

C2.6 Introduction to Schemes

Prof. Alexander F. Ritter
University of Oxford
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ritter@maths.ox.ac.uk

Feedback and corrections are welcome!

References

2018-2019 Course Lecture Notes by Prof. Damian Rössler ← on course page

Ravi Vakil, The Rising Sea, Foundations of Algebraic Geometry ← online

<http://stacks.math.columbia.edu> ← search defns, theorems, proofs in algebra & alg. geometry

Qing Liu, Algebraic Geometry and Arithmetic Curves, OUP 2002 ← modern book, seems rather nice

Eisenbud & Harris, The Geometry of Schemes, Springer GTM 197 ← classic

George R. Kempf, Algebraic Varieties, LMS Lecture notes 172

Classic books by: Mumford (Red Book of Varieties & Schemes)

Hartshorne (Algebraic Geometry)

Shafarevich (Basic Algebraic Geometry 2)

← or my website

My C3.4 Algebraic geometry notes (see C2.6 course webpage) try to fill the gap between classical algebraic geometry (C3.4) and C2.6

For the brave, you can look at the original works by the masters in French:

Grothendieck, "Éléments de géométrie algébrique" series on www.numdam.org

Serre, "Faisceaux Algébriques Cohérents", Annals of Math. 1955.

Prerequisites

← (for more advanced algebra see books by Matsumura, Weibel, Eisenbud)

Commutative algebra (e.g. Atiyah - MacDonald, Introduction to Comm. Alg.)

Category theory — or willingness to read things up as necessary

Homological algebra — or willingness to read things up as necessary

Expectations

That you read the notes regularly after each class.

(This is a 16-lecture course, 2 lectures/week across 8 weeks.)

Not everything can be covered in detail in class, so you need to be willing to look things up as necessary.

Conventions

Diagrams commute unless we say otherwise

Ring means commutative ring with unit 1

Ring homomorphisms are by definition unital i.e. 1 maps to 1

Arrows:

← means injective

→ means surjective

CONTENTS

0. INTRODUCTION

- 0.1 Classical Algebraic Geometry : Affine varieties
- 0.2 Why schemes?
- 0.3 What is a point? (reducible, irreducible)

1. DEFINITION OF SCHEMES

- 1.1 Examples of affine schemes (Spec R , $V(I)$, generic/closed point, Covering Trick, quasi-compact)
- 1.2 Definition of a scheme (ringed space, locally ringed space, affine scheme, scheme)
- 1.3 Pre-sheaves (pre-sheaf, morph of presheaves, sub-presheaf)
- 1.4 Sheaves (sheaf, local-to-global condition, skyscraper sheaf, $Ab(X)$)
- 1.5 Stalks (stalk, direct limits, checking inj/surj at stalk level)
- 1.6 Sheafification (sheafification F^+ , universal property of F^+)
- 1.7 Kernels, cokernels, images (abelian categories, additive categories, additive functor)
- 1.8 Exactness (cochain complex / cohomology in abelian cats, left/right exact)
- 1.9 Push-forward (direct image) and inverse image (f_*F , $f^{-1}F$, $F|_U$, $\Gamma(F, U)$, adjointness of f_* & f^{-1})
- 1.10 Morphisms of ringed spaces
- 1.11 A sheaf defined on a topological basis (B-sheaf, inverse limits, extending morphs defined on basis)
- 1.12 Construction of $\mathcal{O}_{\text{Spec } R}$ (Using $B = \{D_f\}$ for Spec R , structure sheaf \mathcal{O}_X , classical alg. geom.)
- 1.13 Morphisms between Specs (Spec: Rings^{op} $\xrightarrow{\text{equivalence}}$ Aff $\xrightarrow{\text{fully faithful}}$ Locally Ringed Spaces)
- 1.14 Closed affine subschemes (ideal sheaf for $I \subseteq R$ on Spec R , quasi-coherence)
- 1.15 Closed subschemes (sheaf of ideals on a scheme, quasi-coherence, support of a sheaf)

2. GLOBAL SECTIONS AND THE FUNCTOR OF POINTS

- 2.0 Points of Spec R (not necessarily closed) (max ideals in local rings \leftrightarrow points)
- 2.1 Global sections and basic open sets for locally ringed spaces ($X \xrightarrow{\text{canonical}} \text{Spec } \Gamma(X, \mathcal{O}_X)$, D_f)
- 2.2 What it means to be affine
- 2.3 Functor of points h_Y (Yoneda lemma/embedding, $\text{Mor}(X, \text{Spec } R) \cong \text{Hom}(R, \Gamma(X, \mathcal{O}_X))$)

3. PROPERTIES OF SCHEMES

- 3.0 Useful facts from commutative algebra : localisation (localisation of modules, exactness)
- 3.1 Noetherian (locally Noetherian schemes, Useful Trick: basics \subseteq overlap of affines)
- 3.2 Properties that are affine-local (locally of finite type, reduced, Noetherian)
- 3.3 Reduced schemes (stalk-local property, extending morphisms onto closures)
- 3.4 Irreducible schemes (Nilradical as generic point, connectedness, irred. components, primary decomp.)
- 3.5 Integral schemes (integral \Leftrightarrow reduced & irreducible, injectivity of restrictions, function field $K(X)$)
- 3.6 Properties of morphisms (affine, quasi-compact, locally finite type, finite type, closed/open immersion, closed/open subschemes, flat, flatness & deformations, closures in Spec R)

4. GLUING THEOREMS

- 4.1 Gluing sheaves (gluing data, compatibility conditions, morphisms defined by local data)
- 4.2 Gluing schemes (gluing conditions, gluing lemma, functor of points is a sheaf of sets)
- 4.3 Affine n -space by gluing (see Homework for projective space) (A^n and \mathbb{P}^n as representable functors)

5. PRODUCTS

- 5.0 Products in category theory (product, coproduct, category \mathcal{C}/\mathcal{B} , fiber product, pushout)
- 5.1 Fiber products exist in Schemes / \mathcal{B} (A -algebras, tensor products, fiber products in Aff & Sch)
- 5.2 Fibers and preimages (Mumford's picture, underlying topological space of products)
- 5.3 Base change
- 5.4 More properties of schemes (separated, universally closed, proper, projective morphism)
- 5.5 Varieties (abstract varieties, complete, affine and (quasi-)projective vars)
- 5.6 Scheme structure on subsets (induced scheme structure, locally closed subsets)

6. SHEAVES OF MODULES

- 6.1 \mathcal{O}_X -modules ($\mathcal{O}_X\text{-Mods} = \text{Mod}_{\mathcal{O}_X}(X)$, morphs of \mathcal{O}_X -mods)
- 6.2 Modules generated by sections ($\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F) = F(X)$, finite type sheaves)
- 6.3 Vector bundles and coherent modules (locally free, invertible sheaf, coherent, loc. finitely presented)
- 6.4 \mathcal{O}_X -module \tilde{M} on $X = \text{Spec } R$, for $R\text{-mod } M$ ($R\text{-Mods} \rightarrow \mathcal{O}_{\text{Spec } R}\text{-Mods}$ fully faithful exact)
- 6.5 Direct image and inverse image ($f_*F, f^{-1}F$)
- 6.6 Operations on \mathcal{O}_X -mods ($\text{Hom}_{\mathcal{O}_X}(F, G), \oplus F, F \otimes_{\mathcal{O}_X} G$)
- 6.7 Pullback (f^*F , adjointness of f_* and f^*)
- 6.8 \tilde{M} on any scheme ($f^*\tilde{M}$ vs. changing rings)
- 6.9 Classification of \mathcal{O}_X -homs $\tilde{M} \rightarrow F$ ($\text{Hom}_{\mathcal{O}_X}(\tilde{M}, F) = \text{Hom}_R(M, \Gamma(X, F))$ on $X = \text{Spec } R$)
- 6.10 Flatness ($f: X \rightarrow Y$ flat $\Rightarrow f^*: \mathcal{O}_Y\text{-Mod} \rightarrow \mathcal{O}_X\text{-Mod}$ exact, flat resolutions)

7. (QUASI-) COHERENT SHEAVES

- 7.1 $\text{QCoh}(X)$ (locally finitely presented vs. coherence, coherent modules)
- 7.2 Overview of general properties of $\text{QCoh}(X)$ and $\text{Coh}(X)$ for X scheme
- 7.3 Pull-back preserves quasi-coherence
- 7.4 Pushforwards for X Noetherian
- 7.5 Gluing modules (cocycle condition, gluing lemma)
- 7.6 $\text{QCoh}(X), \text{Coh}(X), \text{Vect}(X)$ for $X = \text{Spec } R$ ($R\text{-Mods} \simeq \text{QCoh}(\text{Spec } R), \text{Coh } R\text{-Mods} \simeq \text{Coh}(\text{Spec } R)$)

8. ČECH COHOMOLOGY

- 8.1 Čech complex ($\check{C}^n_{\mathcal{U}}$, Čech differential, $\check{H}^n(X, F)$, chain map, chain homotopy)
- 8.2 Čech complex with ordering (Serre's trick)
- 8.3 Affines have no cohomology except H^0 ($\check{H}^n(\text{Spec } R, F) = 0 \forall n \geq 1$ for $F \in \text{QCoh}$)
- 8.4 Independence of cover (X separated & quasi-compact $\Rightarrow \check{H}^n_{\mathcal{U}}$ indep. of cover for QCoh)
- 8.5 Induced LES on \check{H} ($\Gamma(U, \cdot)$ exact on QCoh for affine U)
- 8.6 Dealing with infinite covers (refinements of covers, \check{H}^* vs. singular cohomology)
- 8.7 Application: line bundles and $\check{H}^1(X, \mathcal{O}_X^*)$ (trivialization, vector bundle, sheaf \mathcal{O}_X^* of invertible fins)
- 8.8 Divisors (Picard group, $\text{Pic}(\mathbb{P}^1), \text{Pic}(\mathbb{P}^n)$)
- 8.9 Čech cohomology computations on \mathbb{P}^n (Cartier divisor vs line bundle, Weil divisors)
- 8.10 Product on Čech cohomology ($\check{H}^*(\mathbb{P}^n, \mathcal{O}(d))$ for $d \in \mathbb{Z}$)

9. SHEAF COHOMOLOGY

- 9.1 Resolutions (injective/projective, left/right-derived functors, "enough injectives")
- 9.2 Acyclic resolutions
- 9.3 Čech cohomology vs Sheaf cohomology (characterization of H^* (separated quasi-compact schemes) for QCoh , separated Noeth. $\Rightarrow \check{H}^* = H^*$ on QCoh , Serre's Theorem)
- 9.4 Product on sheaf cohomology

10. $\text{QCoh}(\mathbb{P}^n)$, GRADED MODULES, $\text{PROJ}(R)$

- 10.1 Graded modules and $\text{QCoh}(\mathbb{P}^n)$ (graded rings/mods, Graded $k[x_0, \dots, x_n]\text{-Mods} \xrightarrow{\text{exact full \& faith.}} \text{QCoh}(\mathbb{P}^n)$)
 - 10.2 $\text{Proj}(R)$ and $\text{QCoh}(\text{Proj } R)$ (line bundles via graded mods)
- $\left(\begin{array}{l} \text{Proj } R, \text{ irrelevant ideal, } V(\text{graded ideal}), \mathcal{O}_{\text{Proj}(R)}, \\ \mathbb{P}^n_R = \text{Proj } R[x_0, \dots, x_n], \text{ Graded } R\text{-Mods} \end{array} \xrightarrow{\text{exact full \& faithful}} \text{QCoh}(\text{Proj } R) \right)$

0.1 Classical Algebraic Geometry : Affine varieties

$R = k[x_1, \dots, x_n]$ polynomial ring over algebraically closed field k

$I \subseteq R$ ideal

$X = V(I) = \{a \in k^n : f(a) = 0 \ \forall f \in I\}$ affine variety

The topological space

Affine space: $\mathbb{A}^n = k^n$ with Zariski topology: $\left\{ \begin{array}{l} \text{closed sets: } V(I) \\ \text{open sets: } U_I = \mathbb{A}^n \setminus V(I) \end{array} \right.$

$X \subseteq \mathbb{A}^n$ subspace topology: $X \cap U_I$

basis of open sets: $= \bigcup_{f \in I} D_f$
 $D_f = \{a \in k^n : f(a) \neq 0\}, f \in R$

The functions on it

$R \cong \text{Hom}(\mathbb{A}^n, \mathbb{A}^1), \quad f \mapsto (a \mapsto f(a))$

$\mathbb{I}(X) = \{f \in R : f(X) = 0\}$

Remark $V(\mathbb{I}(X)) = X$ for affine varieties X

Coordinate ring: $k[X] = R/\mathbb{I}(X)$

← The functions on \mathbb{A}^n are polynomial functions.
 ← The functions on \mathbb{A}^n vanishing on X

← The functions on X are polynomials in the coordinates

Key facts: 1) Hilbert's basis theorem: R Noetherian, so $k[X]$ Noetherian

2) Hilbert's weak nullstellensatz: maximal ideals of R (and of $k[X]$) are $m_a = \mathbb{I}(\{a\}) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$, so correspond to points: $\{a\} = V(m_a)$

3) Hilbert's Nullstellensatz: $\mathbb{I}(V(I)) = \sqrt{I}$ (radical of I) $\left| \begin{array}{l} \text{Hence:} \\ \mathbb{I}V(I) = I \\ \text{if } I \text{ is} \\ \text{radical} \end{array} \right.$
 $\{f : \exists N, f^N \in I\}$

Lemma There are enough functions to separate points

Pf $a \neq b \in X \subseteq \mathbb{A}^n \Rightarrow$ some coordinate $a_i \neq b_i \Rightarrow x_i \in k[X]$ separates a, b . \square

Morphisms between affine varieties

$\text{Hom}(\mathbb{A}^n, \mathbb{A}^m) \cong R^m \leftarrow$ polynomial maps $a \mapsto (f_1(a), \dots, f_m(a))$

$\text{Hom}(X, Y) = \{\text{restriction of a polynomial map } \mathbb{A}^n \rightarrow \mathbb{A}^m \text{ s.t. } X \rightarrow Y\}$

Facts: 1) $k[X] \cong \text{Hom}(X, \mathbb{A}^1) \leftarrow$ "values of functions are enough to determine the abstract function"

2) $\text{Hom}(X, Y) \cong \text{Hom}_{k\text{-alg}}(k[Y], k[X])$

$(F: X \rightarrow Y) \mapsto (F^*: \text{Hom}(Y, \mathbb{A}^1) \rightarrow \text{Hom}(X, \mathbb{A}^1)) \leftarrow$ "pullback"
 $f \mapsto F^*f = f \circ F$
 $\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & & \downarrow f \\ & & \mathbb{A}^1 \\ & \swarrow F^*f & \end{array}$

Equivalence of categories

$\{\text{affine varieties}\} \longleftrightarrow \{\text{finitely generated reduced } k\text{-algebras} \ \& \ \text{homs of } k\text{-algs.}\}$

$X \longmapsto k[X]$
 $(F: X \rightarrow Y) \longmapsto F^*$
 Recall: R/J reduced $\Leftrightarrow J$ radical
 Note: $\mathbb{I}(X)$ is radical
 (f nilpotent if $f^N = 0$ some N)

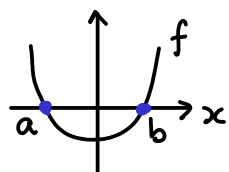
Remark The "same" (up to isomorphism) X can be embedded in various \mathbb{A}^n .

E.g. cuspidal cubic $V(y^2 - x^3) = \begin{array}{c} \diagup \\ \diagdown \end{array} \subseteq \mathbb{A}^2_{x,y}$ is $\cong V(y^2 - x^3, z - x) \subseteq \mathbb{A}^3_{x,y,z}$

0.2 Why schemes?

Some reasons:

- 1) Why always have spaces embedded in \mathbb{A}^n ? (extrinsic)
Can you make sense of X without reference to \mathbb{A}^n ? (intrinsic)
- 2) Why not let R be any ring?
- 3) When you deform varieties, nilpotents arise naturally and should not be ignored:



$$f = (x-a) \cdot (x-b)$$

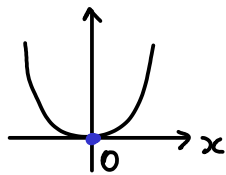
$$X = \mathbb{V}(f) = \{a, b\} \subseteq \mathbb{A}^1$$

← two points

$$k[X] \cong k[x] / (x-a) \oplus k[x] / (x-b) \cong k^2$$

← a value at each point

Deform: a, b become 0:



$$f = (x-0) \cdot (x-0) = x^2$$

$$X = \mathbb{V}(f) = \{0\} \subseteq \mathbb{A}^1$$

$\mathbb{V}(\sqrt{x^2}) = \sqrt{x^2}$ by Hilbert Nullstell.

$$k[X] \cong k[x] / \sqrt{x^2} = k[x] / (x) \cong k$$

notice $k[X]$ is the reduced ring, not $k[x] / (x^2)$

We lost information: classically you cannot tell $x=0$ apart from $x^2=0$

In the theory of schemes, the key role is not played by the topological space. The key role is played by the ring of functions, or rather, the sheaf of functions \mathcal{O} : on each open set $U \subseteq X$ get a ring of functions $\mathcal{O}(U)$.

Example above: $\mathcal{O}(X) = k[x] / (x^2)$ ← we do not reduce the ring of functions

At what cost? Values of functions need not determine the abstract function:

$$\mathcal{O}(X) \ni \alpha + \beta x \longmapsto (\alpha + \beta x : X = \{0\} \rightarrow \mathbb{A}^1) \in \text{Hom}(X, \mathbb{A}^1)$$

$0 \longmapsto \alpha$ do not recover β .

Idea: the abstract " β " remembers that X arose from the collision of

two points, so β records tangential information: $\frac{\partial}{\partial x} (\alpha + \beta x) = \beta$.

0.3 What is a point?

← (and irreducible if not) $(X; \neq X)$

X topological space is reducible if $X = X_1 \cup X_2$ for proper closed $X_i \subseteq X$.

Euclidean world (more generally if X Hausdorff): $Y \subseteq X$ irreducible $\Leftrightarrow Y = \text{point}$ or $Y = \emptyset$

Classical Alg. Geom. $\left\{ \begin{array}{l} \text{point } a \in X \leftrightarrow \text{max ideal } m_a \subseteq k[X] \\ \text{closed } \emptyset \neq Y \subseteq X \text{ irreducible} \Leftrightarrow \mathbb{I}(Y) \subseteq k[X] \text{ prime ideal} \end{array} \right.$

R ring \Rightarrow "points" of R are $\text{Spec}(R) = \{\text{prime ideals of } R\}$ not just max ideals

Categorically a good choice since functorial:

$$\varphi: R \rightarrow S \text{ hom of rings} \Rightarrow \varphi^{-1}(\text{prime ideal}) = \text{a prime ideal}$$

$$\Rightarrow \text{Spec } S \xrightarrow{\varphi^{-1}} \text{Spec } R$$

← fails for max ideals
e.g. $\mathbb{Z} \xrightarrow{\varphi} \mathbb{Q}, \varphi^{-1}(0) = 0$
We were just lucky that homs $k[Y] \rightarrow k[X]$ send max ideal \rightarrow max ideal.

1. DEFINITION OF SCHEMES

1.1 Examples of affine schemes

Motivation: M $n \times n$ matrix over \mathbb{C}
 Then $\mathbb{C}[x] \rightarrow \mathbb{C}[M], x \mapsto M$ has $\text{Ker} = \langle m_A \rangle$
 so $\mathbb{C}[M] \cong \mathbb{C}[x] / \langle m_A \rangle \cong \bigoplus \mathbb{C}[x] / (x-\lambda)^{n_i}$
 $\text{Spec } \mathbb{C}[M] = \{(x-\lambda) : \lambda: \text{eigenvalues of } A\}$

Spec(R) some ring R (always: comm. ring with 1)

- As a set: $\text{Spec}(R) = \{\text{prime ideals } p \subseteq R\}$ ← (prime) spectrum
- Zariski topology:
 closed sets: $V(I) = \{\text{prime ideals containing } I\} \subseteq \text{Spec } R$
 e.g. $V(R) = \emptyset$
 $V(0) = \text{Spec } R$

- sheaf $\mathcal{O}_{\text{Spec } R}$ which we construct later. ← spaces of functions

Rmk The global functions are: $\mathcal{O}_{\text{Spec } R}(\text{Spec } R) = R$. ← so spaces of fns can recover the top. space!

Key exercise
 (⇒ axioms for a topology)

$$V(I) \cup V(J) = V(I \cdot J) = V(I \cap J)$$

$$\bigcap V(I_i) = V(\sum I_i)$$

Rmk
 $(I \cap J) \cdot (I \cap J) \subseteq I \cdot J \subseteq I \cap J$
 so $\sqrt{I \cdot J} = \sqrt{I \cap J}$
 but $I \cdot J$ and $I \cap J$ may be \neq

Key $V(I) = \emptyset \Leftrightarrow I = R \Leftrightarrow 1 \in I$, since any proper ideal \subseteq some max ideal

Topological consequences:

open sets: $U_I = \text{Spec } R \setminus V(I) = \bigcup_{f \in I} D_f$

basis of open sets: $D_f = \{p \in \text{Spec } R : f \notin p\}$
 $f \in R \rightarrow \{p \in \text{Spec } R : f(p) \neq 0\}$

Rmk $D_{f^n} = D_f$
 for $N \geq 1$,
 since $f^N \in p \Leftrightarrow f \in p$

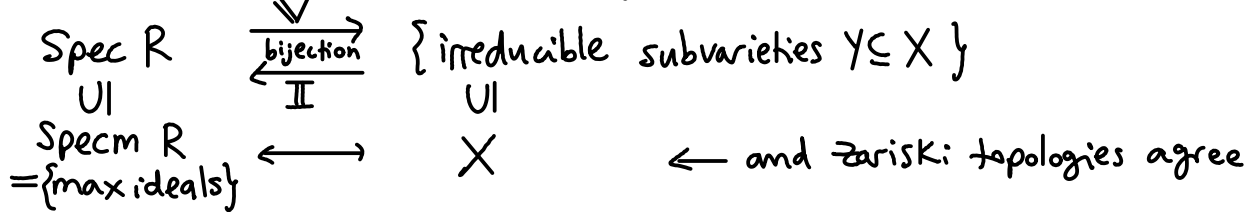
"value of $f \in R$ at p ":
 $R \rightarrow R/p \hookrightarrow K(p) = \text{Frac}(R/p) \cong R_p / p \cdot R_p$
 $f \mapsto f(p)$

localisation of R at p
 target field depends on p !

Rmk:
 p prime
 \updownarrow
 R/p is integral domain

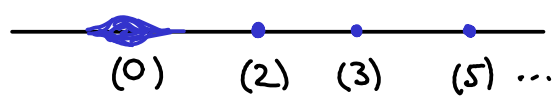
Remark $f(p) = 0 \Leftrightarrow f \in p$

Examples 1) $R = k[X] \leftarrow$ affine variety $X \subseteq \mathbb{A}^n$



Value of $f \in R$ at m_a : $m_a \rightarrow R/m_a \cong k$
 $(m_a = \langle x_1 - a_1, \dots, x_n - a_n \rangle) \quad f \mapsto f(a)$
 ← in this case the target field does not depend on the point

2) $\text{Spec } \mathbb{Z} = \{0\} \cup \{(p) : p \in \mathbb{N} \text{ prime}\}$



value of $f \in \mathbb{Z}$ at (0) :
 $\mathbb{Z} \rightarrow \text{Frac}(\mathbb{Z}/0) = \mathbb{Q}$
 $f \mapsto f$
 so lost no information.

$V((0)) = \{\text{prime ideals containing } (0)\} = \text{Spec } \mathbb{Z}$ so the point (0) is dense!
 $V((p)) = \{(p)\}$ are "closed points". Value of $f \in \mathbb{Z}$: $f((p)) = (f \in \mathbb{Z}/p) = (f \text{ mod } p)$

In general Prime ideals p with $V(p) = \text{Spec } R$ are called generic points
 prime ideals p with $V(p) = \{p\}$ are called closed points

Exercise $\{\text{closed points}\} = \{\text{max ideals of } R\}$

Exercises • a prime ideal \Rightarrow a radical ($a = \sqrt{a}$) ← recall radical of a
 • For a, b radical, $a \subseteq b \Leftrightarrow V(a) \supseteq V(b)$ ← order reversing!

