{M} \\ \Gamma_\bullet(F) \longmapsfrom F \end{array}}$$

$\widetilde{M}$ built by gluing as in 10.1 namely
$\widetilde{M}(D_f) = M_{(f)}$ is homogeneous localizn at f (so localize at f and take 0-th graded part)
Stalk $\widetilde{M}_I = M_{(I)}$ = homogeneous localization at the homog. prime ideal I = 0-th graded part of M_I

where $\Gamma_d(F) := \Gamma(\mathrm{Proj}\, R, F(d))$ ← ($F(d) = F \otimes_{\mathcal{O}_X} \mathcal{O}(d)$ and $\mathcal{O}_X = \widetilde{R}$ on $X = \mathrm{Proj}\, R$)

again, not an equivalence of cats, but $\widetilde{\Gamma_\bullet(F)} \cong F$ and the two functors define an equivalence of cats if we work with saturated graded R-mods $(M_\bullet \cong \Gamma_\bullet(\widetilde{M}))$

Fact If R_0 Noetherian, R generated as R_0-algebra by finitely many elts $\in R_1$

Example: $R = k[x_0,\ldots,x_n]/I$ then $x_0,\ldots,x_n \in R_1$ generate.

then ⊛ $\boxed{\{\text{f.g. } R\text{-mods}\}/\{\text{f.g. "torsion" } R\text{-mods}\} \longrightarrow \mathrm{Coh}(\mathrm{Proj}\, R)}$ is equiv. of cats.
$M \longmapsto \widetilde{M}$ and quasi-inverse $\Gamma_\bullet(F) \longmapsfrom F$

Here "torsion" means $\forall m \in M,\ \exists N \in \mathbb{N} : (R_+)^N \cdot m = 0$. For M f.g. A-mod: this holds $\Leftrightarrow M_k = 0$ for large k
So ⊛ same as working with f.g. R-mods modulo identifying those that "agree" in large degrees.

Exercise M "torsion" $\Rightarrow M_f = 0$ ∀ homogeneous $f \in R_+ \Rightarrow \widetilde{M}(D_f) = M_{(f)} = 0 \Rightarrow \widetilde{M} = 0$.
(homogeneous localisation at f)

Now assume only R Noeth graded ring.

Exercise Show R_0 Noeth, and R generated as R_0-alg. by finitely many $f_1,\ldots,f_a \in R$.
Let $d := \mathrm{lcm}(\deg f_i)$. Call homogeneous $m \in M$ irrelevant if $(R_+ \cdot m)_{N\cdot d} = 0$ for all large N.
M called irrelevant if all homogeneous $m \in M$ are irrelevant. **Fact** ⊛ holds if replace "torsion" by "irrelevant".