Cor $V(I) \subseteq V(J) \Leftrightarrow \sqrt{I} \supseteq \sqrt{J}$

Pf $V(I) = V(\sqrt{I})$, so: $\Leftrightarrow V(\sqrt{I}) \subseteq V(\sqrt{J}) \Leftrightarrow \sqrt{I} \supseteq \sqrt{J}$ by exercise. \square

Cor $V(a) = V(b) \Leftrightarrow \sqrt{a} = \sqrt{b}$

$\sqrt{a} = \{f \in R : f^N \in a \text{ for some } N\}$
 $= \bigcap_{p \in V(a)} p$
 $\sqrt{a} \geq \bigcap_{p \in V(a)} p$
 \parallel
 $\{ \text{nilpotent elements of } R \}$
 \parallel
 $\bigcap_{p \in \text{Spec } R} p$

\Rightarrow $\{ \text{closed sets of } \text{Spec } R \} \xleftrightarrow{1:1} \{ \text{radical ideals of } R \}$ order-reversing correspondence

Proposition $f \in R$ vanishes at all $p \in \text{Spec } R \Leftrightarrow f$ nilpotent ← immediate from

Covering Trick $\text{Spec } R = \bigcup D_{f_i} \Leftrightarrow 1 \in \langle \text{all } f_i \rangle \Leftrightarrow \langle \text{all } f_i \rangle = R$

Pf $\text{Spec } R \setminus \bigcup D_{f_i} = \bigcap V(f_i) = V(\langle \text{all } f_i \rangle)$, now use previous Key. \square

Theorem $\text{Spec } R$ is quasi-compact ← (quasi-compact = compact = open covers have finite subcovers)

Pf $\text{Spec } R = \bigcup_i U_i$. As $U_i = \bigcup_j D_{f_{ij}}$, WLOG $U_i = D_{f_i}$.

Trick $\Rightarrow 1 = \sum_{\text{finite}} r_i f_i$ ← so finitely many f_i generate R , so those D_{f_i} cover. \square

Basic Exercises

1) $\varphi: R \rightarrow S$ ring hom $\Rightarrow \alpha: \text{Spec } S \rightarrow \text{Spec } R$, $p \mapsto \varphi^{-1}(p)$ is continuous
 indeed $\alpha^{-1}(D_f) = D_{\varphi f}$ ← (Hint: $f \notin p \subseteq R \Rightarrow \varphi f \notin \varphi p = q$ has $\varphi f \notin q$)

2) Show that $\text{Spec}(R/I)$ "is" the subspace $V(I) \subseteq \text{Spec } R$ and the quotient map $\pi: R \rightarrow R/I$ induces via (1) the inclusion map on Specs. ← Here "is" means: can be canonically identified with

see my C3.4 Notes about ideals in R and $I^{-1}R$

Example $\text{Spec}(R/(f)) = \{ \text{prime ideals of } R \text{ containing } f \}$
 $= \text{the points of } \text{Spec } R \text{ where } f \text{ vanishes}$
 $= V(f)$

3) Show that $\text{Spec}(S^{-1}R)$ "is" a subspace of $\text{Spec } R$, where $S^{-1}R$ is localisation of R at a multiplicative set $S \subseteq R$, and $R \rightarrow S^{-1}R$, $r \mapsto \frac{r}{1}$ induces via (1) the inclusion

means: $1 \in S$
 $S \cdot S \subseteq S$
 (we do not require $0 \notin S$)

Example $S = \{1, f, f^2, f^3, \dots\}$, so $S^{-1}R = R_f$, then:

$\text{Spec } R_f = \{ \text{prime ideals of } R \text{ not containing } f \}$
 $= \text{the points of } \text{Spec } R \text{ where } f \text{ does not vanish}$
 $= D_f$

4) $D_f \cap D_g = D_{fg}$, so $\text{Spec } R_f \cap \text{Spec } R_g = \text{Spec } R_{fg}$ ← (idea: $f^N = rg \Rightarrow \frac{1}{g} = \frac{r}{f^N}$)

5) $D_f \subseteq D_g \Leftrightarrow V(f) \supseteq V(g) \Leftrightarrow \sqrt{f} \subseteq \sqrt{g} \Leftrightarrow f^N \in (g) \text{ some } N \Leftrightarrow g \in R_f \text{ invertible}$

6) $p \subseteq R$ prime ideal $\Rightarrow R_p = S^{-1}R$ for $S = R \setminus p$, then $\exists!$ closed point $m_p = p \cdot R_p \in \text{Spec } R_p$
 so local ring: $\exists!$ max ideal m (\Leftrightarrow elts outside m are invertible)

Also: $m_p \in U \subseteq \text{Spec } R_p \text{ open} \Rightarrow U = \text{Spec } R_p$.

1.2 Definition of a scheme

RED: WORDS TO BE DEFINED LATER

Def A ringed space is

- a topological space X
- with a sheaf of rings \mathcal{O}_X on X

IDEA

- ← the points
- ← the functions
- ← the germs of functions near point x
- ← the "value" of a function at x lives here

Locally ringed space if also:

- all stalks $\mathcal{O}_{X,x}$ are local rings
- (so \exists unique maximal ideal $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$
- and \exists residue field at x : $k(x) = \mathcal{O}_{X,x} / \mathfrak{m}_{X,x}$)

Def An affine scheme is a locally ringed space isomorphic to $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ for some ring R .

Def A scheme is a locally ringed space which is locally isomorphic to an affine scheme.

means:

$$\forall x \in X \exists \text{ some open neighbourhood } x \in U \subseteq X \text{ s.t. } (U, \mathcal{O}_X|_U) \cong (\text{Spec } R, \mathcal{O}_{\text{Spec } R})$$

\exists some ring R depending on x

1.3 Pre-sheaves

Ab = category of abelian groups and group homs

X = any topological space

$\text{Top } X$ = category with objects: open sets $U \subseteq X$
morphs: inclusion maps

if use category \mathcal{C} get (pre)sheaves with values in \mathcal{C} e.g. $\mathcal{C} = \text{Rings}$ get presheaf of rings

$$\leftarrow (\text{Mor}(U, V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V \\ \{ \text{incl} \} & \text{if } U \subseteq V \end{cases}$$

Def A presheaf (of abelian groups) on X is a contravariant functor $F : \text{Top } X \rightarrow \text{Ab}$

So: \forall open $U \subseteq X$ have an abelian group $F(U)$ ← elements called sections (over U)

• \forall inclusion $U \rightarrow V$ have a "restriction" group hom

$$\boxed{F(V) \rightarrow F(U)} \\ s \mapsto s|_U$$

• $F(\text{id}: U \rightarrow U) : F(U) \xrightarrow{\text{id}} F(U)$ so $s|_U = s$ for $s \in F(U)$.

• $U \subseteq V \subseteq W \Rightarrow F(W) \rightarrow F(V) \rightarrow F(U)$ so: $(s|_V)|_U = s|_U$ for $s \in F(W)$.

Example X topological space, $F(U) = \{ \text{continuous functions } U \rightarrow \mathbb{R} \}$ with obvious restrictions

Morphism of pre-sheaves = natural transformation of such functors: $\varphi : F \rightarrow G$

So: \forall open $U \subseteq X$ have $\varphi_U : F(U) \rightarrow G(U)$ group hom

\forall inclusion $U \rightarrow V$ have $F(U) \xrightarrow{\varphi_U} G(U)$
 $F(V) \xrightarrow{\varphi_V} G(V)$ ← restriction homs

so the homs "are compatible with restrictions"

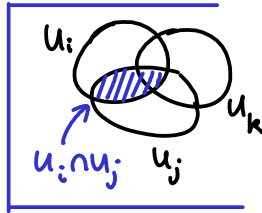
i.e. this diagram with $\varphi_U = \text{inclusion}$

Sub pre-sheaf $F \subseteq G$ means $F(U) \subseteq G(U)$ subgp, compatibly with restrictions

1.4 Sheaves

Def Pre-sheaf F is a sheaf on X if it satisfies the local-to-global condition:

If U_i open, $s_i \in F(U_i)$ agreeing on overlaps:
 $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \in F(U_i \cap U_j)$



idea: can uniquely extend!

Then \exists unique $s \in F(\cup U_i)$ with $s|_{U_i} = s_i$.

Consequences

- two sections $s, t \in F(U)$ equal \Leftrightarrow they equal locally: $s|_{U_i} = t|_{U_i}, U = \cup U_i$
- you can build sections by defining local sections, compatibly on overlaps.
- exact sequence: $0 \rightarrow F(U) \rightarrow \prod_i F(U_i) \rightarrow \prod_{i,j} F(U_i \cap U_j)$
 $s \mapsto (s_i) \quad (s_i) \mapsto (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})$
 (for $U = \cup U_i$)
- $F(\emptyset) = 0$ (Hint. consider empty covering of \emptyset)

Examples

- 1) Sheaf of continuous real functions: $F(U) = \{ \text{continuous maps } U \rightarrow \mathbb{R} \}$
- 2) Skyscraper sheaf at $p \in X$ for group A : $F(U) = \begin{cases} 0 & \text{if } p \notin U \\ A & \text{if } p \in U \end{cases}$
- 3) Presheaf of constant functions for group A :
 $F(U) = \begin{cases} A & \text{if } U \neq \emptyset \\ 0 & \text{if } U = \emptyset \end{cases}$
 (so $f \in F(U)$ is a constant function $f: U \rightarrow A, f \equiv a \in A$)
 (only want one function on \emptyset)
- 4) Sheaf of locally constant functions for group A . So $f \in F(U)$ means $f: U \rightarrow A$ such that $\forall x \in U, \exists$ open $x \in V \subseteq U$ with $f|_V: V \rightarrow A$ constant.
Warning: it implies f constant on connected components but converse can fail. (e.g. consider \mathbb{Q} with usual Euclidean topology)

Exercise (3) is not a sheaf if $X = 2$ points with discrete topology, $R \neq 0$.

Write $Ab(X) =$ category of sheaves on X and morphs of sheaves

$\leftarrow Sh(X)$ if work with category of Sets instead of Ab (morphs of presheaves)

1.5 Stalks

Def stalk at x of presheaf F is the abelian group

$$F_x = \lim_{x \in U} F(U)$$

\leftarrow direct limit over restriction maps induced by inclusions.

Explicitly:

An element of F_x is determined by $s \in F(U)$ some $U \ni x$ open, identify $s \sim t$ for $t \in F(V) \Leftrightarrow s|_W = t|_W$ some $U \cap V \supseteq W \ni x$ open

Rmk • natural map $F(U) \rightarrow F_x, s \mapsto s_x =$ equivalence class of s . (for $x \in U$)
 or write: $s|_x$

• morph $\varphi: F \rightarrow G$ then get $\varphi_x: F_x \rightarrow G_x$ (if $s \in F(U)$)
 or write: $\varphi|_x$ ($\varphi_x(s_x) = \varphi_u(s)|_x$)

Exercise $\varphi, \psi: F \rightarrow G$ morphs of sheaves,
if all $\varphi_x = \psi_x: F_x \rightarrow G_x$ then $\varphi = \psi$.

Hint.
 $\varphi_u(s)|_W = \psi_u(s)|_W$
 \parallel
 $\varphi_w(s|_W) = \psi_w(s|_W)$
 Then use local-to-global
 recall from category theory
 mono:
 $H \rightrightarrows F \rightarrow G \} \Rightarrow H \rightrightarrows F$
 composites equal } equal
 epi:
 $F \rightarrow G \rightrightarrows H \Rightarrow G \rightrightarrows H$

Facts For sheaves F, G in category $Ab(X)$

$F \rightarrow G$ monomorphism $\Leftrightarrow F_x \rightarrow G_x$ injective $\forall x$

$F \rightarrow G$ epimorphism $\Leftrightarrow F_x \rightarrow G_x$ surjective $\forall x$

$F \rightarrow G$ isomorphism $\Leftrightarrow F_x \rightarrow G_x$ iso $\forall x$

Warning mono $\Leftrightarrow F(U) \rightarrow G(U)$ inj. $\forall U$, but fails for epi: $F(U) \rightarrow G(U)$ need not be surj.

Exercise $F_x \xrightarrow{\varphi_x} G_x$ surj $\Leftrightarrow \forall t \in G(U), \exists s \in F(V): \varphi_V(s) = t|_V \in G(V)$ (but V can depend on t !) \uparrow see HWK4

1.6 Sheafification

F pre-sheaf $\Rightarrow F^+$ sheaf (ification): \swarrow so $\forall x \in U, \exists x \in V \subseteq U, t \in F(V)$
 $s(y) = t|_y \in F_y \forall y \in V$

$$F^+(U) = \left\{ s: U \rightarrow \coprod_{x \in U} F_x : \text{locally } s \text{ is a section of } F \right\}$$

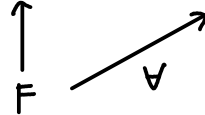
\swarrow in fact by definition $s(x) \in F_x$ so $s: U \rightarrow \coprod_{x \in U} F_x \subseteq \coprod_{x \in X} F_x$

comes with natural morph $F \rightarrow F^+ \leftarrow (s \in F(U) \mapsto (x \mapsto s_x) \in F^+(U))$

Exercise: F^+ is a sheaf, $F_x^+ = F_x$ and it satisfies:

Universal property \forall sheaf G on X , $F^+ \xrightarrow{\exists!} G$

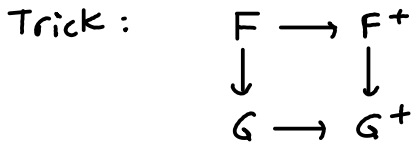
(determines F^+ uniquely up to unique isomorph)



Hint. In our construction:

$$F_x^+ = F_x \longrightarrow G_x$$

so we know locally how sections map but we need to globalise...



finally G is sheaf so $G = G^+$

(natural iso, using $G_x = G_x^+$ and Facts)

Example (pre-sheaf of constant functions) $^+ =$ (sheaf of locally constant functions)

Exercise 1) $F \subseteq G$ sub pre-sheaf, G sheaf $\Rightarrow \exists$ smallest subsheaf $H \subseteq G$ s.t. $F \subseteq H$
 Moreover, $H_x = F_x$.

("sheaf of discontinuous sections")

Hint mimic definition of F^+

2) $(DF)(U) = \prod_{x \in U} F_x$ with obvious restriction maps is a sheaf

3) $i: F \rightarrow DF$ obvious morph, let $F^b =$ presheaf image so $F^b(U) = i(U)$
 then $F^b \subseteq DF$ is a sub pre-sheaf and construction (1) gives $H = F^+$.

1.7 Kernels, Cokernels, Images

$\varphi: F \rightarrow G$ morph of sh.

$(\varphi_u: F(U) \rightarrow G(U))$

• $(\text{Ker } \varphi)(U) = \text{Ker } \varphi_u$ is sheaf

• $\text{Coker } \varphi = (\text{pre-Coker } \varphi)^+$ where $(\text{pre-Coker})(U) = \text{Coker } \varphi_u$

• $\text{Im } \varphi = (\text{pre-Im } \varphi)^+$ where $(\text{pre-Im})(U) = \text{Im } \varphi_u$

A functor of abelian cats is left exact if: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ exact $\Rightarrow 0 \rightarrow FA \rightarrow FB \rightarrow FC$ exact
right exact if $\Rightarrow FA \rightarrow FB \rightarrow FC \rightarrow 0$ exact

(F exact \Leftrightarrow F both left & right exact)

Example $\text{Hom}_R(M, \cdot)$ is left exact, $\cdot \otimes_R M$ is right exact, as functors on R -mods (any R -mod M)

1.9 Push-forward (direct image) and inverse image

$f: X \rightarrow Y$ continuous

\Rightarrow additive functor $f_*: \text{Ab} X \rightarrow \text{Ab} Y$

Def $F \in \text{Ab}(X)$ gives $f_* F \in \text{Ab}(Y)$:

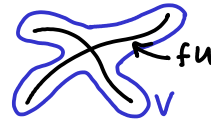
$$(f_* F)(V) = F(f^{-1}(V))$$

Exercise $(g \circ f)_* F = g_*(f_* F)$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$.

\Rightarrow additive functor $f^{-1}: \text{Ab} Y \rightarrow \text{Ab} X$

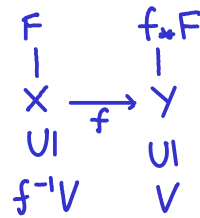
Def $F \in \text{Ab}(Y)$ gives $f^{-1} F \in \text{Ab}(X)$ is $(\text{pre-}f^{-1} F)^+$ where

$$(\text{pre-}f^{-1} F)(U) = \varinjlim_{V \supseteq f(U)} F(V)$$

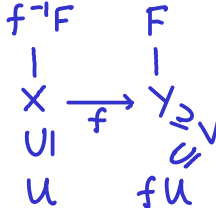


Exercise $(f^{-1} F)_x = F_{f(x)}$ and $(g \circ f)^{-1} \cong_{\text{canonical}} f^{-1} \circ g^{-1}$

PICTURE IN MIND



PICTURE IN MIND



also follows by uniqueness up to unique iso of adjoint functors, see next page.

Examples 1) $i: S \rightarrow X$ inclusion of an open subset :

$$F \in \text{Ab}(S) \quad i_* F: V \mapsto F(V \cap S)$$

$$F \in \text{Ab}(X) \quad i^{-1} F: U \mapsto F(U) \leftarrow \text{denoted } F|_S \text{ called restriction of } F$$

$\text{open } S \subseteq X$

2) $i_x: \text{point} \rightarrow X, i_x(\text{point}) = x$

$$F \in \text{Ab}(X) \quad i_x^{-1} F = F_x$$

\leftarrow (more precisely $(i_x^{-1} F)(U) = \begin{cases} F_x & \text{if } U = \{\text{point}\} \\ 0 & \text{if } U = \emptyset \end{cases}$ will not make such remarks again.)

3) $\pi: X \rightarrow \text{point}$

$$F \in \text{Ab}(X) \quad \pi_* F = \Gamma(X, F) = F(X) \leftarrow \text{global sections functor}$$

Proposition 1) f_* is left exact

2) f^{-1} is exact

\leftarrow in particular $\Gamma(X, \cdot)$ is left exact

For f_* : exercise

proof for f^{-1} : $0 \rightarrow (f^{-1} A)_x \rightarrow (f^{-1} B)_x \rightarrow (f^{-1} C)_x \rightarrow 0$

$$0 \rightarrow A_{f_x} \rightarrow B_{f_x} \rightarrow C_{f_x} \rightarrow 0 \text{ which by assumption is exact } \square$$

Rmk f_* left exact } would follow by category theory from next proposition
 f^{-1} right exact

Proposition f^{-1} is the left adjoint functor of f_* , meaning \exists natural iso

$$\text{Mor}(f^{-1}F, G) \cong \text{Mor}(F, f_*G) \text{ which is natural in } F \text{ and } G$$

Sketch pf

In \rightarrow direction:

$$F(V) \xrightarrow{\text{since } W=V \text{ is allowed}} \lim_{W \supseteq fU} F(W) \xrightarrow{\text{given}} G(U)$$

|| \leftarrow pick $U = f^{-1}V$
 $G(f^{-1}V) = f_*G(V)$

Rmk to get a map into a direct limit, you just need a representative element in one of the groups

In \leftarrow direction:

$$F(V) \xrightarrow{\text{given}} G(f^{-1}V)$$

\leftarrow assume $V \supseteq fU$
take \lim_{\rightarrow} over such V

$$\lim_{V \supseteq fU} F(V) \longrightarrow \lim_{V \supseteq fU} G(f^{-1}V) \xrightarrow{\text{restriction}} G(U)$$

\leftarrow notice $f^{-1}V \supseteq U$

Rmk to get map out of a direct limit, need maps out of all groups, compatibly with maps of lim

Now check these two are natural transformations, inverse to each other, and natural in F, G . \square

Rmk Another example of adjoint functors, for R -modules, are $\text{Hom}(M, \cdot)$ and $\otimes M$:
 $\text{Hom}(F \otimes M, G) \cong \text{Hom}(F, \text{Hom}(M, G))$ for R -mods F, G .

1.10 Morphisms of ringed spaces

Def $(f, \varphi): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ morph of ringed spaces means

often write $\varphi = f^\#$
 $X \xrightarrow{f} Y$ continuous map of topological spaces
 $f_* \mathcal{O}_X \xleftarrow{\varphi} \mathcal{O}_Y$ morph of sheaves of rings (on Y)

work with $\text{Ring}(X)$ instead of $\text{Ab}(X)$, so rings & ring homs instead of ab. grps. & gp. hom

(So: $\mathcal{O}_X(f^{-1}V) \xleftarrow[\text{ring hom}]{\varphi_V} \mathcal{O}_Y(V)$ for $V \subseteq Y$, compatibly with restrictns)

For a morphism of locally ringed spaces want in addition:

$$\mathcal{O}_{X,x} \xleftarrow{\varphi_x} \mathcal{O}_{Y,fx} \text{ is local ring hom}$$

(Explanation: $\varphi_V(\frac{s}{\sum \varphi_{f_x} \varphi_x}) \in \mathcal{O}_X(f^{-1}V)$ is a representative for $\varphi_x(s_{f_x})$)

$\varphi: R \rightarrow S$ local rings is local ring hom if $\varphi(m_R) \subseteq m_S$.
 Equivalently: $\varphi^{-1}(m_S) = m_R$
 since this \mathfrak{m} is prime and contains m_R

Rmk Can compose: $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z)$:
 $(g \circ f)_* \mathcal{O}_X = g_* f_* \mathcal{O}_X \xleftarrow{g_*(f^\#)} g_* \mathcal{O}_Y \xleftarrow{g^\#} \mathcal{O}_Z$

(This ensures that germs of functions vanishing at f_x map to germs vanishing at x)

Rmk Notice in the definition we cannot just talk about a morphism $\mathcal{O}_X \leftarrow \mathcal{O}_Y$ because the sheaves are not defined over the same topological space.

g_* is a functor so $g_*(\varphi)$ means: apply g_* to $f_* \mathcal{O}_X \xleftarrow{f^\#} \mathcal{O}_Y$

\Rightarrow either need a morph $f_* \mathcal{O}_X \leftarrow \mathcal{O}_Y$ of sheaves on Y
 or a morph $\mathcal{O}_X \leftarrow f^{-1} \mathcal{O}_Y$ of sheaves on X

By the proposition, this is the same information since $\text{Mor}(f^{-1} \mathcal{O}_Y, \mathcal{O}_X) \cong \text{Mor}(\mathcal{O}_Y, f_* \mathcal{O}_X)$
 (Notice also the map on stalks $\mathcal{O}_{X,x} = (\mathcal{O}_X)_x \leftarrow (f^{-1} \mathcal{O}_Y)_x = \mathcal{O}_{Y,fx}$ is the φ_x above)

Rmk φ local \Rightarrow also get hom on residue fields: $\varphi_x: k(fx) = \mathcal{O}_{Y,fx}/m_{Y,fx} \hookrightarrow \mathcal{O}_{X,x}/m_{X,x} = k(x)$
 \Rightarrow field extension $\varphi_x: k(fx) \hookrightarrow k(x)$ (in classical algebraic geometry: k alg. closed and x closed point
 get $\text{id}: k \rightarrow k$, $P(fa) \mapsto (f^*P)(a)$ where $\begin{cases} P \in k[Y] \\ a \in X \end{cases}$)

1.11 A sheaf defined on a topological basis

X top. space with a basis B of open subsets \leftarrow (means: basic sets cover X , and: \forall basic $B_1, B_2, x \in B_1 \cap B_2$ \exists basic B with $x \in B \subseteq B_1 \cap B_2$)

Def B-sheaf F means

- $F(U) \in \text{Ab}, \forall$ basic U with homs $F(U) \rightarrow F(V), s \mapsto s|_V \forall$ basic $V \subseteq U$ and as usual: $F(U) \xrightarrow{\text{id}} F(U)$ and $F(U) \rightarrow F(V) \rightarrow F(W)$ for $W \subseteq V \subseteq U$

local-to-global condition:

\forall basic U with $U = \cup U_i$ \leftarrow basic

$\forall s_i \in F(U_i)$ "agreeing locally on overlaps":

$\forall x \in U_i \cap U_j \exists$ basic $x \in U_k \subseteq U_i \cap U_j$ with

$$s_i|_{U_k} = s_j|_{U_k} \in F(U_k)$$



$\Rightarrow \exists$ unique $s \in F(U)$ with $s|_{U_i} = s_i$.

Rmk stalk $F_x = \varinjlim_{x \in \text{basic } V} F(V)$.

(Hence also the stalk is F_x up to canonical iso.)

Theorem 1) B-sheaf F extends uniquely (up to unique iso) to a sheaf \tilde{F} on X . \leftarrow (so $F(\text{basic } U)$ and restrictions for basic sets are same up to canonical isomorphisms.)

2) B-sheaves F, G then morph $F \rightarrow G$ on the extended sheaves is uniquely defined by data:

• homs $F(U) \rightarrow G(U)$ for basic U , commuting with restrictions (for basic opens)

Proof (1):

Uniqueness Such an extension \tilde{F} is unique (if it exists) because we can canonically identify $\tilde{F}(U)$ for any open U in terms of the B-sheaf data:

$$\tilde{F}(U) \xrightarrow{\text{bijection}} \left\{ s_v \in F(V) \text{ for } (\text{basic } V) \subseteq U: s_v|_W = s_{v'}|_W \in F(W) \text{ for basic } W \subseteq V \cap V' \right\}$$

$$s \mapsto (s_v := s|_V \in \tilde{F}(V) = F(V))$$

Explanation: given s , notice that this holds: $s_v|_W = (s|_V)|_W = s|_W = (s|_{V'})|_W = s_{v'}|_W$.

Conversely, given such $s_v \in F(V) = \tilde{F}(V)$, then $s_v|_{V \cap V'} \in \tilde{F}(V \cap V')$ and $s_{v'}|_{V \cap V'} \in \tilde{F}(V \cap V')$ must equal because their restrictions to a covering of $V \cap V'$ by basic W agree ($= s_w$).

\leftarrow (and then use sheaf property of \tilde{F})

Existence

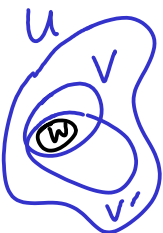
$$F(U) = \varprojlim_{(\text{basic } V) \subseteq U} F(V)$$

\leftarrow inverse limit over restrictions for basics

"compatible families of local sections on basic open sets"

$$= \left\{ (s_v) \in \prod_{(\text{basic } V) \subseteq U} F(V) : s_v|_W = s_w \quad \forall W \subseteq V \subseteq U \right\}$$

with obvious restriction maps (for $U' \subseteq U$ a subset of the $(\text{basic } V) \subseteq U$ are $\subseteq U'$)



Notice: $F(\text{basic } U)$ has not changed up to canonical identification:

$$F(U) \xrightarrow{\cong} \varprojlim_{(\text{basic } V) \subseteq U} F(V)$$

$$s \longmapsto (s|_V) \quad \text{which includes } s|_U = s.$$

and for stalks:

$$\varinjlim_{x \in (\text{basic } V)} F(V) \xrightarrow{\cong} \varinjlim_{x \in U} F(U)$$

← easy check: if sections agree on $x \in W$ then agree on $x \in V \subseteq W$ some basic V .

← includes basic $U=V$

Proof (2): by functoriality of \varprojlim :

$$\varprojlim_{(\text{basic } V) \subseteq U} F(V) \longrightarrow \varprojlim_{(\text{basic } V) \subseteq U} G(V). \quad \square$$

Rmk Equivalently, it is enough to remember germs around each point:

$$F(U) = \left(\varprojlim_{(\text{basic } V) \subseteq U} F(V) \right) \xrightarrow{\cong} \left\{ s: U \rightarrow \prod_{x \in X} F_x : s(x) \in F_x \text{ which are "locally compatible":} \right.$$

↑ take germs

with obvious restriction maps for these (just restrict the map $U \rightarrow \prod F_x$).

$$\left. \begin{array}{l} \forall x \in U, \exists x \in (\text{basic } V) \subseteq U \\ \exists t \in F(V) \\ \exists \text{ open } x \in W \subseteq V \end{array} \right\} \text{with } t_y = s(y) \quad \forall y \in W$$

Rmk Can simplify • WLOG W also basic (just pick $x \in \text{basic } \subseteq W$) } so: $\forall x \in U \exists x \in (\text{basic } V) \subseteq U$
 • WLOG replace V by W , so $V=W$ basic. } $\exists t \in F(V)$ with $t_y = s(y) \quad \forall y \in V$

Inverse: have cover $U = \cup (\text{basic } x \in V^x)$ and $t^x \in F(V^x)$ s.t. t^x agree locally (since germs agree) } so \star holds so can extend to unique global section.

1.12 Construction of $\mathcal{O}_{\text{Spec } R}$

$X = \text{Spec } R$, we define \mathcal{O}_x first on basic open sets:

$$\mathcal{O}_x(D_f) = R \text{ localised at multiplicative set } \{g : g \text{ does not vanish on } D_f\}$$

$$\cong R_f$$

↑ natural

Motivation: $\frac{1}{g}$ should be an acceptable function on D_f provided we don't divide by zero!

(Recall exercise: $\begin{array}{l} \updownarrow \\ V(g) \subseteq V(f) \Leftrightarrow D_f \subseteq D_g \\ \Leftrightarrow f^N \in (g) \Leftrightarrow g \in R_f \text{ invertible} \end{array}$)

For $D_f \subseteq D_g$ define natural restriction homs: (which are compatible under composition)

$$\mathcal{O}_x(D_g) \longrightarrow \mathcal{O}_x(D_f)$$

$$\parallel \quad \parallel$$

$$R_g \longrightarrow R_f$$

← "localise further"

← explicitly: $f^N = rg$ so

$$\frac{x}{g^m} \longmapsto \frac{x r^m}{(rg)^m} = \frac{x r^m}{f^N m}$$

Lemma 1 This is a B-sheaf on X for $B = \{ \text{basic open sets } D_f, f \in R \}$

Pf Uniqueness: $\alpha, \beta \in R_f = \mathcal{O}_X(D_f)$ and $D_f = \cup D_{f_i}$

(in \star) if $\alpha|_{D_{f_i}} = \beta|_{D_{f_i}} \forall i$ then $\alpha = \beta$

Proof By redefining X, R by D_f, R_f we can assume $f=1, R_f=R, D_f=X$.

$\alpha - \beta = 0 \in R_{f_i} \Rightarrow f_i^N \cdot (\alpha - \beta) = 0$ some $N \in \mathbb{N} \leftarrow N$ may depend on i , but

$\Rightarrow \langle \text{all } f_i^N \rangle \cdot (\alpha - \beta) = 0$ (quasi-compactness) \rightarrow WLOG finite subcover D_{f_i} so pick maximal N

recall "Covering Trick" $\rightarrow \cong R$ since $X = D_{f_1} \cup \dots \cup D_{f_n} = D_{f_1^N} \cup \dots \cup D_{f_n^N} \leftarrow$ (recall $D_f = D_{f^N}$)

$\Rightarrow 1 \cdot (\alpha - \beta) = 0$ so $\alpha = \beta \quad \square$

Existence in \star : as before WLOG $U = D_f, R_f$ become X, R .

Uniqueness \Rightarrow in \star can assume sections $s_i \in \mathcal{O}_X(D_{f_i})$ agree on overlaps $D_{f_i} \cap D_{f_j} = D_{f_i f_j}$

(apply Uniqueness to $D_{f_i f_j}$)

$$s_i|_{D_{f_i f_j}} = s_j|_{D_{f_i f_j}} \in R_{f_i f_j}$$

WLOG $X = D_{f_1} \cup \dots \cup D_{f_n}$ finite cover, $s_i = \frac{g_i}{f_i^{n_i}}$ since $D_{f_i} = D_{f_i^{n_i}}$, WLOG $n_i = 1$, so $s_i = \frac{g_i}{f_i}$

$s_i = s_j$ on $D_{f_i f_j} \Rightarrow (f_i f_j)^N (f_j g_i - f_i g_j) = 0 \in R \leftarrow N$ depends on i, j but can pick largest N over finitely many i, j so N works $\forall i, j$

rewrite: $\underbrace{(f_j^{N+1})}_{b_j} \cdot \underbrace{(f_i^N g_i)}_{a_i} - \underbrace{(f_i^{N+1})}_{b_i} \cdot \underbrace{(f_j^N g_j)}_{a_j} = 0$ notice $s_i = \frac{a_i}{b_i}, D_{f_i} = D_{b_i}$ so WLOG $N=0!$ so $f_j g_i = f_i g_j$

"Covering Trick": $X = D_{f_1} \cup \dots \cup D_{f_n}$ so $1 = \sum r_i f_i \leftarrow$ ("partition of unity" trick)

$$1 \cdot g_j = \left(\sum_i r_i f_i \right) g_j = \sum_i r_i (f_i g_j) = \sum_i r_i (f_j g_i) = f_j \left(\sum_i r_i g_i \right)$$

$\Rightarrow s_j = \frac{g_j}{f_j} = \frac{\sum_i r_i g_i}{1} \in R_{f_j} \forall j$ so we globalised the $s_j \in \mathcal{O}_X(D_{f_j})$ to $\sum_i r_i g_i \in \mathcal{O}_X(X) = R \quad \square$

Corollary \mathcal{O}_X extends uniquely to a sheaf on $X = \text{Spec } R$ called structure sheaf (or sheaf of regular functions)

stalk $\mathcal{O}_{X, p} := \lim_{D_f \ni p} \mathcal{O}_X(D_f)$

Messy unpacking of definitions: we identify $\frac{r}{f^m} \in R_f \cong \mathcal{O}_X(D_f)$ and $\frac{s}{g^n} \in R_g \cong \mathcal{O}_X(D_g)$ iff $\frac{r}{f^m} = \frac{s}{g^n} \in R_h$ some $h \in R$ with $p \in D_h \subseteq D_f \cap D_g$ (iff $h^N (r g^n - s f^m) = 0 \in R$ some N) $D_{f g}$

Lemma 2

$$\begin{array}{ccc} \mathcal{O}_{X, p} & \cong & R_p \\ \text{rest. } \uparrow & & \uparrow \text{localise} \\ \mathcal{O}_X(X) & \cong & R \end{array}$$

Pf $\lim_{D_f \ni p} \mathcal{O}_X(D_f) \cong \lim_{f \notin p} R_f \cong R_p \quad \square$

straightforward algebra exercise \leftarrow (Recall in R_p you invert all elements $f \notin p$)

$\Rightarrow \mathcal{O}_X(U) = \{s: U \rightarrow \bigsqcup_{p \in X} R_p : s(p) \in R_p \text{ which are locally compatible:}$

$\forall p \in U, \exists \text{ open nbhd } p \in D_f \subseteq U \text{ with } s(x) = t_x \}$
 $\exists t \in \mathcal{O}_X(D_f)$
 $\stackrel{f}{=} \frac{f}{f \cdot m} \stackrel{f}{=} \frac{f}{f} \text{ some } f \in R$
 $\forall x \in D_f \parallel \frac{f}{f \cdot m} \in \mathcal{O}_{X,x}$

with the obvious restriction maps.

Rmk could assume $t = \frac{f}{f}$ since can replace D_f with $D_{fm} (= D_f)$.

could just ask $s(x) = t_x$ on a smaller open $p \in V \subseteq D_f$.

is image via natural $\mathcal{O}_X(D_f) \rightarrow \mathcal{O}_{X,x}$

Comparison with classical algebraic geometry

recall $X \subseteq k^n$
 $k = \text{alg. closed field}$
 $k[X] = k[x_1, \dots, x_n]$
 $\mathbb{A}^1(X)$
 so look at scheme:
 $X = \text{Spec } k[X]$
 But classically just study closed points:
 $X = \text{Spec } k[X] \subseteq k^n$
 $m_a \leftrightarrow a$
 $\langle x_1 - a_1, \dots, x_n - a_n \rangle$

X affine variety, $p \in U \subseteq X$ open nbhd

$f: U \rightarrow k$ is regular at p if \exists open nbhd $p \in W \subseteq U$ with $f = \frac{g}{h}$ on W , $g, h \in k[X]$, $h(w) \neq 0 \forall w \in W$

Rmk In fact can assume $W = D_h$ basic open (if $f = \frac{g}{h^n}$, replace D_h by $D_{h^n} = D_h$)

$\mathcal{O}_X(U) = k$ -algebra of functions $U \rightarrow k$ regular at all $p \in U$

$\mathcal{O}_{X,p} = k$ -algebra of germs of functions near p , regular at p

(so pairs (U, f) with $p \in U \subseteq X$ open, $f: U \rightarrow k$ regular at p
 (and identify $(U, f) \sim (V, g) \Leftrightarrow f|_W = g|_W$ on some open $p \in W \subseteq U \cap V$)

Theorem $\mathcal{O}_X(X) \cong k[X]$ \leftarrow (Rmk This theorem is not obvious in C3.4 course. $X = \text{Spec } k[X]$ so by Lemma 1 get $\mathcal{O}_X(X) = k[X]$)

$X \subseteq \mathbb{A}^n$ affine variety

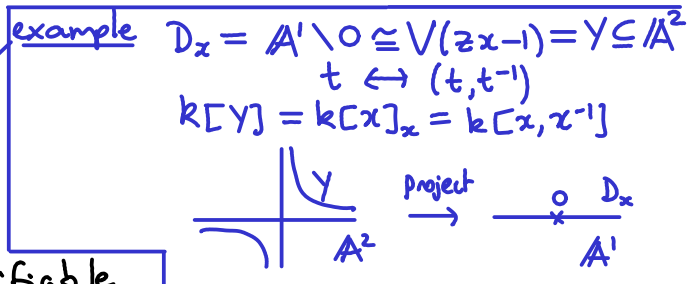
$f \in R = k[x_1, \dots, x_n]$ polynomial

$V(f) = \{f = 0\} \subseteq X$ hypersurface

$D_f = \{f \neq 0\} \subseteq X$ open, but identifiable

with affine variety $Y = V(zf - 1) \subseteq \mathbb{A}^{n+1}$ ($D_f \rightarrow Y, a \mapsto (a, \frac{1}{a})$)

and $k[Y] = k[X]/(zf - 1) \cong k[X]_f$ via $z \leftrightarrow \frac{1}{f}$



fact $\mathcal{O}_X(D_f) \cong k[X]_f$

$\mathcal{O}_{X,p} \cong k[X]_{m_p}$

\leftarrow where $m_p = \mathbb{I}(p) = \{f \in k[X] : f(p) = 0\}$ is max ideal corresponding to p .

local ring

\cup
 $m_{X,p} = m_p \cdot k[X]_{m_p} = \text{germs of functions near } p \text{ vanishing at } p$

residue field $K(p) = \mathcal{O}_{X,p} / m_{X,p} \cong k, \frac{g}{h} \mapsto \frac{g(p)}{h(p)}$ (for $p \in X$ closed point, otherwise more complicated e.g. $\mathbb{A}_k^1 = \text{Spec } k[x] : \bullet 0 \in \mathbb{A}_k^1$ is closed point ($x \in k[x]$), $K((x)) = k(x)$. $\bullet (0) \subseteq k[x]$ not closed point, $K((0)) = k(x)$.)

Morphs:

$\alpha: X \rightarrow Y \Rightarrow \alpha^\#: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\alpha^{-1}U), \alpha^\#(f: U \rightarrow k) = (\alpha^\#(f) = f \circ \alpha: \alpha^{-1}U \rightarrow k)$
 (usual pullback on functions in classical alg. geom)

1.13 Morphisms between Specs

$$\varphi: R \rightarrow S \text{ hom of rings} \Rightarrow \boxed{\text{Spec}(\varphi): \text{Spec } S \rightarrow \text{Spec } R}$$

$$p \mapsto \varphi^{-1}(p)$$

Example $\varphi: R \rightarrow R_f, r \mapsto \frac{r}{1}$ localisation

$\text{Spec } R \leftarrow \text{Spec } R_f$ is an "inclusion" with image = D_f .

$$\alpha = \text{Spec}(\varphi): Y \rightarrow X, p \mapsto \varphi^{-1}(p)$$

Lemma $\alpha^{-1}(D_f) = D_{\varphi(f)}$

automatically true!

Pf $\alpha^{-1}\{q \in X: f \notin q\} = \{p \in Y: \varphi^{-1}(p) = q \text{ some } q \in X, f \notin \varphi^{-1}(p)\}$
 $= \{p \in Y: \varphi(f) \notin p\}. \square$

Claim $\exists \varphi^\#: \mathcal{O}_X \rightarrow \alpha_* \mathcal{O}_Y$ such that $\varphi^\#_X: \mathcal{O}_X(X) = R \xrightarrow{\varphi} S = \alpha_* \mathcal{O}_Y(X)$

Pf Enough to build $\varphi^\#$ on basic opens, compatibly with restrictions

(By Theorem on B-sheaves)

$$\begin{array}{ccc} \varphi^\#: \mathcal{O}_X(D_f) & \rightarrow & \alpha_* \mathcal{O}_Y(D_f) = \mathcal{O}_Y(\alpha^{-1}D_f) = \mathcal{O}_Y(D_{\varphi(f)}) \\ \parallel & \xrightarrow{\text{natural hom}} & \parallel \\ R_f & & S_{\varphi(f)} \\ \frac{r}{f^n} \mapsto & & \frac{\varphi(r)}{\varphi(f^n)} = \frac{\varphi(r)}{\varphi(f)^n} \end{array}$$

Easy check: compatible with restriction maps for $D_g \subseteq D_f. \square$

Claim $\mathcal{O}_{X,p}$ is local and $\varphi^\#_p$ is local

Pf Lemma 2: $\mathcal{O}_{X,p} \cong R_p$ so local with max ideal $\mathfrak{m}_p = p \cdot R_p$.

For $p \in Y, \varphi^\#_p: \mathcal{O}_{X,\alpha p} \rightarrow \mathcal{O}_{Y,p}$

$$\begin{array}{ccc} \mathcal{O}_{X,\alpha p} & \rightarrow & \mathcal{O}_{Y,p} \\ \parallel & & \parallel \\ R_{\varphi^{-1}p} & \rightarrow & S_p \end{array}$$

(easy exercise: this is local. Hint: $\varphi(r) \notin p \Rightarrow r \notin \varphi^{-1}p$)

is direct limit of maps hence: natural map: $\frac{r}{t} \mapsto \frac{\varphi(r)}{\varphi(t)}$
 $t \notin \varphi^{-1}p$ so $\varphi(t) \notin p$

\Rightarrow Theorem (ring R) \rightarrow locally ringed space $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$
 (ring hom $R \xrightarrow{\varphi} S$) $\rightarrow ((\text{Spec } \varphi, \varphi^\#): (\text{Spec } S, \mathcal{O}_{\text{Spec } S}) \rightarrow (\text{Spec } R, \mathcal{O}_{\text{Spec } R}))$

Contravariant functor $\boxed{\text{Spec}: \text{Rings} \rightarrow \text{Locally Ringed Spaces}}$ (easy to check)

Claim The functor is fully faithful \leftarrow i.e. surj & inj. (so iso) on morphism spaces

Pf Given a hom of loc. ringed spaces $(f, f^\#): (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ $X = \text{Spec } R, Y = \text{Spec } S$

Let $\varphi := f^\#_X: R \cong \mathcal{O}_X(X) \xrightarrow{f^\#_X} f_* \mathcal{O}_Y(X) = \mathcal{O}_Y(Y) \cong S$ ring hom.

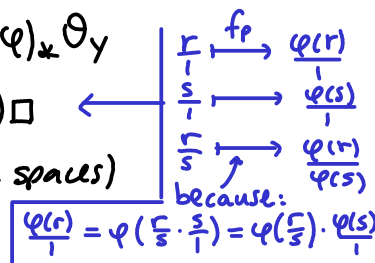
$$\begin{array}{ccc} \mathcal{O}_X(X) & \xrightarrow{f^\#_X} & \mathcal{O}_Y(Y) \\ \downarrow \ell_{f,p} & & \downarrow \ell_p \\ R_{f_p} \cong \mathcal{O}_{X,f_p} & \xrightarrow{f^\#_p} & \mathcal{O}_{Y,p} \cong S_p \supseteq \mathfrak{m}_p = p \cdot S_p \end{array}$$

\leftarrow localisation maps (Lemma 2) for $\mathcal{O}_X, \mathcal{O}_Y$

$\Rightarrow \varphi^{-1}(p) = \varphi^{-1}(\underbrace{\ell_p^{-1}(\mathfrak{m}_p)}_p) = \underbrace{\ell_{f_p}^{-1}}_{\text{diagram}}(\underbrace{f^\#_p^{-1}(\mathfrak{m}_p)}_{\mathfrak{m}_{f_p}}) = f(p)$
 since $f^\#_p$ local ring hom

$\Rightarrow f(p) = \varphi^{-1}(p)$ so $f = \text{Spec}(\varphi)$ is the map on Specs induced by $\varphi: R \rightarrow S$.

Upshot: have two morphs of sheaves $f^\#, \varphi^\# : \mathcal{O}_X \rightarrow \text{Spec}(\varphi)_* \mathcal{O}_Y$ and $f^\# = \varphi^\#$ since equal on stalks (by the diagram have $f^\#_p = \varphi^\#_p$) \square



Def Aff = category of affine schemes (and morphs of locally ringed spaces)
 (locally ringed spaces $\cong (\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ some ring R)

$\Rightarrow \text{Spec} : \text{Rings}^{\text{op}} \rightarrow \text{Aff}$ is an equivalence of categories.

(op = opposite category = reverse arrows so artificially make Spec covariant)

full, faithful, essentially surjective functor

1.14 Closed affine subschemes

$X = \text{Spec } R$, $I \subseteq R$ ideal \leftarrow (Rmk same as specifying a surj. ring hom $\varphi: R \rightarrow S$, then $\frac{I}{R/I} \cong S$)

each object in target category is iso to an object in image

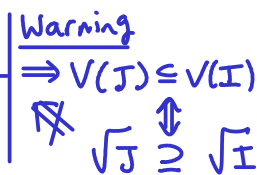
$Y = V(I) \cong \text{Spec}(R/I)$ are called closed (affine) subschemes of X

$(p \subseteq R \text{ prime } \supseteq I) \mapsto p \cdot R \subseteq R/I$

(as top. space, $V(I) = V(\sqrt{I})$ but sheaf remembers $I: \mathcal{O}_Y(Y) = R/I$)

Example $I = \mathfrak{m}$ max ideal \Rightarrow get a closed point $\{\mathfrak{m}\} = \text{Spec } R/\mathfrak{m} \hookrightarrow X$.

Rmk $\text{Spec}(R/J)$ is closed subscheme of $\text{Spec}(R/I)$ means $J \supseteq I$



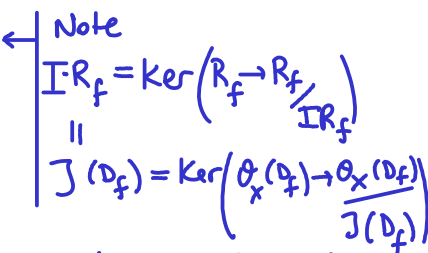
Def $\text{Spec } R/I \cap \text{Spec } R/J := \text{Spec}(R/I+J)$, $\text{Spec } R/I \cup \text{Spec } R/J := \text{Spec } R/I \cap J$

Define sheaf of ideals $\mathcal{J} = \mathcal{J}_{X/Y}$ on X :

Classical Alg. Geom: $\mathcal{J}(U)$ are the regular functions vanishing on $Y \cap U$

(also: ideal sheaf) $\mathcal{J}(D_f) = I \cdot R_f \subseteq R_f = \mathcal{O}_X(D_f)$ ideal

Notice $\mathcal{O}_Y(D_f) = (R/I)_f \cong R_f/I \cdot R_f = \mathcal{O}_X(D_f) / \mathcal{J}(D_f)$



$$\begin{array}{l}
 \mathcal{J} = \text{Ker}(\mathcal{O}_X \rightarrow j_* \mathcal{O}_Y) \\
 \mathcal{O}_Y = \mathcal{O}_X / \mathcal{J}
 \end{array}$$

where $j: Y \rightarrow X$ inclusion.

more precisely this is $j_* \mathcal{O}_Y$

1.15 Closed subschemes

(later in course: sheaves of R -modules and quasi-coherence)

Think of these as the regular functions which "vanish" on Y .

(X, \mathcal{O}_X) scheme, sheaf of ideals \mathcal{J} means $\mathcal{J}(U) \subseteq \mathcal{O}_X(U)$ ideal compatibly with restrictions.

see 1.14

Def A sheaf of ideals on $X = \text{Spec } R$ is quasi-coherent if it arises as \mathcal{J} as above, some ideal $I \subseteq R$ on $X = \text{scheme}$ " " if \forall affine open U , $\mathcal{J}|_U$ is quasi-coherent.

(later revisit these in Sec. 3.6)

Rmk $\mathcal{J} = \text{Ker}$ of surjection $\mathcal{O}_X \rightarrow j_* \mathcal{O}_Y$

closed subscheme means $Y \subseteq X$ closed topological space

$\mathcal{O}_Y = \mathcal{O}_X / \mathcal{J}$ some quasi-coherent sheaf of ideals \mathcal{J} on X ,

s.t. $Y \cap (\text{affine open } U) \subseteq U$ is closed affine subscheme for the ideal $\mathcal{J}(U) \subseteq \mathcal{O}_X(U)$.

Rmk \exists 1:1 correspondence $\{\text{closed subschemes of } X\} \leftrightarrow \{\text{quasi-coh. sheaves of ideals on } X\}$

Can recover $Y \subseteq X$ from \mathcal{J} from the support of $\mathcal{O}_X / \mathcal{J}$: \leftarrow if $I \subseteq \mathfrak{p} \subseteq R$ then $I \cdot R_{\mathfrak{p}} \neq R_{\mathfrak{p}}$ since $I \cdot R_{\mathfrak{p}} \subseteq \mathfrak{m}_{\mathfrak{p}}$

$$Y = \text{Supp } \mathcal{O}_X / \mathcal{J} = \{x \in X : (\mathcal{O}_X / \mathcal{J})_x \neq 0\} = \{x \in X : \mathcal{J}_x \neq \mathcal{O}_{X,x}\}$$

Example closed point $p \in X$ (so $\overline{\{p\}} = \{p\}$) \Rightarrow pick affine $p \in \text{Spec } R \hookrightarrow X$ then $p \leftrightarrow (\text{max ideal}) \subseteq R$

\Rightarrow sheaf \mathcal{J} on $\text{Spec } R \Rightarrow$ extend \mathcal{J} to X by $\mathcal{J}(V) = \mathcal{O}_X(V)$ if $p \notin V$ (so $\mathcal{O}_Y(V) = 0$)

2. GLOBAL SECTIONS AND THE FUNCTOR OF POINTS

2.0 Points of Spec R (not necessarily closed)

$$R \xrightarrow{\text{loc}} R_p \xrightarrow{\text{quotient}} K(p) = R_p/m_p \Rightarrow \text{Spec } K(p) \hookrightarrow \text{Spec } R_p \hookrightarrow \text{Spec } R$$

$$\text{loc}^{-1}(m_p) = P \leftarrow P \cdot R_p = m_p \leftarrow (0) \quad \{0\} \xrightarrow{(0)} m_p \xrightarrow{(0)} P$$

So points of Spec R correspond to the max ideals in the local rings.

2.1 Global sections and basic open sets for locally ringed spaces

(X, \mathcal{O}_X) locally ringed space $\Gamma(\cdot, \mathcal{O}_X) : \text{Top}(X)^{\text{op}} \rightarrow \text{Rings}$, sections functor

$$U \xrightarrow{\Gamma} \mathcal{O}_X(U) \xrightarrow{\text{restrict}} \mathcal{O}_X(V) \xleftarrow{\Gamma} V$$

global sections functor: $\text{Locally Ringed Spaces}^{\text{op}} \rightarrow \text{Rings}$, $(X, \mathcal{O}_X) \mapsto \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$

\exists canonical map $X \rightarrow \text{Spec } \mathcal{O}_X(X)$, $x \mapsto \text{res}_x^{-1}(m_{x,x})$ where $\text{res}_x: \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}$ restricts.

Trick $f \in \mathcal{O}_X(X)$ then $f_x \in \mathcal{O}_{X,x}$ invertible $\Leftrightarrow f(x) \neq 0 \in K(x) = \mathcal{O}_{X,x}/m_x$

Pf $f_x \in \mathcal{O}_{X,x} \setminus m_x = \{\text{invertibles of } \mathcal{O}_{X,x}\} \Leftrightarrow f_x \notin m_x \square$

image of f via $\mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x} \rightarrow K(x)$
 $f \mapsto f_x \mapsto f(x)$

Lemma $f \in \mathcal{O}_X(X) \Rightarrow D_f = \{x \in X : f(x) \neq 0 \in K(x)\}$ is open in X . $\Leftrightarrow f \notin m_x \Leftrightarrow (f_x \in \mathcal{O}_{X,x} \text{ invertible})$

Pf Trick $\Rightarrow \exists g \in \mathcal{O}_{X,x} : f \cdot g = 1$ so \exists open $x \in U \subseteq X$ s.t. $f, g \in \mathcal{O}_X(U)$, $f \cdot g = 1 \in \mathcal{O}_X(U)$
 $\Rightarrow x \in U \subseteq D_f$ since $\forall y \in U, f_y \cdot g_y = (f \cdot g)_y = 1 \in \mathcal{O}_{X,y}$ so $f_y \in \{\text{invertibles of } \mathcal{O}_{X,y}\}$ so $f(y) \neq 0$, so $y \in D_f \square$

Lemma $f|_{D_f} \in \mathcal{O}_X(D_f)$ is invertible

Pf Lemma $\Rightarrow f$ is locally invertible. If $f \cdot h = 1$ on U , $f \cdot g = 1$ on V then $h = g$ on $U \cap V$. So can globalise. \square

uniqueness of inverses ($h = h \cdot 1 = hfg = 1 \cdot g = g$)

2.2 What it means to be affine

(X, \mathcal{O}_X) locally ringed space affine $\Leftrightarrow \exists$ ring $R : \exists X \xrightarrow{\alpha} Y = \text{Spec } R$ homeomorph, and $\exists \mathcal{O}_Y \xrightarrow{\varphi} \alpha_* \mathcal{O}_X$ local on stalks

But $\mathcal{O}_Y(Y) = R$ so $R \xrightarrow{\varphi} \mathcal{O}_X(X)$ so $\text{Spec } \mathcal{O}_X(X) \xrightarrow{\cong} Y$.

$\varphi_x \text{ local} \Rightarrow \mathcal{O}_{Y, \alpha(x)} = R_{\alpha(x)} \xrightarrow{\varphi_x} \mathcal{O}_{X,x}$ $R \xrightarrow{\cong} \mathcal{O}_X(X) \xrightarrow{\varphi} \mathcal{O}_{X,x} \xrightarrow{\varphi_x} \mathcal{O}_{X,x}$ $R \supseteq \alpha(x) \xrightarrow{\cong} \text{res}_x^{-1}(m_x) \subseteq \mathcal{O}_X(X)$ $\alpha(x) \cdot R_{\alpha(x)} \rightarrow m_x$ so $X \xrightarrow{\text{canonical}} \text{Spec } \mathcal{O}_X(X) \cong Y$ $x \mapsto \text{res}_x^{-1}(m_x) \mapsto \alpha(x)$

via $\varphi^{-1}(\cdot)$

So a locally ringed space (X, \mathcal{O}_X) is affine precisely if:

- the canonical map $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ is homeomorph
- $\mathcal{O}_X(D_f) \cong (\Gamma(X, \mathcal{O}_X))_f \forall f \in \Gamma(X, \mathcal{O}_X)$ and restrictions are localisations \leftarrow (by Sec. 1.12)

2.3 Functor of points by

MOTIVATION Y set, you recover set Y from $\text{Mor}(\text{point}, Y)$
 Y group, " " set " " $\text{Mor}(\mathbb{Z}, Y)$

Functor of points $h_Y : \text{Sch}^{\text{op}} \rightarrow \text{Sets}$, $h_Y(X) = \text{Mor}(X, Y)$
 on morphs: $h_Y(X \xleftarrow{f} Z) = (\text{Mor}(X, Y) \xrightarrow{\circ f} \text{Mor}(Z, Y))$
 MOTIVATION $Y = \text{Spec } \mathbb{C}[x]/(x^2+1)$. \mathbb{C} -valued points of Y ?
 $\mathbb{C}[x]/(x^2+1) \rightarrow \mathbb{C}, x \mapsto i \Rightarrow \text{morph } X = \text{Spec } \mathbb{C} \rightarrow Y \text{ so } i \in h_Y(X) \leftarrow (\text{often write } Y(\mathbb{C}))$

op = opposite category = reverse arrows
 Think: "X-valued points of Y"

HWK 1 natural transformations
Yoneda lemma $\text{Nat}(h_Y, F) \cong F(Y)$
 take image of $\text{id}_Y \in \text{Mor}(Y, Y) = h_Y(Y)$ given $F(Y)$
 Conversely given $\alpha \in F(Y), \varphi \in h_Y(X)$ get $F(\varphi)(\alpha) \in F(X)$

Yoneda embedding $h_{\bullet} : \text{Sch} \rightarrow \text{Sets}^{\text{Sch}^{\text{op}}}$ $Y \mapsto h_Y$ is fully faithful
 (iso on morphisms: $\text{Nat}(h_Y, h_W) \cong \text{Mor}(Y, W)$)

UPSHOT ① $h_Y \cong h_W \iff Y \cong W$
 (Sets^{Sch^{op}} = category: {Obj are functors Sch^{op} → Sets, Morph are natural transformations})

② Can now ask which functors $\text{Sch}^{\text{op}} \rightarrow \text{Sets}$ are $\cong h_Y$, i.e. represented by a scheme Y .

Example Will show that $\mathbb{A}^n = \text{Spec } \mathbb{Z}[x_1, \dots, x_n]$ represents $\text{Sch}^{\text{op}} \rightarrow \text{Sets}, X \mapsto \{\text{morphs } \bigoplus_{i=1}^n \mathcal{O}_X \rightarrow \mathcal{O}_X \text{ which are } \mathcal{O}_X\text{-linear}\}$
 ("tell me who your friends are and I will tell you who you are")

Example 1
 $h_{\text{Spec } R} : Y \text{ affine} \implies \text{Mor}(X, \text{Spec } R) \rightarrow \text{Hom}(R, \Gamma(X, \mathcal{O}_X))$ bijective
 $= \text{Spec } R \implies g \mapsto g_{\#}$
 $\implies \text{Spec} \text{ \& global sec. are adjoint functors}$

KEY EXAMPLE
 $Y = \mathbb{A}^1 = \text{Spec } \mathbb{Z}[x]$
 $\text{Mor}(X, \mathbb{A}^1) \cong \mathcal{O}_X(X)$
 (since $\mathbb{Z}[x] \rightarrow \mathcal{O}_X(X)$ determined by image of x)
 pf. $\mathcal{O}_Y(Y) \xrightarrow{\varphi} \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}$ preimage of \mathfrak{m}_x gives $p \in \text{Spec } R = Y$
 $\parallel \leftarrow Y = \text{Spec } R$ defines $g: X \rightarrow Y, g(x) = p$
 $\cup \leftarrow \mathfrak{m}_x$
 • g is continuous (check $g^{-1}(D_f) = D_{\varphi f}$). (see 2.1 for basic opens of locally ringed spaces)
 • $\mathcal{O}_Y(D_f) = R_f \xrightarrow{\varphi_f} \mathcal{O}_X(D_{\varphi f}) \rightarrow \mathcal{O}_X(D_{\varphi f}) = \mathcal{O}_X(g^{-1}D_f) = g_{\#} \mathcal{O}_X(D_f)$
 (natural map induced by restriction $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(D_{\varphi f})$ since φ_f invertible in $\mathcal{O}_X(D_{\varphi f})$ see 2.1)
 These are compatible with restrictions \square
 (Universal property of localisation: $R_1 \xrightarrow{\varphi} R_2$ and $\varphi(S) \subseteq \text{invertibles of } R_2 \implies \exists ! R_1 \rightarrow S^{-1}R_1 \rightarrow R_2$. Obvious: $\frac{1}{s} \mapsto \varphi(s)$)

Cor 1 (X, \mathcal{O}_X) scheme \implies canonical morph $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$
 Obvious: $\frac{1}{s} \mapsto \varphi(s)$

Explicitly: on sets $x \mapsto \text{res}^{-1}(\mathfrak{m}_{X,x}) \subseteq \mathcal{O}_X(X)$
 on sheaves over $D_f \subseteq X: \mathcal{O}_X(X)_f \xrightarrow{\text{rest}} \mathcal{O}_X(D_f)$
 Rmk often not useful if X has few global sections (e.g. \mathbb{P}^n only has constants)

Rmk Canonical morph is injective if global sections separate points meaning:
 $x \neq y \in X \implies \exists f \in \Gamma(X, \mathcal{O}_X), f(x) \neq f(y)$ (equivalently $\exists f: f(x)=0, f(y) \neq 0$)

Classical algebraic geom. $X \subseteq \mathbb{A}^n$ affine variety ($X = \mathbb{V}(I), I \subseteq k[x_1, \dots, x_n]$)
 so $\Gamma(X, \mathcal{O}_X) = k[X], \mathcal{O}_X(D_f) = k[X]_f, \mathcal{O}_X(U) = \{\text{regular functions } u \mapsto k\}, \mathcal{O}_{X,a} = k[X]_{\mathfrak{m}_a}$
 separates points, and $X \xrightarrow{\text{inj.}} \{\text{closed points}\} \subseteq \text{Spec } k[X]$
 $a \mapsto \text{max ideal } \mathfrak{m}_a \subseteq k[X] \iff \text{max ideal of } \mathcal{O}_{X,a}$
 in fact get embedding {Category of Affine Varieties} \hookrightarrow Sch

Example 2 $X = \text{Spec } R \Rightarrow \left\{ \begin{array}{l} f \in \text{Mor}(\text{Spec } R, Y) \\ \text{with } f(m) = y \end{array} \right\} \xrightarrow{1:1} \text{Hom}_{\text{local rings}}(\mathcal{O}_{Y,y}, R)$ via $f \mapsto f^\#$
 $h_y(\text{Spec } R)$ $m \subseteq R = \text{local ring}$

Pf (\Rightarrow) $\text{Spec } R \xrightarrow{f} Y$
 $\downarrow \quad \quad \downarrow$
 $m \longmapsto y$
 $R = \mathcal{O}_{\text{Spec } R, m} \xleftarrow{f^\#} \mathcal{O}_{Y,y}$ local hom of rings
 (if $m \in U \subseteq \text{Spec } R$ open then $U = \text{Spec } R$, since $\text{Spec } R \setminus U$ closed so if $\neq \emptyset$ then would find another max ideal)

(\Leftarrow) **Affine case** $Y = \text{Spec } S$
 $\varphi: S_y \rightarrow R \Rightarrow S \xrightarrow{\text{loc}} S_y \rightarrow R \Rightarrow \text{Spec } R \rightarrow \text{Spec } S = Y$
 $\varphi^{-1}(m) = y \cdot S_y \quad \quad m \longmapsto (\text{preimage of } \varphi^{-1}(m)) = y$
 (via $S \rightarrow S_y \cong y \cdot S_y$)

General case
 $y \in U \subseteq Y$ open affine, then $\mathcal{O}_{U,y} = \mathcal{O}_{Y,y} \xrightarrow{\varphi} R$ gives $\text{Spec } R \rightarrow U \subseteq Y$
uniqueness: suppose $f: \text{Spec } R \rightarrow Y$ gives same φ
 $m \longmapsto y$
 pick $y \in V \subseteq Y$ affine open $\Rightarrow f^{-1}(V)$ open $\ni m = (\text{unique closed point of } \text{Spec } R) \Rightarrow f^{-1}(V) = \text{Spec } R$
 (exercise 6 in 1.1, so trick)
 so $f: \text{Spec } R \rightarrow V \subseteq Y$ so reduce to affine case. \square

Cor 2 $x \in X \Rightarrow \exists$ canonical morph $\text{Spec } \mathcal{O}_{X,x} \rightarrow X$.
 (By Example 2 for $(\text{id}: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x})$)
 Any $\text{Spec } R \rightarrow X$ factors as $\text{Spec } R \rightarrow \text{Spec } \mathcal{O}_{X,x} \rightarrow X$ some $x \in X$.
 (local ring) $\quad \quad \quad$ induced by a local ring hom
 Any $f: X \rightarrow Y$ of schemes get $\text{Spec } \mathcal{O}_{X,x} \rightarrow X \xrightarrow{f} Y$
 $x \longmapsto y$ $\quad \quad \quad$ induced by $f^\#_x$ $\quad \quad \quad$ $\begin{array}{ccc} \mathcal{M}_{X,x} & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \mathcal{M}_{Y,y} & \xrightarrow{y} & Y \end{array}$

Example Case $X = \text{Spec } K$ for field K .
 R local \Rightarrow residue field $K = R/m$
 A local hom $R \xrightarrow{\varphi} K = \text{field}$ factors $R \xrightarrow{\text{quot}} K \rightarrow K$
 (since $\ker \varphi = \varphi^{-1}(0) = m$) (since local hom)

Thus: $\left\{ \begin{array}{l} f \in \text{Mor}(\text{Spec } K, Y) \\ \text{with } f(0) = y \end{array} \right\} \xrightarrow{1:1} \text{Hom}(K(y), K)$ and any $\text{Spec } K \rightarrow Y$ factors:
 $(0) \mapsto y$
 $h_y(\text{Spec } K) \leftarrow$ (also written $Y(K)$) $\mathcal{O}_{Y,y}/\mathcal{M}_{Y,y}$ $\text{Spec } K \rightarrow \text{Spec } K(y) \rightarrow Y$

UPSHOT: Morphs from local rings or fields don't give more information than already know from $\text{Spec } \mathcal{O}_{X,x} \rightarrow X$ and $\text{Spec } K(x) \rightarrow X$.

Non-examinable:
Rmk $y \in Y$ called K -valued point if $K(y) \cong K$, then id_K defines a morph $\text{Spec } K \rightarrow Y$.
 (or K -point, or K -rational point)
 If Y comes with a morph $Y \xrightarrow{\pi} \text{Spec } K$ (hence $\mathcal{O}_y(U)$ are K -algebras) and above require morphs to commute with π , then get $\text{Hom}_K(K(y), K)$, and if $K(y) \cong K$ then $\text{Hom}_K(K, K) = \{\text{id}_K\}$. E.g. $\text{Spec } (\mathbb{C})$ has many \mathbb{C} -points: one for each automorphism of \mathbb{C} (e.g. $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto \bar{z}$) but if work over \mathbb{C} get only one \mathbb{C} -point. if work over \mathbb{R} get two \mathbb{C} -points.
 (say "Y is a scheme over K")

3. PROPERTIES OF SCHEMES

mod = module

3.0 Useful facts from commutative algebra: localisation

$1 \in S, S \cdot S \subseteq S$

R ring, M R -mod, $S \subseteq R$ multiplicative set

write " $\frac{m}{s}$ "

some $u \in S$

\Rightarrow localisation $S^{-1}M = M \times S / \text{relation } (m,s) \sim (n,t) \Leftrightarrow u \cdot (tm - sn) = 0$

which is an $S^{-1}R$ -mod and have R -mod hom $M \rightarrow S^{-1}M$ localisation map.
 $m \mapsto \frac{m}{1}$

Fact $S^{-1}M \cong M \otimes_R S^{-1}R$ canonically \leftarrow (via $\frac{m}{s} \mapsto m \otimes \frac{1}{s}$ and $\sum \frac{r_i m_i}{s_i} \mapsto \sum m_i \otimes \frac{r_i}{s_i}$)

Exercise $\alpha: M \rightarrow N$ hom (of R -mods) $\Rightarrow \exists$ natural $S^{-1}\alpha: S^{-1}M \rightarrow S^{-1}N$

Fact Localisation is an exact functor.

$\leftarrow (\frac{m}{s} \mapsto \frac{\alpha(m)}{s})$

Cor $S^{-1}(M/N) \cong S^{-1}M / S^{-1}N$

Pf apply S^{-1} to exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ \square

Fact Submods of $S^{-1}M$ have form $S^{-1}N$ for submods $N \subseteq M$ (indeed take $N = \text{preimage via } M \rightarrow S^{-1}M$)

Fact $S^{-1}M = \varinjlim M_f$ via localisation maps $M_f \rightarrow M_g$ whenever $g = fh$
 (e.g. proof: $\varinjlim M \otimes R_f = M \otimes \varinjlim R_f = M \otimes S^{-1}R$) $\frac{m}{f^n} \mapsto \frac{m h^n}{g^n}$ (induced by $R_f \rightarrow R_g$ via $M \otimes R_f \rightarrow M \otimes R_g$)

Local algebra theorem

multiplicative set $S = R \setminus p$

same results hold if only use max ideals p .

- ① $x \in M: x = 0 \Leftrightarrow x_p = 0 \in M_p \quad \forall p \in \text{Spec } R$
- ② $M = 0 \Leftrightarrow M_p = 0 \quad \forall p \in \text{Spec } R$
- ③ $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ exact $\Leftrightarrow M_p \xrightarrow{\alpha_p} M'_p \xrightarrow{\beta_p} M''_p$ exact $\forall p \in \text{Spec } R$
- ④ $f: M \rightarrow N$ inj. $\Leftrightarrow f_p: M_p \rightarrow N_p$ inj. $\forall p \in \text{Spec } R$
 " surj. " surj. "
 " iso. " iso. "

$1 \notin \text{Ann } x$ if $x \neq 0$

Pf ① $\Leftrightarrow \text{Ann}(x) = \{r \in R : rx = 0\}$ ideal \subseteq max ideal m (unless $x=0$)
 $x_m = 0 \in R_m \Rightarrow \exists r \in R \setminus m$ s.t. $rx = 0 \in R \Rightarrow$ (since $r \notin \text{Ann}(x)$)

② by ①

③ $\Leftrightarrow H := \text{Ker } \beta / \text{Im } \alpha \Rightarrow H_p \cong (\text{Ker } \beta)_p / (\text{Im } \alpha)_p = \text{Ker } \frac{\beta_p}{\text{Im } \beta_p} = 0$ now use ②
 (exact $M_p \xrightarrow{\alpha_p} M'_p \xrightarrow{\beta_p} M''_p$)

(\Rightarrow holds since localisation is exact) (since $0 \rightarrow \text{Ker } \beta \xrightarrow{\text{ind}} M' \xrightarrow{\beta} \text{Im } \beta \rightarrow 0$ exact $\Rightarrow 0 \rightarrow (\text{Ker } \beta)_p \rightarrow M'_p \xrightarrow{\beta_p} (\text{Im } \beta)_p \rightarrow 0$ exact, so $\text{Ker}(\beta_p) = (\text{Ker } \beta)_p$ $\text{Im}(\beta_p) = (\text{Im } \beta)_p$)

④ by ③ \leftarrow (e.g. inj means $0 \rightarrow M \xrightarrow{f} N$ exact) \square

Rmk $\text{Spec } R = \cup D_f$ ($1 \in \langle \text{all } f_i \rangle$) then above results hold \Leftrightarrow hold when localise at each f_i

Pf $x_i = 0 \in M_{f_i} = M \otimes R_{f_i} \Rightarrow$ localise further at $p \in \text{Spec } R_{f_i}: M_{f_i} = M \otimes_R R_{f_i} \rightarrow M \otimes_R R_p = M_p$
 (Note every $p \in \text{Spec } R$ is in some $D_{f_i} = \text{Spec } R_{f_i}$) $0 = x_i \mapsto x_p$, so 0 . \square

3.1 Noetherian

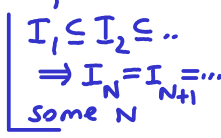
f.g. = finitely generated

Recall: ring R is Noetherian \Leftrightarrow ideals of R are f.g. \Leftrightarrow submods of f.g. R -mods are f.g. \Leftrightarrow ascending family of ideals in R stabilise ("ascending chain condition" ACC)

Rmk localisation and quotients preserve Noetherian property

Def An affine open (for the ring R) means an open subset $U \subseteq X$ admitting an isomorphism

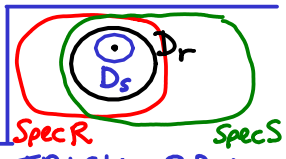
$$(U, \mathcal{O}_X|_U) \cong (\text{Spec } R, \mathcal{O}_{\text{Spec } R}) \text{ for some ring } R. \leftarrow [\text{Note: } \mathcal{O}_X(U) \cong R$$



Def scheme (X, \mathcal{O}_X) is Noetherian if quasi-compact and locally Noetherian:

Claim The following are equivalent definitions for (X, \mathcal{O}_X) to be locally Noetherian

- 1) every point has an affine open neighbourhood U with $\mathcal{O}_X(U)$ Noetherian
- 2) $X = \cup U_i$ for open affines U_i with $\mathcal{O}_X(U_i)$ Noetherian
- 3) given any open affine for a ring R , R must be Noetherian



Pf (1) \Leftrightarrow (2) and (3) \Rightarrow (1) since schemes are locally affine.

(1) & (2) \Rightarrow (3): consider $\text{Spec } R \cong U \subseteq X$

$\forall p \in U, \exists$ affine open $p \in V = \text{Spec } S \subseteq X$ with S Noetherian (by (1))

$\Rightarrow \exists$ basic open $p \in D_g \subseteq U$ for $\text{Spec } S$, some $g \in S$
 $= \text{Spec } (S_g)$ and S_g Noeth. (since S Noeth.)

By the USEFUL TRICK, WLOG D_g is basic also for $\text{Spec } R$, say $\text{Spec } R_f$.

Since $\text{Spec } S_g \cong \text{Spec } R_f$ get $S_g \cong R_f$ so Noetherian. Get cover for U ,

so need: Algebra Lemma R_{f_i} Noeth. $\forall i \} \Rightarrow R$ Noeth.
 $\langle \text{all } f_i \rangle \ni 1 \leftarrow \text{by "Covering Trick"}$

proof $I \subseteq R$ ideal (aim: I is f.g.)

$\Rightarrow I_{f_i} := I \cdot R_{f_i} \subseteq R_{f_i}$ ideal, f.g. since R_{f_i} Noeth., say generators $g_{ij} = \frac{h_{ij}}{f_i^N}$ (some $h_{ij} \in I$)

$\Rightarrow f_i^N \cdot g_{ij} = \frac{h_{ij}}{1}$ also generate (since $\frac{1}{f_i^N} \in R_{f_i}$)

$\Rightarrow \bigoplus_{ij} R \xrightarrow{\varphi} I, e_{ij} \mapsto h_{ij}$ satisfies φ_{f_i} surjective $\forall f_i$ so φ surj. \square

USEFUL TRICK R, S rings
 $p \in \text{Spec } R \cap \text{Spec } S = Y \subseteq X$
 $\Rightarrow \exists$ open $p \in D \subseteq Y$
 which is basic for both R, S .
 Pf \exists basic $p \in D_r \subseteq Y$ for R and
 \exists basic $p \in D_s \subseteq D_r$ for S as
 $\Rightarrow s \in S = \Gamma(\text{Spec } S, \mathcal{O}_X)$
 \downarrow restrict
 $s|_{D_r} =: h \in \Gamma(D_r, \mathcal{O}_X) \cong R_r$
 $\Rightarrow h = \frac{a}{r^n}$ so $(R_r)_r = R_{ar}$
 $\Rightarrow D_s = \{x \in D_r : s(x) \neq 0\}$
 $\stackrel{\text{Spec } S_s}{=} \{x \in D_r : h(x) \neq 0\}$
 $= \text{Spec } R_{ar} = D_{ar}$

Exercise give an alternative proof of algebra lemma by proving the ACC for R

(Key trick: $I = \bigcap \varphi_i^{-1}(I_{f_i})$ where $\varphi_i: R \rightarrow R_{f_i}$ is localisation.
 You may need the famous Trick: $\text{Spec } R = D_{f_1^N} \cup \dots \cup D_{f_n^N}$ so $\sum r_i f_i^N = 1$)

Lemma (Hwk 3 ex 1(v), (vi)) X Noeth. scheme \Rightarrow every subset of X is quasi-compact.

3.2 Properties that are affine-local

Above we had a property \star of affine opens ("ring is Noetherian") satisfying Affine-local conditions

- 1) $\text{Spec } R \hookrightarrow X \star \Rightarrow \text{Spec } R_{f_i} \hookrightarrow X \star \quad \forall f_i \in R$
- 2) $\text{Spec } R = \cup D_{f_i}, \text{Spec } R_{f_i} \hookrightarrow X \star \Rightarrow \text{Spec } R \hookrightarrow X \star$

so property is preserved by localisation
 can globalise from basic affines to affine

Claim $X = \cup \text{Spec } R_i$: each has $\star \implies$ every open affine in X has $\star \leftarrow$ "if holds for a cover, it holds \forall affine open"

Pf $\text{Spec } R \rightarrow X \implies \text{Spec } R = \bigcup_{\text{finite}} D_{f_{ij}}, D_{f_{ij}} \subseteq \text{Spec } R_i \xrightarrow{(1)} D_{f_{ij}} \star \xrightarrow{(2)} \text{Spec } R \star \square$

Examples of \star : "ring is reduced", "ring is Noeth.", "ring is f.g. B-algebra" (use USEFUL TRICK in 3.1)
 "locally of finite type over B" \leftarrow some fixed ring B ("base")

3.3 Reduced schemes

(X, \mathcal{O}_X) **reduced** if all $\mathcal{O}_X(U)$ reduced rings (=no nilpotents $\neq 0$)

Hwk 1 reduced \iff stalks $\mathcal{O}_{X,x}$ are reduced \leftarrow (so "stalk-local property")
 $\iff \forall p \in X$ has an open affine neighbourhood for a reduced ring

Rmk By 3.2: $\text{Spec } R$ reduced $\iff R$ reduced

Lemma X reduced, $f, g \in \mathcal{O}_X(U)$ take same values $f(x) = g(x) \in K(x) = \mathcal{O}_{X,x} / \mathfrak{m}_x \implies f = g$

Pf. Take $f - g$, wlog $g = 0$. On affine, $K(p) \cong \text{Frac}(R_p)$ so $f \in \bigcap p = \text{Nilradical}(R) = \{\text{nilpotents}\} = \{0\}$.

(Don't confuse this with general fact \forall scheme: $f_x = g_x \in \mathcal{O}_{X,x} \forall x \in U \implies f = g \in \mathcal{O}_X(U)$)

Claim (not that strong a condition e.g. $f, g: \mathbb{C} \rightarrow \mathbb{C}, f(z) = z, g(z) = \bar{z}$ different, but $f(0) = g(0), \text{Spec } \mathbb{C} = \{0\}$)

X reduced, $f, g: X \rightarrow Y, f = g$ as topological maps, $f = g$ on open dense set $\implies f = g$. $g^{-1}(\text{Spec } R)$

Pf enough show $f = g$ locally by sheaf property. wlog $Y = \text{Spec } R, X = \text{Spec } S$ (pick $\text{Spec } S \subseteq f^{-1}(\text{Spec } R)$)

$\varphi := f^\# - g^\#: R \rightarrow S$: to show φ vanishes it is enough to show $s = \varphi(1) \in S$ is zero $\leftarrow (\varphi(r) = \varphi(r \cdot 1) = \varphi(r) \cdot \varphi(1))$

$\{p \in \text{Spec } S : s(p) = 0 \in K(p)\} = \mathcal{V}(s)$ closed & contains an open dense set, hence $s = 0$ by Lemma \square
 \leftarrow since $\{p : s_p = 0 \in \mathcal{O}_{X,p}\}$ contains open dense set by assumption (means $\neq X$)

3.4 Irreducible schemes

Def Topological space X is **irreducible** if X is not a union of 2 proper closed sets:

$$X = C_1 \cup C_2 \implies X = C_1 \text{ or } X = C_2 \quad (\text{where } C_i \text{ closed})$$

Easy exercise If X irreducible:
 • Any non-empty open $U \subseteq X$ is dense and irreducible
 • Any two " " U_1, U_2 have $U_1 \cap U_2 \neq \emptyset$ (open, dense, irred)

Recall: $\text{Nil}(R) = \text{nilradical}(R) = \{\text{nilpotent elements}\} = \sqrt{(0)} = \bigcap \{p \in \text{Spec } R\}$ (R ring)

Hwk 2 (X, \mathcal{O}_X) irreducible \iff all affine opens are irreducible

Hwk 1 $\text{Spec } R$ irreducible $\iff \text{Nil}(R)$ prime ideal \implies
 $\iff R/\text{Nil}(R)$ integral domain
 $\iff \exists!$ generic point, namely $\text{Nil}(R)$

Recall $p \in X$ **generic point** if closure $\bar{p} = X$ (p is dense)

Example $\mathcal{V}(I) = \text{Spec}(R/I) \subseteq \text{Spec } R$
 irreducible $\iff \sqrt{I}$ prime ideal.
 Since $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$ as sets,
 irred. closed subsets of $\text{Spec } R$
 are: $\mathcal{V}(p)$ for $p \in \text{Spec } R$. So:
 irred. components: if p minimal
 \leftarrow (irred. & max w.r.t. \subseteq) \leftarrow (w.r.t. \subseteq)

Claim (X, \mathcal{O}_X) irreducible $\implies \exists!$ generic point y , and $y \in$ every affine open $\neq \emptyset$

Pf affine open $\emptyset \neq U \subseteq X \xrightarrow{\text{ex. above}} U$ irred. $\xrightarrow{\text{Hwk 1}} \exists!$ generic pt $x \in U \implies \bar{x} \supseteq \bar{U} = X$ (\bar{x} in X closed and $\geq U$)

Suppose $y \in X$ generic \implies if $y \in X \setminus U$ then $\bar{y} \subseteq \overline{X \setminus U} = X \setminus U$ not dense, so $y \in U$, so $y = x$. \square

Hwk 2 irreducible \iff connected $\leftarrow (X \neq U_1 \cup U_2 \text{ for disjoint open } U_i \neq \emptyset)$
 \leftarrow **Fact** $\text{Spec } R$ connected \iff no idempotents $\neq 0, 1$
 \leftarrow Classifies connected components of $\text{Spec } R$ in terms of idempotents $\leftarrow r \in R$ with $r^2 = r$

Exercise R Noetherian $\implies \exists!$ sequence of prime ideals p_1, \dots, p_n (up to reordering): $\begin{cases} \bigcap p_i = \text{Nil}(R) \\ p_i \not\subseteq \bigcap_{j \neq i} p_j \end{cases}$
 (Same Pf. as in C3.4) \leftarrow (in fact they are the minimal prime ideals of R)

$\implies \exists!$ sequence of irred. closed subsets $C_i = \mathcal{V}(p_i)$ (up to reordering): $\text{Spec } R = \bigcup C_i, C_i \not\subseteq \bigcup_{j \neq i} C_j$
 \leftarrow (which as top. subspaces are the irreducible components) \leftarrow as topological spaces

Warning: $\mathfrak{q} = (x^2) \subseteq k[x] = R \implies \mathfrak{p} = \text{Nil}(R_{\mathfrak{q}}) = (x), C = \text{Spec}(R/\mathfrak{p}) = \{0\} = \text{Spec}(R/\mathfrak{q})$ as top. spaces, not as schemes

Non-examinable (see C3.4 Notes on Lasker-Noether theorem)

To recover the scheme $\text{Spec}(R) = \bigcup \mathbb{V}(q_i)$, $\mathbb{V}(q_i) \not\subseteq \bigcup_{j \neq i} \mathbb{V}(q_j)$ need primary decomposition (like "unique factorization" but for ideals)

(so "irredundant": can't omit q_i)

$\{0\} = q_1 \cap q_2 \cap \dots \cap q_n \cap \dots \cap q_m$ where q_i are primary ideals s.t. $q_i \not\subseteq \bigcap_{j \neq i} q_j$

$q \subseteq R$ primary ideal if zero divisors of R/q are nilpotent
(Equivalently: $ab \in q \Rightarrow a \in q$ or $b \in q$ or $b^N \in q$ some N (\Leftrightarrow if $a, b \notin q$ then $a, b \in \sqrt{q}$))

Rmk $p = \sqrt{q}$ is prime ideal ("associated prime ideal") and is smallest prime ideal containing q .
So: $ab \in q, a \notin q \Rightarrow b \in p$

Example p^n is primary if p prime ideal, e.g. $(3^4) \subseteq \mathbb{Z}$

Example $(18) = (2 \cdot 3^2) = (2) \cap (3^2) \subseteq \mathbb{Z}$ is primary decomposition.

The q_i are not unique, but the $p_i = \sqrt{q_i}$ are unique (up to reordering)

(the p_i are precisely the prime ideals arising as radicals of annihilators of elts of R)

$(\mathbb{V}(q_i) = \mathbb{V}(p_i))$
(as closed sets)

The $\mathbb{V}(q_i)$ are called primary components: not unique as schemes, but are unique topologically.


• WLOG $p_1 = \sqrt{q_1}, \dots, p_n = \sqrt{q_n}$ are as in previous exercise: the minimal prime ideals

↓ (so $\text{Nil}(R) = p_1 \cap \dots \cap p_n$, which is the primary decomposition for $R/\text{Nil}(R)$)
give the isolated components $\mathbb{V}(q_i)$ (as top. subspace = $\mathbb{V}(p_i)$: irreducible comp.). These q_1, \dots, q_n are unique.

• The other q_{n+1}, \dots, q_m give rise to the embedded components $\mathbb{V}(q_j), j > n+1$ (not unique).

(Note $p_j \supseteq p_i$ some i , so $\mathbb{V}(p_j) \subseteq \mathbb{V}(p_i) \subseteq \mathbb{V}(q_i)$ are closed subschemes, but $\mathbb{V}(q_j) \not\subseteq \mathbb{V}(p_i)$ as scheme)

Rmk Can apply above to R/I to get $\sqrt{I} = p_1 \cap \dots \cap p_n, I = q_1 \cap \dots \cap q_n \cap \dots \cap q_m$, etc.

Example $I = (y^2, xy) \subseteq k[x, y] = R, X = \text{Spec}(R/I) =$ 

$\sqrt{I} = q_1, I = q_1 \cap q_2$ for $q_1 = (y), p_1 = (y)$ min prime, $\mathbb{V}(q_1)$ is isolated, irreducible
 $q_2 = (x, y)^2, p_2 = (x, y)$ embedded prime, $\mathbb{V}(q_2)$ = "fattened origin" is embedded
Think: functions vanishing on x -axis in \mathbb{A}^2 , and "order 2 at 0."
annihilator of $x \in R/I$
order 2, 2 = max length of ideals in \mathcal{O}_{X, p_2} (max length of chain of ideals $\mathcal{O}_{X, p} \not\subseteq \mathcal{I}_1 \not\subseteq \dots \not\subseteq \mathcal{I}_2 = 0$)
In example: $\mathcal{I}_1 = (x, y) \supseteq 0 = \mathcal{I}_2$
notice $p_2 \supseteq p_1$, so not minimal.
not unique, e.g. could also pick (y^2, x) .
// ← as top. space

3.5 Integral schemes

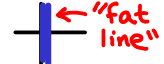

(X, \mathcal{O}_X) integral if all $\mathcal{O}_X(U)$ ID (integral domain = no zero divisors $\neq 0$)

Hwk 2 $\Leftrightarrow \mathcal{O}_X(U)$ ID \forall affine open U

Fact Localisation } preserve ID property
Direct limits \varinjlim

Cor X integral $\Rightarrow \mathcal{O}_{X, x}$ ID (but not \Leftarrow)

2 Key Non-Examples

 ← "fat line"
 $k[x, y]/(x^2)$ not reduced

 $k[x, y]/(xy) \cong k[x] \oplus k[y]$ reducible: union of two axes

nonexaminable fact if X is locally Noeth: X integral \Leftrightarrow $\begin{cases} \bullet$ connected
 \bullet $X = \bigcup \text{Spec } R_i$
 R_i integral

Hwk 2 X integral \Leftrightarrow reduced and irreducible

$\text{Spec } R$ integral $\Leftrightarrow R$ integral domain ← Example All irreducible affine varieties $X \subseteq \mathbb{A}^n$ ($\text{Spec } k[X]$)

Claim (X, \mathcal{O}_X) integral \Rightarrow restrictions $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ are injective (for $V \neq \emptyset$)

\Rightarrow • all sections can be compared in $\mathcal{O}_{X, y} \leftarrow y = \text{generic point}$

• $k(y) \cong \mathcal{O}_{X, y} \cong \text{Frac } \mathcal{O}_X(U)$ via restriction (any $U \neq \emptyset$) ← called function field $K(X)$

Pf $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V) \rightarrow \mathcal{O}_{X, y}$ so enough show $s_y = 0 \Rightarrow s = 0$.

If show $s = 0$ on every open affine $\subseteq U$ then $s_x = 0$ all $x \in U$ so $s = 0 \in \mathcal{O}_X(U)$.

\Rightarrow WLOG $U = \text{Spec } R, y = \text{Nil}(R) = \{0\}$ (since R is ID), so $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X, y}$ becomes

$R \hookrightarrow R_{(0)} = \text{Frac } R, r \mapsto \frac{r}{1}$ inj. since R is ID. Thus $s_y = 0 \Rightarrow s = 0$ □

Classical Alg. Geometry $X \subseteq \mathbb{A}^n$ irred. affine var $\Rightarrow \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(D_f) \rightarrow \mathcal{O}_{X, p} \xrightarrow{\parallel} k(X) \xrightarrow{\parallel} \text{Frac } k[X]$
(so $\text{Spec } k[X]$) $k[X] \subseteq k[X]_f \subseteq k[X]_p \subseteq \text{Frac } k[X]$

3.6 Properties of morphisms ← all properties we list are preserved when compose such morphisms

A morph of schemes $f : X \rightarrow Y$ is: (will suppress $f^\#, \mathcal{O}_X, \mathcal{O}_Y$ from notation)

① affine: equivalent conditions:

- f^{-1} (affine open) is **affine**
- \exists affine open cover V_i of Y , $f^{-1}(V_i)$ **affine**
- \forall affine open cover V_i of Y , $f^{-1}(V_i)$ **affine**

② quasi-compact: replace **affine** by **quasi-compact**

③ locally of finite type:

- \forall affine opens $U \subseteq X, V \subseteq Y$ with $f(U) \subseteq V$, $f^\# : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)$ finite type
- (meaning: $\mathcal{O}_Y(V) \xrightarrow{f^\#} \mathcal{O}_X(f^{-1}V) \xrightarrow{\text{rest}} \mathcal{O}_X(U)$)
- \exists open affine covers $Y = \cup V_i, f^{-1}(V_i) = \cup U_{ij}$
- $f^\# : \mathcal{O}_Y(V_i) \rightarrow \mathcal{O}_X(U_{ij})$ finite type

(Rings: $A \rightarrow B$ finite type means B f.g. as A -alg., i.e. \exists surj $A[x_1, \dots, x_n] \rightarrow B$ of A -algs)

④ finite type: ② + ③: quasi-compact & locally finite type

⑤ closed immersion: iso onto a closed subscheme.

Explicitly: $f : X \xrightarrow{\text{homeo}} f(X) \subseteq_{\text{closed}} Y$
 $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ surjective (so ideal sheaf $\mathcal{J} = \ker f^\#$)
 \forall aff. open $U = \text{Spec } R \subseteq Y \exists$ ideal $I \subseteq R$ s.t. $f^{-1}(U) \cong \text{Spec}(R/I)$
 \exists aff. cover $Y = \cup \text{Spec } R_i$, ideals $I_i \subseteq R_i, f^{-1}(\text{Spec } R_i) = \text{Spec}(R_i/I_i)$

Idea: functions on X are restrictions of functions of Y
 automatically quasi-coherent.
 Rmk Can specify an ideal $I \subseteq R$ by a surjective ring hom $R \rightarrow S$ (get $I = \ker$)
 Conversely given I consider $S = R/I$

Example $X = Y_{\text{red}} \subseteq Y$ closed subscheme: $X = Y$ as topological space and

(reduction of Y : it's reduced) sheaf of ideals $\mathcal{J}(U) = \{s \in \mathcal{O}_Y(U) : s(p) = 0 \in K(p), \forall p \in U\}$ (so $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{J}$)
 Note locally: on $U = \text{Spec } R, \mathcal{J}(U) = \{s \in R : s \in \cap p = \text{Nil}(R) = \{\text{nilpotents}\}\}$, so locally \mathcal{J} agrees with $\text{Nil}(\mathcal{O}_Y)$, indeed \mathcal{J} is the sheafification of $\text{Nil}(\mathcal{O}_Y)$ ← need not be sheaf, e.g. $Y = \bigsqcup_n Y_n, Y_n = \text{Spec}(\mathbb{Z}/2^n)$
 $2 \in \mathcal{O}_Y(Y), 2 \notin \text{Nil}(\mathcal{O}_Y(Y))$ but $2 \in \text{Nil}(\mathcal{O}_Y(Y_n)), 2 \in \mathcal{J}(X)$

⑥ open immersion: iso onto an open subscheme ← $U \subseteq_{\text{open}} Y, \mathcal{O}_U = \mathcal{O}_Y|_U$
 (idea: functions on X are the same as " " Y locally)

Explicitly: $f : X \xrightarrow{\text{homeo}} f(X) \subseteq_{\text{open}} Y$
 $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ iso (\Leftrightarrow iso on stalks $f^\#_x : \mathcal{O}_{Y, f_x} \rightarrow \mathcal{O}_{X, x}$)

⑦ flat: all $\mathcal{O}_{Y, f_x} \rightarrow \mathcal{O}_{X, x}$ are flat ring homs

Not intuitively clear, but ensures that fibers of f vary in a controlled way:
 Many invariants of fibers like dimension, do not change unless you "expected" it!
 It is weaker than saying the fibers are locally iso e.g. it allows two points to collide as vary fiber.

Algebra: R -mod M is flat if $M \otimes_R \cdot$ is exact functor on R -mods
 $\varphi : R \rightarrow S$ flat ring hom means S flat R -mod (using $r \cdot s = \varphi(r)s$)

Basic facts

1) $M \otimes_R \cdot$ always right exact, so M flat R -mod $\Leftrightarrow N_1 \hookrightarrow N_2$ implies $M \otimes_R N_1 \hookrightarrow M \otimes_R N_2$

Fact Enough to check $M \otimes_R I \hookrightarrow M \otimes_R R \forall$ f.g. ideal $I \subseteq R$.

2) M free $\Rightarrow M$ flat (Pf. $M \cong \bigoplus_{i \in I} R \Rightarrow M \otimes N \cong \bigoplus_{i \in I} N$)

(so no elts $\neq 0$ of finite order)

Example $\prod_{\text{infinite}} \mathbb{Z}$ is not free \mathbb{Z} -mod, but it is flat. An abelian gp is flat \mathbb{Z} -mod \Leftrightarrow torsion free

Non-example \mathbb{Z}/n is not flat \mathbb{Z} -mod: $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ then $\cdot \otimes \mathbb{Z}/n$ get $\mathbb{Z}/n \xrightarrow{\cdot n} \mathbb{Z}/n$ not inj.

Fact (Lazard) R -mod M is flat $\Leftrightarrow M = \varinjlim M_i$ some f.g. free R -mods M_i

3) R local, M finite R -mod (so $M = \sum_{\text{finite}} R m_i$): M flat $\Leftrightarrow M$ free local but $\theta_{x,x}$ is rarely finite over it

4) $A \rightarrow B$ flat, $B \rightarrow C$ flat $\Rightarrow A \rightarrow C$ flat
 Pf $N_1 \hookrightarrow N_2$ A -mods $\Rightarrow B \otimes_A N_1 \hookrightarrow B \otimes_A N_2$ B -mods $\Rightarrow C \otimes_B B \otimes_A N_1 \hookrightarrow C \otimes_B B \otimes_A N_2$ \square

5) $A \rightarrow B$ flat $\Rightarrow A_p \rightarrow B_p = B \otimes_A A_p$ flat $\forall p \in \text{Spec } A$
 Pf $N_1 \hookrightarrow N_2$ A_p -mods $\Rightarrow N_1 \hookrightarrow N_2$ A -mods (via $A \rightarrow A_p$) $\Rightarrow B \otimes_A N_1 \hookrightarrow B \otimes_A N_2$ \square

6) Ring hom $\psi: A \rightarrow B$, multiplicative sets $S \subseteq A, T \subseteq B$ with $\psi(S) \subseteq T$, then $\psi: S^{-1}B \rightarrow T^{-1}B$, $\frac{a}{s} \otimes b \mapsto \frac{\psi(a)b}{\psi(s)}$ factorizes as $S^{-1}B \xrightarrow{\cong} (\psi(S))^{-1}B \xrightarrow{\psi} T^{-1}B$
 Since isos of rings and localisation are exact functors, get ψ flat.

Example: $\mathfrak{p} \subseteq B$ prime ideal, $\mathfrak{q} = \psi^{-1}\mathfrak{p} \subseteq A$ prime ideal, $S = A \setminus \mathfrak{q}, T = B \setminus \mathfrak{p} \Rightarrow B_{\mathfrak{q}} = B \otimes_A A_{\mathfrak{q}} \rightarrow B_{\mathfrak{p}}$ flat

Theorem $\psi: A \rightarrow B$ flat ring hom $\Leftrightarrow \psi^\#: \text{Spec } B \rightarrow \text{Spec } A$ flat

Pf \Rightarrow $A \rightarrow B$ flat $\Rightarrow A_{\mathfrak{q}} \rightarrow B_{\mathfrak{q}}$ flat for $\mathfrak{q} = \psi^{-1}\mathfrak{p}$ by (5), $B_{\mathfrak{q}} \rightarrow B_{\mathfrak{p}}$ flat by (6) $\stackrel{(4)}{\Rightarrow} A_{\mathfrak{q}} \rightarrow B_{\mathfrak{p}}$ flat.

\Leftarrow Recall $\text{Ker}(B \otimes_A N_1 \xrightarrow{\psi} B \otimes_A N_2) \neq 0 \Leftrightarrow \text{Ker } \psi_p \neq 0 \forall p \in \text{Spec } B$.
 $\text{Ker}(N_1 \rightarrow N_2) = 0 \Rightarrow \text{Ker}(A_{\mathfrak{q}} \otimes_A N_1 \rightarrow A_{\mathfrak{q}} \otimes_A N_2) = 0 \Rightarrow \text{Ker}(B_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A_{\mathfrak{q}} \otimes_A N_1 \rightarrow B_{\mathfrak{p}} \otimes_{A_{\mathfrak{q}}} A_{\mathfrak{q}} \otimes_A N_2) = 0$ \square

Motivation: Deformations (see Homework 2 ex. 6)

Flatness \Rightarrow 1-parameter families of schemes have "limits".
 Fact $B = \text{Spec } k[t]$
 $B^* = B \setminus 0 = \text{Spec } k[t, t^{-1}]$
 $X \subseteq \mathbb{A}_B^n$ closed subscheme
 $\pi: X \rightarrow B$
 π flat over 0 \Leftrightarrow fiber X_0 is "limit" $\lim_{b \rightarrow 0} X_b$
 (fiber over 0 of closure of $X^* = \pi^{-1}(B^*)$)
 so $\Leftrightarrow \overline{X^*} = X$
 (see 5.1: $B^* \times_B X$)
 defined rigorously later in 5.1, for now $X_b = \pi^{-1}(b) = \text{Spec } K(b) \times_B X = \text{Spec}(K(b) \otimes_{k[t]} R)$ if $X = \text{Spec } R$

Fact Another nice properties of flat morphs $f: X \rightarrow B$, for B, X locally Noeth.:
 $\dim_x f^{-1}(b) = \dim_x X - \dim_b B$ where $b = f(x)$
 So dimensions of fibers don't "jump" unexpectedly.

Geometrical motivation (very loosely)

$X_t = \mathbb{V}(xy-t) \subseteq \mathbb{A}^2$ $X_0 = \mathbb{V}(xy)$

$X = \mathbb{V}(xy-t) \subseteq \mathbb{A}^3 = \text{Spec } k[t, x, y]$
 \downarrow
 $\mathbb{A}^1 = \text{Spec } k[t]$

how many times does a line in \mathbb{A}^2 intersect fiber?

if have a family for which intersection number is constant, it may be easy to calculate for a degenerate fiber

example: \mathbb{A}^2 has $\dim=2$
 $\{p\} \subseteq \text{line} \subseteq \text{plane}$
 $\parallel \subseteq \parallel \subseteq \parallel$
 $z_0 \subseteq z_1 \subseteq z_2$

in such theorems you will almost always see the flatness assumption

Remarks about calculating closures of sets in $X = \text{Spec } R$

1) $p \in \text{Spec } R \Rightarrow \boxed{\bar{p} = V(p)}$

Pf $p \in V(p) \Rightarrow \bar{p} \subseteq V(p)$ (since $V(p)$ closed)

Converse: $p \in \bar{p} = \bigcap_{I \subseteq p} V(I) \Rightarrow I \subseteq p \Rightarrow I \subseteq p \subseteq q \Rightarrow q \in V(I) \square$
 $q \in V(p) \Rightarrow p \subseteq q$

Example $X^* = V_*(p_1 \cdot p_2 \cdots p_k) \subseteq \mathbb{A}_{B^*}^n$, $B^* = \text{Spec } R[t, t^{-1}]$, $p_i \in R[x_1, \dots, x_n, t, t^{-1}]$ prime ideals
 $= V_*(p_1) \cup \dots \cup V_*(p_k)$ where $V_*(\cdot)$ is $V(\cdot)$ calculated in $\mathbb{A}_{B^*}^n$
 $\Rightarrow \bar{X}^* = V(p_1) \cup \dots \cup V(p_k) \subseteq \mathbb{A}_B^n$ since $p_i \in X^* \subseteq \bar{X}^*$ and $p_i \in V_*(p_i) \subseteq V(p_i) = \bar{p}_i$
 $= V(p_1 \cdot p_2 \cdots p_k)$

Recall topology:
 X topological space
 $Y \subseteq X$ top. subspace
 $\bar{Y} = \bigcap_{C \text{ closed}, Y \subseteq C} C$
 so any closed $C \supseteq Y$ satisfies $\bar{Y} \subseteq C$. Also:
 $\overline{Y_1 \cup \dots \cup Y_n} = \bar{Y}_1 \cup \dots \cup \bar{Y}_n$
Pf $Y_i \subseteq Y_1 \cup \dots \cup Y_n \Rightarrow \bar{Y}_i \subseteq \overline{Y_1 \cup \dots \cup Y_n}$
 converse:
 $Y_1 \cup \dots \cup Y_n \subseteq \underbrace{\bar{Y}_1 \cup \dots \cup \bar{Y}_n}_{\text{closed}}$
 $\Rightarrow \overline{Y_1 \cup \dots \cup Y_n} \subseteq \bar{Y}_1 \cup \dots \cup \bar{Y}_n$

2) For $\varphi: R \rightarrow S$ ring hom, $\alpha: \text{Spec } S \rightarrow \text{Spec } R$, $\alpha(p) = \varphi^{-1}p$:

Given $C = V(J) \subseteq \text{Spec } S$, $\boxed{\overline{\alpha(C)} = V(\varphi^{-1}J)}$
radical

Pf $J = \sqrt{J} = \bigcap_{\substack{J \subseteq p \\ p \in \text{Spec } S}} p \Rightarrow \varphi^{-1}J = \bigcap_{J \subseteq p} \varphi^{-1}p$
 $\alpha(p) = \varphi^{-1}p \in V(\varphi^{-1}J)$ since $\alpha(C) \subseteq \overline{\alpha(C)} = V(I)$, $I \subseteq \varphi^{-1}p \Rightarrow I \subseteq \varphi^{-1}J$
 $\alpha C \subseteq V(\varphi^{-1}J)$ since $\varphi^{-1}p \in V(\varphi^{-1}J)$ for all $p \in C$.
 $V(I) \supseteq V(\varphi^{-1}J) \square$
 $\parallel \alpha C$ $\cup I$
 αC

Example $S = R_f$ localisation, $f \in R$, if $\varphi: R \hookrightarrow R_f$ injection then $\varphi^{-1}J = R \cap J$
 e.g. $X^* = V(J) \subseteq \mathbb{A}_{B^*}^n$ for $B = \text{Spec } R[t]$, $B^* = \text{Spec } R[t, t^{-1}]$
 so $\mathbb{A}_B^n = \text{Spec } R[x_1, \dots, x_n, t]$, $\mathbb{A}_{B^*}^n = R[x_1, \dots, x_n, t, t^{-1}]$
 $\Rightarrow \boxed{\bar{X}^* = V(R[x_1, \dots, x_n, t] \cap J) \subseteq \mathbb{A}_B^n}$ is the closure

Rmk Also know inverse images of closed sets: $\boxed{\alpha^{-1}(V(I)) = V(\langle \varphi I \rangle)}$

Pf $I = \langle f_i \rangle$, $\text{Spec } R \setminus V(I) = \bigcup D_{f_i}$,
 $\bigcup D_{\varphi f_i} = \alpha^{-1}(\bigcup D_{f_i}) = \alpha^{-1}(\text{Spec } R \setminus V(I)) = \text{Spec } S \setminus \alpha^{-1}V(I)$
 $\Rightarrow \alpha^{-1}V(I) = \text{Spec } S \setminus \bigcup D_{\varphi f_i} = V(\langle \varphi f_i \rangle) \square$
(recall $\alpha^{-1}D_f = D_{\varphi f}$)

4. GLUING THEOREMS

4.1 Gluing sheaves

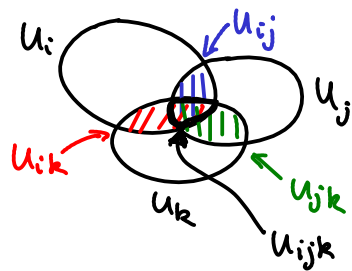
$X = \cup U_i$ open cover, abbreviate $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$

F_i sheaf on U_i

$$\varphi_{ij} : F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$$

Compatibility conditions

- 1) $\varphi_{ii} = \text{id}$
- 2) $\varphi_{ji} = \varphi_{ij}^{-1}$
- 3) $\varphi_{ik}|_{U_{ijk}} = \varphi_{jk} \circ \varphi_{ij}|_{U_{ijk}}$



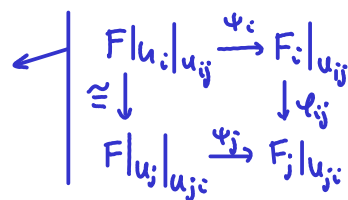
Example F sheaf on X , $F_i := F|_{U_i}$ (so $F_i(V) = F|_{U_i}(V) = F(U_i \cap V)$, \forall open $V \subseteq U_i$)

φ_{ij} = isos induced by double restrictions (iso of functors $\cdot|_{U_i}|_{U_{ij}} \cong \cdot|_{U_j}|_{U_{ij}}$)

Theorem \exists , up to unique iso, a sheaf F on X with isos

$$\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$$

s.t. $\psi_j^{-1} \circ \varphi_{ij} \circ \psi_i|_{U_{ij}}$ is the natural iso $F|_{U_i}|_{U_{ij}} \cong F|_{U_j}|_{U_{ij}}$



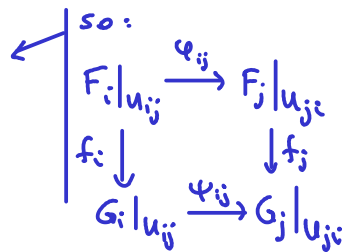
Pf Let $E = \bigsqcup_i \bigsqcup_{x \in U_i} (F_i)_x$ / equivalence relation $(F_i)_x \xrightarrow[\varphi_{ij}]{\cong} (F_j)_x$ for $x \in U_{ij}$ (using conditions)

$F(U) = \{s : U \rightarrow E : s \text{ is locally a section of some } F_i\}$. \square $(F_i)_y$
 ($\forall x \in U, \exists i, \exists \text{ open } x \in V_i \subseteq U_i, \exists t \in F_i(V_i), s(y) = t_y \forall y \in V_i$)

Theorem Given sheaves F, G constructed as above from local data F_i, φ_{ij} on U_i ; G_i, ψ_{ij}

a morph $f : F \rightarrow G$ can be uniquely defined from data:

- morphs $f_i : F_i \rightarrow G_i$
- compatibility condition: $\psi_{ij} \circ f_i|_{U_{ij}} = f_j|_{U_{ij}} \circ \varphi_{ij}$



s.t. via identifications $F|_{U_i} \cong F_i$, $G|_{U_i} \cong G_i$ recover $f|_{U_i} = f_i$

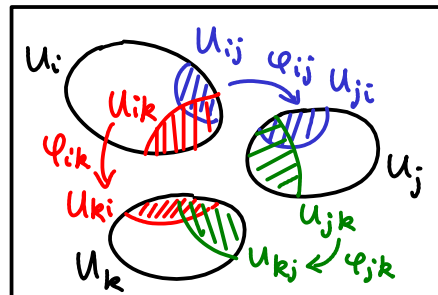
4.2 Gluing schemes

U_i schemes, $U_{ij} \subseteq U_i$ open subschemes ($U_{ii} = U_i$)

$\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}$ isos \leftarrow (think "go from U_i to U_j ")

gluing conditions

- 1) $\varphi_{ii} = \text{id}$
- 2) $\varphi_{ij}(U_{ij} \cap U_{ik}) \subseteq U_{ji} \cap U_{jk}$
- 3) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ when restricted as maps $U_{ij} \cap U_{ik} \rightarrow U_k$



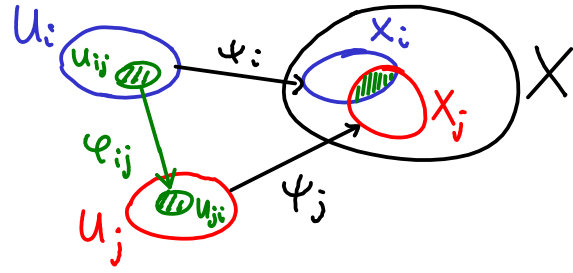
(case $k=i$)
 $\varphi_{ji}^{-1} = \varphi_{ij}$

Example if $U_i \subseteq X$ open subschemes, can take $U_{ij} = U_i \cap U_j \subseteq X$ with $\varphi_{ij} = \text{id}$

Claim (exercise) \exists unique (up to iso) scheme X with open cover $X = \cup X_i$:

• isos of schemes $U_i \xrightarrow[\varphi_i]{\cong} X_i$

• $U_{ij} \xrightarrow[\cong]{\varphi_i} X_i \cap X_j$
 $\varphi_{ij} \downarrow \cong \quad \downarrow \text{id}$
 $U_{ji} \xrightarrow[\varphi_j]{\cong} X_i \cap X_j$



Gluing Lemma Suppose we built X as above

$\Rightarrow f: X \rightarrow Y$ morph can be uniquely defined from morphs $f_i: X_i \rightarrow Y$ s.t.

compatibility condition:

$$\begin{array}{ccc} X_i \cap X_j & \longrightarrow & X_i & \xrightarrow{f_i} & Y \\ \text{id} \downarrow & & \downarrow & & \downarrow \\ X_j \cap X_i & \longrightarrow & X_j & \xrightarrow{f_j} & Y \end{array} \quad (*)$$

Pf Continuous map: $f: X \rightarrow Y$ defined by $f|_{X_i} = f_i$ (compatibly)

on sheaves need $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X \leftarrow$ (recall get $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ by adjunction)

$(f^{-1}\mathcal{O}_Y)|_{X_i} = f|_{X_i}^{-1}\mathcal{O}_Y = f_i^{-1}\mathcal{O}_Y \leftarrow (X_i \xrightarrow{\varphi_i} X \text{ inclusion, then } \varphi_i^{-1}f^{-1}\mathcal{O}_Y = (f \circ \varphi_i)^{-1}\mathcal{O}_Y$

$f_i^\# \in \text{Mor}(\mathcal{O}_Y, (f_i)_*\mathcal{O}_{X_i}) \cong \text{Mor}(f_i^{-1}\mathcal{O}_Y, \mathcal{O}_{X_i})$ and $\mathcal{O}_{X_i} = \mathcal{O}_X|_{X_i}$ since open subs.

Finally we can glue the $f_i^\#: f_i^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X|_{X_i}$ by $(*)$ to get $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$. \square

Consequence $h_Y|_{\text{Top}(X)^{\text{op}}}: \text{Top}(X)^{\text{op}} \rightarrow \text{Sets}$ is a sheaf of sets.
 $(X, Y \text{ schemes}) \quad U \longmapsto h_Y(U) = \text{Mor}(U, Y)$

4.3 Affine space by gluing (see Homework for projective space)

Affine n-space over Spec R: $\mathbb{A}_R^n := \text{Spec } R[x_1, \dots, x_n] (=:\mathbb{A}_{\text{Spec } R}^n)$

Rmk $R \rightarrow S$ ring hom \Rightarrow hom on polys (so: $R[x_1, \dots, x_n] \rightarrow S[x_1, \dots, x_n]$) $\xrightarrow{\text{Spec}} \mathbb{A}_S^n \rightarrow \mathbb{A}_R^n$

Example $R \rightarrow R_f \Rightarrow \mathbb{A}_{R_f}^n \rightarrow \mathbb{A}_R^n$ is the basic open set of \mathbb{A}_R^n for $f \in R \setminus \{0\}$

If $U \subseteq \text{Spec } R$ open $\Rightarrow U = \cup D_{f_i} \Rightarrow \mathbb{A}_U^n := \cup \mathbb{A}_{R_{f_i}}^n \subseteq \mathbb{A}_R^n$ (glued along $\text{Spec } R_{f_i} = \cup D_{f_i} \cap D_{f_j}$)

X scheme, affine n-space over X: $\mathbb{A}_X^n := \cup \mathbb{A}_{X_i}^n$ where $X = \cup X_i$ affine open cover
 (notice $\mathbb{A}_{X_i}^n = \cup_j \mathbb{A}_{X_i \cap X_j}^n$, then identify these copies) (glued along $\mathbb{A}_{X_i \cap X_j}^n$) (open in affine X_i)

Claim $\mathbb{A}^n = \text{Spec } \mathbb{Z}[x_1, \dots, x_n]$ represents functor $F: \text{Sch}^{\text{op}} \rightarrow \text{Sets}, X \mapsto \left\{ \begin{array}{l} \text{Morphs } \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{O}_X \text{ s.t. } \forall U, \\ \mathcal{O}_X(U)^{\oplus n} \rightarrow \mathcal{O}_X(U) \text{ is hom of } \mathcal{O}_X(U)\text{-mod} \end{array} \right\}$

Pf $F|_{\text{Top}(X)^{\text{op}}}$ is a sheaf of sets (easy to check: can glue morphs since \mathcal{O}_X sheaf)

$h_{\mathbb{A}^n}|_{\text{Top}(X)^{\text{op}}}$ " by consequence above. Thus if the two functors agree on affines then

by sheaf property they agree everywhere. For affine $X = \text{Spec } R$ just need compare global sections

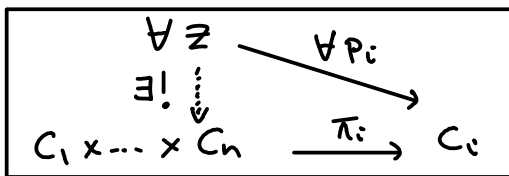
$F(\text{Spec } R) = \text{Hom}_R(R^n, R) \leftarrow$ (here: R -mod homs!) $\left. \begin{array}{l} \text{in both cases just} \\ \text{need specify} \\ \text{where generators go} \end{array} \right\} \left\{ \begin{array}{l} e_i = (0, \dots, 1, \dots, 0) \mapsto r_i \\ x_i \mapsto r_i \end{array} \right.$

5. PRODUCTS

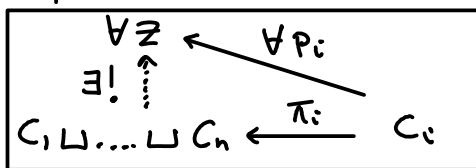
5.0 Products in category theory

Category theory: \mathcal{C} cat., $C_i \in \mathcal{C}$

product $C_1 \times \dots \times C_n$ (if exists) is an object with morphs π_i to C_i , s.t.



coproduct $C_1 \sqcup \dots \sqcup C_n$:



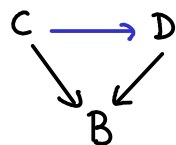
← Yoneda / functor of points interpretation: \leftarrow product of sets
 $F: \mathcal{C}^{op} \rightarrow \text{Sets}, F(Z) = \prod \text{Mor}_{\mathcal{C}^{op}}(C_i, Z) = \prod h_{C_i}(Z)$
 Is it representable? if so, call the object $\prod C_i, h_{\prod C_i} \cong F = \prod h_{C_i}$
 Explicitly: $(p_i) \in \prod h_{C_i}(Z)$ gives unique $\in h_{\prod C_i}(Z) = \text{Mor}(Z, \prod C_i)$
 Why \exists maps π_j ? \exists projections of sets $h_{\prod C_i}(Z) \cong \prod h_{C_i}(Z) \rightarrow h_{C_j}(Z)$
 but $\text{Mor}(h_{\prod C_i}, h_{C_j}) \cong \text{Mor}(\prod C_i, C_j) \ni \pi_j$.

Examples Sets / Top.spaces: \times = product, π_i = projections, \sqcup = disjoint union, π_i are inclusions
 Vector spaces/abelian gps/modules: " , \sqcup = direct sum, π_i are inclusions.
 Rings: " , \sqcup = tensor product, $\pi_i(r) = 1 \otimes \dots \otimes r \otimes \dots \otimes 1$

Fix $B \in \mathcal{C}$ ("base")

Category of B-objects: \mathcal{C}/B

obj: morphs $C \rightarrow B$, morphs: in \mathcal{C}

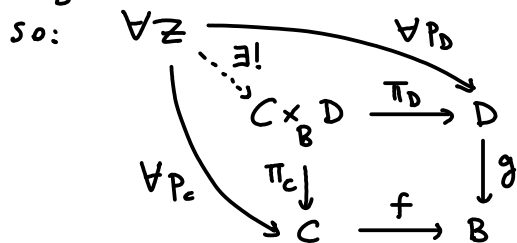


(think of B as a parameter space and C as a family parametrised by B)

fiber product

$C \times_B D$ is the product in \mathcal{C}/B of $C \xrightarrow{f} B, D \xrightarrow{g} B$ (if exists)

(or pullback, or Cartesian square)



similarly get $C_1 \times_B \dots \times_B C_n$

IMPORTANT EXAMPLES:

All schemes X have canonical $X \rightarrow \text{Spec } \mathbb{Z}$ by giving canonical maps on affines:
 $\text{Spec } R \rightarrow \text{Spec } \mathbb{Z}$ from $\mathbb{Z} \rightarrow R, 1 \mapsto 1$

Schemes over field k means have $X \rightarrow \text{Spec } k$, same as saying all $\mathcal{O}_X(U)$ are k -algebras and restrictions are k -alg.homs

← Functor of points interpretation:

$\text{Hom}(Z, C \times_B D) \cong \text{Hom}(Z, C) \times_{\text{Hom}(Z, B)} \text{Hom}(Z, D)$
 So we are asking whether $h_C \times_{h_B} h_D$ is representable

Example for Sets or Top.spaces: $C \times_B D = \{ (c, d) \in C \times D : f(c) = g(d) \in B \}$

for example if f, g are inclusions of subsets (subspaces) then $C \times_B D = C \cap D$

Pushout The opposite diagram (reverse arrows)

Example: for Rings the pushout of $B \rightarrow C, B \rightarrow D$ is the tensor product $C \otimes_B D$ sec. 4.2

Example: $B \xrightarrow{f} C, B \xrightarrow{g} D$ inclusions of open subschemes, then pushout $C \sqcup_B D$ is the gluing!

Exercise: (co)product, fiber product, pushout are unique up to unique iso if they exist.

(Hint: compose unique maps between them (s.t. diagram commutes) then composites = id by uniqueness of self-map)

Examples of fiber products in cat. of Sets or TopSpaces: $C \times_B D = \{ (c, d) : f(c) = g(d) \} \subseteq C \times D$

$B = \text{point} \Rightarrow C \times_B D = C \times D$

$C \xrightarrow{\subseteq} B, D \xrightarrow{\subseteq} B \Rightarrow C \times_B D \cong C \cap D$

$D \xrightarrow{\subseteq} B \Rightarrow C \times_B D \cong f^{-1}(D) \subseteq C$ for example $D = \text{point} = b \in B$ get fiber $f^{-1}(b)$

$C = D \Rightarrow C \times_B D = \{ (x, y) : f(x) = g(y) \} \subseteq C \times D$ ("equaliser")

5.1 Fiber products exist in Schemes/B

Rmk $B = \text{Spec } \mathbb{Z}$ gives $X \times_B Y = X \times Y$

Fix scheme B, consider category Schemes/B

Theorem fiber products $X_1 \times_B \dots \times_B X_n$ exist

Inductively suffices to do case $n=2$. First need some algebraic preliminaries

An A-algebra R is a ring R together with a ring hom $A \xrightarrow{\psi} R$
(A ring) $(\Rightarrow R \text{ is } A\text{-mod via } a \cdot r = \psi(a)r)$

R, S A-algebras $\Rightarrow (R \otimes_A S) = \frac{\text{free R-alg. on } R \times S}{\text{relations}}$

\leftarrow (so general element is $\sum r_i \otimes s_i$
so "generators" are $r \otimes s$)

relations: i) \otimes is bilinear

ii) $a \cdot (r \otimes s) = (\psi_R(a) \cdot r) \otimes s = r \otimes (\psi_S(a) \cdot s)$

\leftarrow (often drop ψ_R, ψ_S from notation.)

In particular $A \rightarrow R \otimes_A S$ is $a \mapsto a \cdot (1 \otimes 1) = \psi_R(a) \otimes 1 = 1 \otimes \psi_S(a)$

The product on generators: $(r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'$.

Rmk R, S rings $\Rightarrow R \otimes S = R \otimes_{\mathbb{Z}} S$

Facts

1) $R \otimes_R S \cong S$ (via $\sum r_i \otimes s_i \mapsto \sum r_i s_i$)

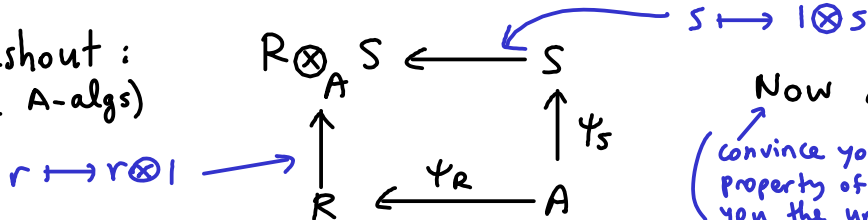
2) $R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] \cong R[x_1, \dots, x_n, y_1, \dots, y_m]$

3) $(S/I) \otimes_R T \cong (S \otimes_R T) / (I \otimes 1) \cdot (S \otimes_R T)$ where S, T are R-algebras

4) k field, A k-alg, for A-algs R, S get: $R \otimes_A S \cong (R \otimes_k S) / \langle \psi_R(a) \otimes 1 - 1 \otimes \psi_S(a) : a \in A \rangle$

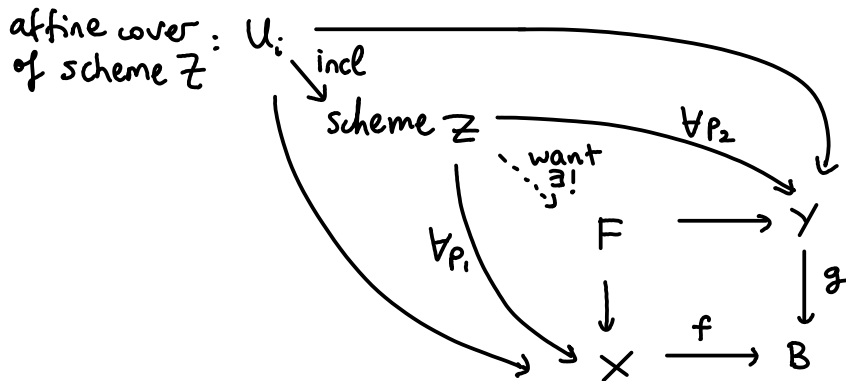
Affine case: $\text{Spec } R \times_{\text{Spec } A} \text{Spec } S = \text{Spec } (R \otimes_A S)$ exists in $\text{Aff}/\text{Spec } A$:

have pushout:
(in category of A-algs)



Now apply Spec. \square
(convince yourself that universal property of pushout for A-algs gives you the univ. prop. for fiber prod. for $\text{Aff}/\text{Spec } A$)

Claim: this is fiber product also in $\text{Sch}/\text{Spec } A$: let $X = \text{Spec } R$
 $Y = \text{Spec } S$
 $B = \text{Spec } A$
 $F = \text{Spec } (R \otimes_A S)$



Recall fiber products are unique up to unique iso if they exist.

By construction (as U_i affine) $\exists!$ $U_i \rightarrow F$ making diagram commute

(used universal property in Aff/B)

If can show these agree on overlaps $U_{ij} = U_i \cap U_j$, then glue to unique $Z \rightarrow F$.

If U_{ij} were affine, this would have been immediate.

$U_{ij} \subseteq \text{affine } U_i$, so running same argument with Z replaced by U_{ij} ,

we can cover U_{ij} by basic open affines $D_{f_k} \subseteq U_i$ and now $D_{f_k} \cap D_{f_l} = D_{f_k f_l}$ affine!

\Rightarrow glue uniquely to give $U_{ij} \rightarrow F$

"USEFUL TRICK" in 3.1

Recall trick that can pick open cover of U_{ij} that are basic opens simultaneously for U_i, U_j

$\Rightarrow U_{ij} \rightarrow F$ and $U_{ji} \rightarrow F$ agree.

General case build schemes/morphs by 3 gluing procedures (tedious!)

- | | |
|---|---|
| 1) case $U_i \times_B Y$ with B, Y affine, $X = \cup U_i$ affine open cover | $\Rightarrow \exists X \times_B Y$ affine |
| 2) case $X \times_B V_j$ with B affine, $Y = \cup V_j$ " " " | $\Rightarrow \exists X \times_B Y$ affine |
| 3) case $X \times_{W_k} Y$ with $B = \cup W_k$ " " " | $\Rightarrow \exists X \times_B Y$ |

Gluing works because agreement on overlaps is ensured by uniqueness up to iso of fiber products. Sketch:

preimage of open set viewed as open subscheme of U_i
(easy check by category theory)

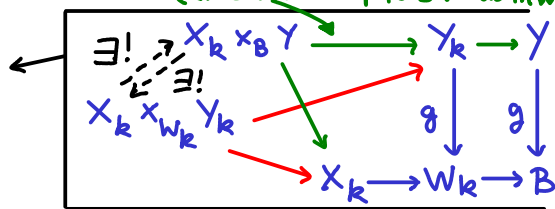
① if know $U_i \times_B Y$ exist, then $\pi_i^{-1}(U_{ij})$ is fiber product $U_{ij} \times_B Y$ so by uniqueness \exists iso $\pi_i^{-1}(U_{ij}) \rightarrow \pi_i^{-1}(U_{ji})$, so glue & get $X \times_B Y$
(indeed a natural identification since $U_{ij} = U_{ji}$ with sheaf $\mathcal{O}_{U_{ij}}$)

② as in ①, swapping roles X, Y .

again: open subschemes since preimages of opens

③ let $X_k = f^{-1}(W_k), Y_k = g^{-1}(W_k) \Rightarrow X_k \times_{W_k} Y_k$ exists by ② *(W_k affine, X_k, Y_k general)*
(exists as map to B lands in W_k)

Key trick: notice $X_k \times_{W_k} Y_k = X_k \times_B Y$
 "because images are trapped in W_k, Y_k anyway"
 Then use argument in ① to glue the $X_k \times_B Y$. \square



Rmk Proof shows that $X \times_B Y$ has affine open cover by $\cup (U_i \times_{W_k} V_j)$

where $X = \cup U_i, Y = \cup V_j$ are " " "

$B = \cup W_k$ with $U_i \rightarrow W_k \subseteq B, V_j \rightarrow W_k \subseteq B$

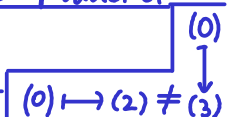
e.g. $(x-y) \in \text{Spec } \mathbb{Z}[x,y]$

more points than fiber product of sets

Examples

1) $\mathbb{A}_R^n \times_{\text{Spec } R} \mathbb{A}_R^m = \text{Spec } R[x_1, \dots, x_n, y_1, \dots, y_m] = \mathbb{A}_R^{n+m}$

2) $\text{Spec } \mathbb{Z}/2 \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}/3 = \text{Spec } (\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3) = \text{Spec } (0) = \emptyset$



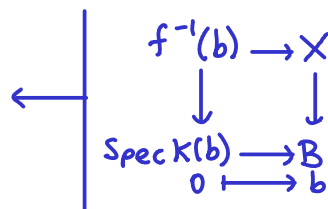
Exercise $X \times_Y Y \cong X, X \times_B Y \cong Y \times_B X, (X \times_B Y) \times_B Z \cong X \times_B (Y \times_B Z), X \times_A B \times_B Y \cong X \times_A Y$.

5.2 Fibers and preimages

$f: X \rightarrow B$ morph of schemes

fiber over point $b \in B$: $f^{-1}(b) = \text{Spec } k(b) \times_B X$

preimage of closed subscheme $Y \subseteq B$: $f^{-1}(Y) = Y \times_B X$



Examples

$k =$ algebraically closed field \leftarrow (so classical alg. geometry)

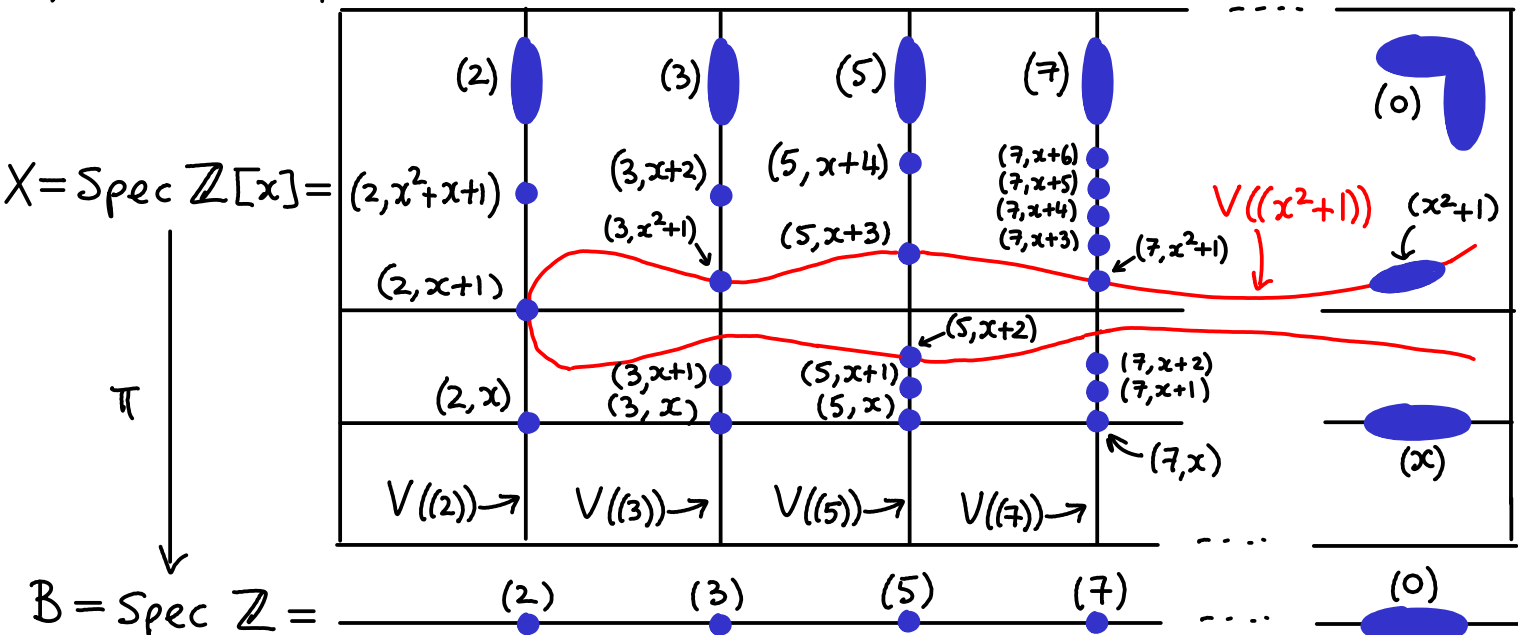
$f: \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ induced by $f^\#: k[x] \rightarrow k[y], x \mapsto y^2$

fiber over 0: (view point 0 as $\text{Spec } k \rightarrow \mathbb{A}_k^1$ so $k \cong k[x]/(x)$)

fiber = $\text{Spec } k \times_{\text{Spec } k[x]} \mathbb{A}_k^1 = \text{Spec}(k \otimes_{k[x]} k[y])$
 $= \text{Spec}(k[y^2]/(y^2) \otimes_{k[y^2]} k[y]) \cong \text{Spec}(k[y]/(y^2))$ where $f(x) = y^2$
 (e.g. use facts about \otimes from 5.1)

Rmk Notice how a product of affine varieties gave a scheme that was not an affine variety.

4) Mumford's picture of $\text{Spec } \mathbb{Z}[x]$:



π is induced by inclusion $\mathbb{Z} \rightarrow \mathbb{Z}[x]$
 $\Rightarrow \pi^{-1}((p)) = V((p)) = \{(p), (p, f(x)) : f(x) \text{ mod } p \text{ is irreducible in } \mathbb{F}_p[x]\}$
 (so (p) is a dense point in $\pi^{-1}((p))$).
 (if $p \in \mathcal{I}$ then $\mathbb{Z}[x]/\mathcal{I} \cong \mathbb{F}_p[x]/\mathcal{I}$ where $\mathbb{F}_p = \mathbb{Z}/p$ PID, so (f) prime $\Leftrightarrow f$ irred or 0

Rmk curve $V(x^2+1)$ passes through $(p, x+j)$ iff x^2+1 vanishes at that point, so iff $x^2+1=0$ in $\mathbb{F}_p[x]/(x+j) \cong \mathbb{F}_p, x \mapsto -j$, so iff $j^2 = -1$.
 Classical number theory says a square root of -1 exists in $\mathbb{F}_p \Leftrightarrow (p \equiv 1 \pmod{4} \text{ or } p=2)$

fiber over (p) : $k(p) = \mathbb{Z}_{(p)}/p \cdot \mathbb{Z}_{(p)} = (\mathbb{Z}/p)_{(p)} = \mathbb{F}_p = \mathbb{Z}/p$
 $\Rightarrow \pi^{-1}(p) = \text{Spec}(k(p) \otimes_{\mathbb{Z}} \mathbb{Z}[x]) = \text{Spec } \mathbb{F}_p[x] = \{(0), (\bar{f}(x))\}$ irred in $\mathbb{F}_p[x]$ nonconstant

fiber over (0) : $k(0) = \mathbb{Z}_{(0)} = \mathbb{Q}$
 $\Rightarrow \pi^{-1}(0) = \text{Spec}(k(0) \otimes_{\mathbb{Z}} \mathbb{Z}[x]) = \text{Spec } \mathbb{Q}[x] = \{(0), (f(x))\}$
 [Gauss's Lemma: for $f \in \mathbb{Z}[x]$ primitive (gcd(coeffs)=1) f irred. $\in \mathbb{Z}[x] \Leftrightarrow f$ irred. $\in \mathbb{Q}[x]$] irred in $\mathbb{Q}[x]$ nonconstant so WLOG irred in $\mathbb{Z}[x]$ nonconstant

Consequence $\text{Spec } \mathbb{Z}[x] = \{(0), (p), (f), (p, f)\}$ $f \in \mathbb{Z}[x]$ irred. mod p nonconstant
 \uparrow $p \in \mathbb{Z}$ prime \uparrow $f \in \mathbb{Z}[x]$ irred, nonconstant

Forgetful functor $|\cdot|: \text{Sch} \rightarrow \text{TopSpaces}$, $X \mapsto |X| = \text{underlying topological space}$,
 $\text{morph} \mapsto \text{underlying continuous map}$

Claim $f: X \rightarrow B$ morph schemes $\Rightarrow |f^{-1}(b)| = |f|^{-1}(b)$

← fiber is homeomorphic to topological fiber

Pf WLOG B affine = $\text{Spec } S$ and b is prime ideal $p \subseteq S$

$f^{-1}(B) = \cup \text{Spec } R_i$ given by $\varphi_i: S \rightarrow R_i$

WLOG just consider one affine, so $R = R_i$, so WLOG $X = \text{Spec } R$

$$\Rightarrow \text{Spec } k(b) \times_B X = \text{Spec } (k(b) \otimes_S R)$$

$$k(b) = (S/p)_p \Rightarrow k(b) \otimes_S R = (S/p)_p \otimes_S R = S_p \otimes_S S/p \otimes R = S_p \otimes_S R / \varphi(p)R = R_{\varphi(p)} / \varphi(p) \cdot R_{\varphi(p)}$$

$$\Rightarrow \text{Spec } (k(b) \otimes_S R) \xleftarrow{|\cdot|} \left\{ \begin{array}{l} q \subseteq R \text{ prime ideal containing } \varphi(p) \text{ but not intersecting } \varphi(S \setminus p) \\ q \cdot R_p \leftrightarrow q \end{array} \right. \quad (= \text{preimage of } qR_p \text{ via localisation } R \rightarrow R_p = S_p \otimes_S R)$$

$$q \subseteq R \setminus \varphi(S \setminus p) \Rightarrow \varphi^{-1}q \subseteq S \setminus (S \setminus p) = p \quad \text{so get } \{q \in \text{Spec } R : \varphi^{-1}q = p\}.$$

$$q \supseteq \varphi(p) \Rightarrow \varphi^{-1}q \supseteq p$$

Cor Given $f: X \rightarrow B$, $g: Y \rightarrow B$,

(apply 1.1 to diagram defining $X \times_B Y$ then by universal property in category of topological spaces get unique map \otimes)

fiber of $|X \times_B Y| \xrightarrow{\otimes} |X| \times_{|B|} |Y|$ over (x, y) is $|\text{Spec } (k(x) \otimes_{k(b)} k(y))|$ where $f(x) = g(y) = b$

Pf fiber of $X \times_B Y \rightarrow X$ over x : $\text{Spec } k(x) \times_X (X \times_B Y) = \text{Spec } k(x) \times_B Y$

fiber of $\text{Spec } k(x) \times_B Y \rightarrow Y$ over y : $\text{Spec } k(x) \times_B Y \times_Y \text{Spec } k(y) = \text{Spec } k(x) \times_B \text{Spec } k(y)$

fiber of $\text{Spec } k(x) \times_B \text{Spec } k(y) \rightarrow B$ over b : $\text{Spec } k(x) \times_{\text{Spec } k(b)} \text{Spec } k(y) = \text{Spec } k(x) \otimes_{k(b)} k(y)$.
 lands in $\{b\} \subseteq B$

(by Claim can work with fiber in Sch before applying 1.1)

<p>at algebra level: if A_1, A_2 are modules over $S = R_p/pR_p$ then</p> $S \otimes_R (A_1 \otimes_R A_2) \cong A_1 \otimes_S A_2$ <p>namely:</p> $\frac{r}{t} \otimes a_1 \otimes a_2 \mapsto \frac{r}{t} \cdot (a_1 \otimes a_2)$	<p>or at category level, with abuse of notation:</p> <p>hence isos $\rightarrow \exists!$</p>
--	--

Examples $|\text{Spec } \mathbb{Z}_2 \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}_3| = |\text{Spec } \mathbb{Z}_2| \times_{|\text{Spec } \mathbb{Z}|} |\text{Spec } \mathbb{Z}_3| = \emptyset$ since 1st factor $\mapsto (2) \in \text{Spec } \mathbb{Z}$ and 2nd factor $\mapsto (3) \in \text{Spec } \mathbb{Z}$

$\mathbb{A}_k^2 = \mathbb{A}_k^1 \times_{\text{Spec } k} \mathbb{A}_k^1 = \text{Spec } k[x, y]$ then $(x+y) \mapsto (0)$ via both projections to \mathbb{A}_k^1 but $(x+y) \neq (0)$
 so $|\mathbb{A}_k^2| \neq |\mathbb{A}_k^1| \times |\mathbb{A}_k^1|$: the fiber over $(0, 0)$ is complicated.
 (field k) note $\text{Spec } k = \text{point} = \{(0)\}$ so often omit "Spec k " from notation.

Rmk If x, y closed points of schemes X, Y finite type over k , k algebraically closed, then fiber over (x, y) of $X \times_{\text{Spec } k} Y$ is $\text{Spec } (k(x) \otimes_k k(y)) = \text{Spec } (k \otimes_k k) = \text{Spec } k = (0)$
 so over closed points you get the product of sets. ← (so classical alg. geom.)

Warning $\mathbb{A}_k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1$ does not have the product topology, e.g. consider $\mathbb{V}(x-y)$

Non-examinable Rmk Working over an algebraically closed field k , the stalk of $X \times_{\text{Spec } k} Y$ at (x, y) is $\mathcal{O}_{X, x} \otimes_k \mathcal{O}_{Y, y}$ localised at max ideal $\mathfrak{m}_{X, x} \otimes \mathfrak{m}_{Y, y} + \mathcal{O}_{X, x} \otimes \mathfrak{m}_{Y, y}$

5.3 Base change

$$X_A := X \times_B A \rightarrow X \quad \text{is base-change of } X \rightarrow B \text{ to } A$$

all schemes \rightarrow

$$\begin{array}{ccc} & & \downarrow \\ & & A \longrightarrow B \\ & & \downarrow \end{array}$$

Example $\mathbb{A}_X^n = \mathbb{A}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} X$ is base change of $\mathbb{A}_{\mathbb{Z}}^n$ to X via $X \rightarrow \text{Spec } \mathbb{Z}$

Motivation This generalises the idea of changing the "base coefficients"

example: $X = \text{Spec } \mathbb{R}[x_1, \dots, x_n] / (f_1, \dots, f_n)$ real affine variety $\subseteq \mathbb{R}^n$

$B = \text{Spec } \mathbb{R}$
 $A = \text{Spec } \mathbb{C}$ } and $A \rightarrow B$ via $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ inclusion

$X \times_B A$ is Spec of: $\frac{\mathbb{R}[x_1, \dots, x_n]}{(f_1, \dots, f_n)} \otimes_{\mathbb{R}} \mathbb{C} \cong \frac{\mathbb{C}[x_1, \dots, x_n]}{(\varphi(f_1), \dots, \varphi(f_n))}$ so affine var $\subseteq \mathbb{C}^n$
 (same polys but viewed over \mathbb{C})

Same works if replace $\mathbb{R} \rightarrow \mathbb{C}$ by any ring hom $S \rightarrow R$.

FACT Many properties of $A \rightarrow B$ are inherited by the base change $X_A \rightarrow X$:

- ① affine, ② quasi-compact, ③ locally finite type, ④ finite type, ⑤/⑥ closed/open immersion, ⑦ flat as well as properties from 5.4: ⑧ separated, ⑨ universally closed, ⑩ proper

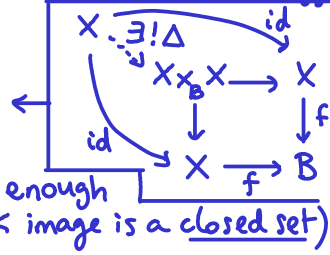
5.4 More properties of schemes (all properties we list are preserved when compose such morphs)

Motivation Topological space X is Hausdorff \Leftrightarrow diagonal $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ closed for product topology

⑧ $f: X \rightarrow B$ morph of schemes is separated if

$\Delta = \Delta_{X/B}: X \rightarrow X \times_B X$ is a closed immersion.

$\forall \exists$ open cover U_i of B , $f^{-1}(U_i) \rightarrow U_i$ separated



(HWK 3: enough to check image is a closed set)

Rmk often write Δ to mean image $\subseteq X \times_B X$ of morphism Δ .

Rmk Any subscheme $S \subseteq X$ over B is also separated since $\Delta_{S/B} = \Delta_{X/B} \cap (S \times_B S)$

Rmk X separated means separated over $\text{Spec } \mathbb{Z}$ so $\Delta \subseteq X \times X$ closed

Example for affine varieties (similar for projective varieties) work over $B = \text{Spec } k$:

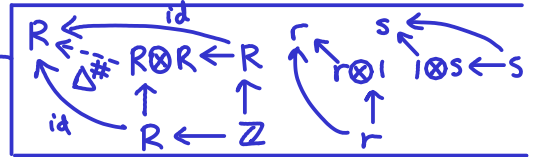
$$\text{Spec } k[X] \times_k \text{Spec } k[X] = \text{Spec } k[X] \otimes_k k[X] \supseteq \Delta \text{ has ideal } \langle f \otimes 1 - 1 \otimes f : f \in k[X] \rangle$$

Why good? It disallows pathologies like "affine line with two origins" (Hwk 1 ex. 5) arising by gluing $\text{Spec } \mathbb{R}[s, s^{-1}] \rightarrow \text{Spec } \mathbb{R}[x]$ by $x \rightarrow s$ (if do $x \rightarrow t^{-1}$ then get $\mathbb{P}_{\mathbb{R}}^1$: Hwk. 2, ex 1)

Claim Affine opens are separated (same proof for $\text{Spec } R \rightarrow \text{Spec } S$)

Pf $\Delta: \text{Spec } R \rightarrow \text{Spec } R \times \text{Spec } R$ comes from $R \otimes R \xrightarrow{m} R$,
 $= \text{Spec } (R \otimes R)$

surjective: $m(r, 1) = r$ (and $\ker = \langle r \otimes 1 - 1 \otimes r : r \in R \rangle$). \square



Claim X separated $\Leftrightarrow \forall$ affine opens U_1, U_2 $\left\{ \begin{array}{l} \text{i) } U_1 \cap U_2 \text{ affine} \\ \text{ii) } \Gamma(U_1, \mathcal{O}_X) \otimes \Gamma(U_2, \mathcal{O}_X) \xrightarrow{\text{surj}} \Gamma(U_1 \cap U_2, \mathcal{O}_X) \end{array} \right.$ (enough if holds for cover U_i)

Pf \Rightarrow $U_1 \cap U_2 \cong (U_1 \times U_2) \cap \Delta$, so $U_1 \cap U_2 \subseteq U_1 \times U_2$ closed inside affine $U_1 \times U_2$ so affine

U_i affine $\Rightarrow \Gamma(U_1) \otimes \Gamma(U_2) \cong \Gamma(U_1 \times U_2)$. Say $U_1 \times U_2 = \text{Spec } A$, then:

$$U_1 \cap U_2 \cong \Delta \cap \text{Spec } A = \text{Spec } A_{\mathbb{I}}$$

\Leftarrow Cover $X \times X = \cup U_i \times U_j$ by products of affine opens. $A \xrightarrow{\cong} A/\mathbb{I}$

$\Gamma(U_1 \times U_2) \cong \Gamma(U_1) \otimes \Gamma(U_2) \xrightarrow{\text{surj}} \Gamma(U_1 \cap U_2)$ so $\Delta^{-1}(U_i \times U_j) \cong U_i \cap U_j \subseteq X$ closed (its ideal is ker of hom (ii))
 So Δ closed immersion (use 3rd definition in ⑤ Sec. 3.6)

HWK 3 Claim holds also in case $\Delta_{X/B}$, after tweaking conditions slightly.

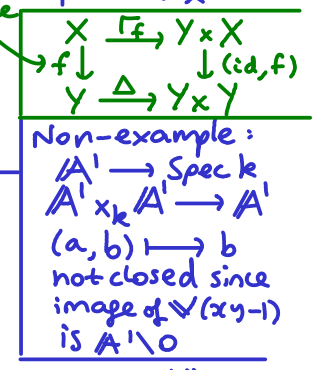
Claim X separated $\Leftrightarrow \forall \varphi_1, \varphi_2: Y \rightarrow X$ if $\varphi_1 = \varphi_2$ on dense subset \leftarrow "equalizers are closed"
 then $\varphi_1 = \varphi_2$ as topological maps (so if Y reduced then $\varphi_1 = \varphi_2$ as morphisms)

Pf $\Rightarrow \varphi_1 \times \varphi_2: Y \rightarrow X \times X, (\varphi_1 \times \varphi_2)^{-1}(\Delta) \subseteq Y$ is closed & dense so $= Y$. \leftarrow see 3.3

\Leftarrow Let $Y = \overline{\Delta \cap (U_i \times U_j)}$ $\subseteq U_i \times U_j$ (affine) and $\varphi_1, \varphi_2: Y \rightarrow X$ projections $\Rightarrow \varphi_1 = \varphi_2$ is precisely the set $\Delta \cap (U_i \times U_j)$.
 \leftarrow aff. cover of $X \times X$

Claim $X \xrightarrow{f} Y, Y$ separated \Rightarrow graph $\Gamma_f: X \rightarrow X \times Y$ closed imm.
Pf $f \times id: X \times Y \rightarrow Y \times Y, \Gamma_f \cong (f \times id)^{-1} \Delta$ closed \square \leftarrow Non-examinable Rmk: Can also view this as a base change

9 Motivation For top. spaces, X compact $\Leftrightarrow (\forall Y, X \times Y$ is closed map i.e. sends closed sets to closed sets)
 $f: X \rightarrow B$ universally closed: $X_y = X \times_B Y \rightarrow X$ every base change is closed map $\rightarrow \downarrow$ $\downarrow f$ is closed map
Fact f univ. closed $\Rightarrow f$ quasi-compact. $Y \xrightarrow{\Delta} B$
10 $f: X \rightarrow B$ proper \Leftrightarrow (4), (8), (9) (finite type, separated and universally closed)



Non-example: $\mathbb{A}^1 \rightarrow \text{Spec } k, \mathbb{A}^1 \times_k \mathbb{A}^1 \rightarrow \mathbb{A}^1$
 $(a, b) \mapsto b$ not closed since image of $\forall (xy-1)$ is $\mathbb{A}^1 \setminus \{0\}$

Motivation Analogue in smooth world is "preimages of compact sets are compact"
Example Projective n-space $\mathbb{P}_B^n = \mathbb{P}_Z^n \times B$ (build \mathbb{P}_Z^n by gluing in Hwk 2)

$f: X \rightarrow Y$ is a projective morphism if factors $X \xrightarrow{\text{closed immersion}} \mathbb{P}_Y^n \xrightarrow{\text{projection}} Y$
Fact if X, Y Noetherian this is proper.

Non-examinable Rmk
 Quasi-projective morph $X \rightarrow Y$ if X open imm. $\rightarrow Z$ Proj. morph. $Z \rightarrow Y$ if X, Y Noeth. this is (4) & (8) (finite type & separated)

5.5 Varieties \leftarrow or abstract variety

Def A variety is a scheme over k s.t.
 (i) integral
 (ii) $X \rightarrow \text{Spec } k$ finite type (4)
 (iii) $X \rightarrow \text{Spec } k$ separated (8)
 \leftarrow algebraically closed field \leftarrow means we're given a morph $X \rightarrow \text{Spec } k \Rightarrow \mathcal{O}_X(U)$ is k -algebra and restrictions are k -algebra homs. By 2.3 same as giving a hom $k \rightarrow \Gamma(X, \mathcal{O}_X)$ i.e. a k -algebra structure on $\Gamma(X, \mathcal{O}_X)$

(i) $\Leftrightarrow X$ irreducible, $\mathcal{O}_X(U)$ reduced \leftarrow Sometimes don't require irreducibility, just require reduced. But can study one irreducible component at a time.
 (ii) $\Leftrightarrow X$ quasi-compact, $\mathcal{O}_X(U)$ are f.g. k -algebras

The definition includes all quasi-projective varieties from classical algebraic geom. but \exists more: Nagata (1956) \exists variety can't embed into any \mathbb{P}_k^n (Rmk finite union of quasi-compacts is quasi-compact)

You get varieties by gluing together finitely many affine varieties along common open sets (the separated assumption prevents pathologies, see (8))

A variety is complete if $X \rightarrow \text{Spec } k$ proper (10), so extra condition: (iv) universally closed (9)

Motivation Over \mathbb{C} for "holomorphic spaces" you ask whether a holomorphic map $D^* \rightarrow X$ on the punctured disc, meromorphic at 0, can be extended to a holomorphic map $D \rightarrow X$ i.e. there are no "missing points in X ". (Made rigorous by "valuative criterion for properness")

Hwk 3: \blacksquare integral closed subsch. of variety is variety \leftarrow exclude e.g. irred. closed subsch. $\text{Spec}(k[x]/(x^2)) \subseteq \mathbb{A}_k^1$
 \square irreducible open subsch. of variety is variety

Examples Complete Varieties: \mathbb{P}_k^n , projective varieties ($\blacksquare \subseteq \mathbb{P}_k^n$), Nagata's 1956 example
 Varieties: \mathbb{A}_k^n , affine varieties ($\blacksquare \subseteq \mathbb{A}_k^n$), quasi-projective varieties ($\square \subseteq$ proj. variety)

\leftarrow not complete (except point, \emptyset) \leftarrow (uses that k is alg. closed)
Rmk A point $x \in X$ of a variety is closed $\Leftrightarrow K(x) \cong k$. E.g. $\mathbb{A}_k^1 = \text{Spec } k[x], K((x-a)) \cong k, K((0)) = k(x)$

5.6 Scheme structure on subsets ← Motivation: classically, a projective variety is a closed subset of \mathbb{P}^n_k . A quasi-proj. var. is an open \subseteq Proj. var., so \Leftrightarrow locally closed subset of \mathbb{P}^n_k .

Claim Any closed subset $C \subseteq X$ of a scheme $\Rightarrow \exists!$ closed reduced subscheme $(C, \mathcal{O}_C) \rightarrow X$

Pf $\mathcal{J}(U) := \{s \in \mathcal{O}_X(U) : s(p) = 0 \in K(p) \forall p \in C \cap U\}$ is sheaf of ideals

Locally: $U = \text{Spec } R, C \cap U = \mathbb{V}(I)$ for unique radical ideal I
 then $s(p) = 0 \in K(p) = (R/p)_p \forall p \in \mathbb{V}(I) \Leftrightarrow s \in \bigcap_{p \in \mathbb{V}(I)} P = \sqrt{I} = I \Rightarrow \mathcal{J}(\text{Spec } R) = I$

Same trick shows $\mathcal{J}(D_f) = I_f$, so \mathcal{J} is the quasi-coherent ideal sheaf corresponding to I
 Note: $C = \text{supp}(\mathcal{O}_X/\mathcal{J})$ and $C \cap U = \text{Spec } R/I$, and we define $\mathcal{O}_C = \mathcal{O}_X/\mathcal{J}$. \square

Def call this the induced reduced scheme structure on C . (so sheafify $C \cap U \mapsto \mathcal{O}_X(U)/\mathcal{J}(U)$)

Example When we consider an irreducible component $Z \subseteq X$, we use this scheme structure

Exercise For $C = X \subseteq X$ get the reduced scheme X_{red} (see ⑤ in Sec. 3.6)

Def $Z \subseteq X$ locally closed means $\forall z \in Z, \exists$ open $z \in U$ s.t. $Z \cap U$ is closed in U . (i.e. \exists closed $C \subseteq X$ with $Z \cap U = C \cap U$)

Lemma Z locally closed $\Leftrightarrow Z$ open in \bar{Z} (i.e. $Z = \bar{Z} \cap U$ some open $U \subseteq X$) \leftarrow by Lemma, $C = \bar{Z}$ works

Pf \Leftarrow : $Z = \bar{Z} \cap U$ for open $U \subseteq X \Rightarrow Z \cap U = Z = \bar{Z} \cap U$
 \Rightarrow : $Z \cap U$ closed in U so equals its closure in U which is: $\text{Cl}_U(Z \cap U) = \bar{Z} \cap U$.

$\Rightarrow z \in Z \cap U = \bar{Z} \cap U \subseteq Z$ so Z contains an open neighbourhood of z in \bar{Z} . \square

Rmk $\bar{Z} \subseteq X$ closed, so $\exists!$ induced reduced scheme structure $\mathcal{O}_{\bar{Z}}$ on \bar{Z}
 $Z \subseteq \bar{Z}$ is open so get " " " $\mathcal{O}_Z = \mathcal{O}_{\bar{Z}}|_Z$ (so $\mathcal{O}_Z(V) = \mathcal{O}_{\bar{Z}}(V)$)

The local description is the same as above: $Z \cap U = \bar{Z} \cap U = \text{Spec}(R/I), \mathcal{O}_Z|_U \cong \mathcal{O}_{\text{Spec}(R/I)}$

Rmk If Z irreducible ($\Rightarrow \bar{Z}$ irreducible) then $I = p \in \text{Spec } R$ where p is a generic point for both Z, \bar{Z}

Hwk 3 Z irred. locally closed \subseteq variety $(X, \mathcal{O}_X) \Rightarrow (Z, \mathcal{O}_Z)$ variety

Hwk 3 (X, \mathcal{O}_X) variety, $Z \subseteq X$ irreducible subspace \leftarrow (the irreducibility is not so important if allow varieties to be reducible)

Define sheaf \mathcal{O}_Z on Z : for open $V \subseteq Z$,
 $\mathcal{O}_Z(V) = \{s : V \rightarrow \bigsqcup_{x \in V} K(x) : \forall x \in V \exists$ open $x \in U \subseteq X, t \in \Gamma(U, \mathcal{O}_X)\}$
 such that $s(x) = t(x) \in K(x), \forall x \in V \cap U$

Prove that: (Z, \mathcal{O}_Z) variety $\Rightarrow Z$ locally closed and \mathcal{O}_Z is the induced reduced scheme structure

(universal property for the above sheaf)

Lemma With that definition, if Y reduced scheme, $f: Y \rightarrow X$ morph of sch. if $f(Y) \subseteq Z$ (as topological spaces) then f factorizes $f: Y \rightarrow Z \rightarrow X$

Pf Need check sheaves: $s \in \mathcal{O}_Z(U \cap Z)$ for $U \subseteq X$ open then \exists open cover $U \cap Z = \cup U_i \cap Z$ and $s_i \in \mathcal{O}_X(U_i), s_i(x) = s(x) \in K(x) \forall x \in U_i \cap Z$
 $\Rightarrow f^*(s_i) \in \mathcal{O}_Y(f^{-1}U_i), f^*(s_i)(y) = f^*(s_j)(y) \in K(y), \forall y \in f^{-1}(U_i \cap U_j)$ \leftarrow (since both are equal to $f^*(s)$ where by 1.10: $f^*_y: K(f_y) \rightarrow K(y)$)
 \Rightarrow by Sec. 3.3 since Y reduced: $f^*(s_i)_y = f^*(s_j)_y \in \mathcal{O}_{Y,y} \forall y \in f^{-1}(U_i \cap U_j)$
 $\Rightarrow f^*(s_i)$ glue to a unique section $r \in \mathcal{O}_Y(f^{-1}U)$. Define $\mathcal{O}_Z(U) \rightarrow \mathcal{O}_Y(f^{-1}U), s \mapsto r$
 and note $\mathcal{O}_X(U_i) \rightarrow \mathcal{O}_Z(U_i \cap Z) \rightarrow \mathcal{O}_Y(f^{-1}U_i), s_i \mapsto s|_{U_i \cap Z} \mapsto r|_{f^{-1}U_i}$. \square

Rmk Applying the Lemma to the case $Y =$ locally closed $Z \subseteq X$ with induced reduced sheaf, implies $\mathcal{O}_Y \cong \mathcal{O}_Z$.

Non-examinable Pf

Z has unique generic point p (see 3.4) so $Z \subseteq \bar{p} \subseteq \bar{Z}$ so $\bar{p} = \bar{Z} = \mathbb{V}(p)$

Idea: We ensure functions on Z are locally restrictions of local functions of X , in classical sense of k -valued functions, rather than germs (recall $K(x) \cong k$ if x is closed point, k alg. closed)

6. SHEAVES OF MODULES

6.1 \mathcal{O}_X -modules

Def \mathcal{O}_X -module is : • sheaf $F \in \text{Ab}(X)$
 (or sheaf of/in \mathcal{O}_X -mods) • $F(U)$ is an $\mathcal{O}_X(U)$ -module
 • restrictions are compatible with module structure

(X, \mathcal{O}_X) ringed space
 (often abbreviate)
 $\mathcal{O}_U := \mathcal{O}_X|_U$

EXAMPLE:
 $F = \bigoplus_{i \in I} \mathcal{O}_X$
 free \mathcal{O}_X -mod

Morphism $F \rightarrow G$ of \mathcal{O}_X -module is : • morph $F \xrightarrow{\varphi} G$ of sheaves

(if monomorph, i.e. φ_U injective, F is \mathcal{O}_X -submod of G) • $F(U) \xrightarrow{\varphi_U} G(U)$ is hom of $\mathcal{O}_X(U)$ -mods

Rmk stalk F_x is $\mathcal{O}_{X,x}$ -mod, and for morphs $F \rightarrow G$ get $F_x \rightarrow G_x$ is $\mathcal{O}_{X,x}$ -mod hom.

Example A sheaf of ideals is an \mathcal{O}_X -submod of \mathcal{O}_X ← (just like R -submods of R are ideals)

Fact $\mathcal{O}_X\text{-Mods} = (\text{category of } \mathcal{O}_X\text{-mods on } X)$ is an abelian cat ← (proof similar to $\text{Ab}(X)$ abelian)

Indeed notions of submod, quotient mod, ker, coker, Im agree with what get in $\text{Ab}(X)$
 e.g. $F \rightarrow G \rightarrow H$ exact \iff exact in $\text{Ab}(X)$ \iff exact on stalks

Will write $\text{Hom}_{\mathcal{O}_X}$ for morphisms in this category.

6.2 Modules generated by sections

$\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F) \xrightarrow{\cong} F(X) \quad \forall F \in \mathcal{O}_X\text{-Mods}$ ← analogue of $\text{Hom}_R(R, M) \cong M$
 $(\varphi: \mathcal{O}_X \rightarrow F) \longleftrightarrow s = \varphi(1)$ since $\varphi_U(r) = \varphi_U(r \cdot 1) = r \cdot s|_U \quad \forall r \in \mathcal{O}_X(U)$

Similarly $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, F) \xrightarrow{\cong} F(X)^{\oplus n}$ defined by n global sections $s_1, \dots, s_n \in F(X)$

Def F is generated by global sections if

\exists surjection $\bigoplus_{i \in I} \mathcal{O}_X \rightarrow F$ of \mathcal{O}_X -mods (\iff $s_i|_x$ generate $\mathcal{O}_{X,x}$ -mod $F_x \quad \forall x \in X$)
 same as picking sections $s_i \in F(X)$ (as \mathcal{O}_U -module, $\bigoplus \mathcal{O}_U \rightarrow F|_U$)

Def F is locally generated by sections if $\forall x \in X \exists$ open $x \in U$ s.t. $F|_U$ generated by global sections

Rmk Can produce \mathcal{O}_X -submods from given local sections $s_i \in F(U_i)$ ← sheafify $U \mapsto$ possible $\mathcal{O}_X(U)$ -linear combos of $\{s_i|_U : U \subseteq U_i\}$

Def A sheaf has finite type if locally generated by finitely many sections.

6.3 Vector bundles and coherent modules

Def \mathcal{O}_X -mod F is locally free \mathcal{O}_X -mod of finite rank ("or" vector bundle) if

$\forall x \in X \exists$ open $x \in U : F|_U \cong \mathcal{O}_U^{\oplus n}$ ← (rank n can depend on U unless we say "of rank n ")
 as \mathcal{O}_U -mods

so $\mathcal{O}_U^{\oplus n} \rightarrow F|_U$
 some open $x \in U$
 some $n \in \mathbb{N}$
 (not fixed)

i.e. locally generated by finite # of "independent sections"

Def X invertible sheaf ("or" line bundle) if $n=1$ (fixed) ← locally $\mathcal{O}_U \xrightarrow{\cong} \mathcal{O}_U \cdot s = F(U)$ generated by one section $s \in F(U)$

Question Is it enough to ask $F_x \cong \mathcal{O}_{X,x}^{\oplus n} \quad \forall x$ some $n \in \mathbb{N}$ depending on x ? (\implies : clear, \impliedby : can fail)
 ← (X, \mathcal{O}_X) locally ringed space (as $\mathcal{O}_{X,x}$ -mods)

Lemma F finite type, $\mathcal{O}_{X,x}^{\oplus n} \xrightarrow{\varphi_x} F_x$ surj $\implies \exists x \in U \subseteq X$ with surj $\mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} F|_U, \varphi|_x = \varphi_x$.

Pf finite type $\implies \exists$ surj $\mathcal{O}_U^{\oplus m} \xrightarrow{\psi} F|_U$. Let $s_i = \psi(e_i) \in F_x$ ($e_i = 1$ in i th copy of $\mathcal{O}_{X,x}$) so $F_x = \sum \mathcal{O}_{X,x} \cdot s_i$. Now $s_i \in F(U_i)$ some $x \in U_i$. Replace U by $U \cup U_1 \cup \dots \cup U_n$ so wlog $s_i \in F(U)$. Let $f_j = 1 \in (j$ -th copy of $\mathcal{O}_U) \implies \psi(f_j)|_x = \sum r_j \cdot s_i$ some $r_j \in \mathcal{O}_{X,x}$. So $\psi(f_j)|_{V_j} \in \sum \mathcal{O}_{V_j} \cdot s_i|_{V_j}$ some $V_j \subseteq U$, again wlog $V_j = U$ (replace U by $U \cup V_1 \cup \dots \cup V_m$) $\implies \psi(f_j) \in \text{Im } \psi$ for $\psi: \mathcal{O}_U^{\oplus n} \rightarrow F|_U$ with $\psi(e_i) = s_i$ on U . So ψ hits \mathcal{O}_U -mod generators $\psi(f_j)|_U$.

Continuing above Question: We know φ_x is inj at x , but we don't know if the same φ works also for y close to x , so we do not know whether φ_y inj (recall φ inj $\iff \varphi_y$ inj at all stalks at $y \in U$).

Lemma In previous Lemma, if $\text{Ker } \varphi$ finite type, $\varphi_x \text{ iso} \Rightarrow \varphi : \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U \text{ iso}$, some U .

Pf Shrinking U , $\exists \text{ surj } \mathcal{O}_U^{\oplus m} \xrightarrow{\psi} \text{Ker } \varphi$, hence $\mathcal{O}_U^{\oplus m} \xrightarrow{\psi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \rightarrow 0$ exact. Apply Lemma to $\text{Ker } \varphi$: using $(\text{Ker } \varphi)_x = 0$ deduce $(\text{Ker } \varphi)|_U = 0$ possibly after shrinking U further. So φ is also injective. \square

This motivates the definition:

Def $F \in \mathcal{O}_X\text{-Mods}$ is coherent if $\left\{ \begin{array}{l} F \text{ finite type} \\ \text{Ker } (\mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U) \text{ finite type} \end{array} \right.$ $\forall \mathcal{O}_U\text{-mod homs}$
 $\forall \text{ open } U, \forall n \in \mathbb{N}$

Rmk $F \in \text{Coh}(X) \Rightarrow F$ locally finitely presented

Pf F finite type $\Rightarrow \exists \text{ surj } \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U$, then consider Ker . \square

$\text{Vect}(X) = \{\text{vector bundles on } X\} \subseteq \mathcal{O}_X\text{-Mods}$, but not an abelian cat (Ker, Coker need not be free)

$\text{Coh}(X) = \{\text{coherent } \mathcal{O}_X\text{-mods}\} \leftarrow$ Fact abelian category! (explains partly its importance)

Claim $F \in \text{Coh}(X)$ and $F_x \cong \mathcal{O}_{X,x}^{\oplus n} \forall x \Rightarrow F \in \text{Vect}(X)$ ($\forall x \in X$, some $n \in \mathbb{N}$ depending on x unless we fix the rank)

Claim follows by Lemmas. Converse of Claim?

Cor X locally Noetherian scheme $\Rightarrow \text{Vect}(X) = \{F \in \text{Coh } X : \forall x, F_{x,x} \cong \mathcal{O}_{x,x}^{\oplus n} \text{ some } n\} \subseteq \text{Coh}(X)$

Pf $F \in \text{Vect}(X) \Rightarrow F$ finite type, in general

$\text{Ker} (\mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U)$ (need show finite type for small U) shrinking U WLOG U affine = $\text{Spec } R$ \leftarrow Noetherian

In sections below we will prove that because $\mathcal{O}_U^{\oplus n}, \mathcal{F}|_U$ are "quasi-coherent" the problem reduces to taking global sections: $\text{Ker} (R^n \xrightarrow{\varphi} F(U))$ and this is finitely generated since R Noeth (so get exact sequence $R^m \rightarrow R^n \xrightarrow{\varphi} F(U) \rightarrow 0$ and this will imply $\mathcal{O}_U^{\oplus m} \rightarrow \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F} \rightarrow 0$ exact). \square

6.4 \mathcal{O}_X -module \tilde{M} on $X = \text{Spec } R$, for R -mod M

sheaf \tilde{M} on $X = \text{Spec } R$ by Sec. 1.12 method:

- $\tilde{M}(D_f) = M_f$ (so $\tilde{M}(X) = \tilde{M}(D_1) = M$)
- $D_g \subseteq D_f \Rightarrow M_f \rightarrow M_g$ induced by $R_f \rightarrow R_g$
- stalk $\tilde{M}_p = \varinjlim_{D_f \ni p} \tilde{M}(D_f) = \varinjlim_{D_f \ni p} M_f \cong M_p$
- $\tilde{M}(U) = \{s : U \rightarrow \coprod_{p \in \text{Spec } R} M_p : s(p) \in M_p \text{ which are locally compatible:}$

$M_f = \text{localisation of } M \text{ at } f \cong M \otimes_R R_f$
 $M_p = S^{-1}M \text{ localisation of } M \text{ at } S = R \setminus p \cong M \otimes_R R_p$

$\left(\varinjlim M \otimes_R R_f \cong M \otimes \varinjlim R_f \cong M \otimes R_p \right)$

$\forall p \in U, \exists \text{ open nbhd } p \in D_f \subseteq U$ with $s(x) = t_x$
 $\exists t \in \tilde{M}(D_f) \cong M_f$ some $f \in R$ $\forall x \in D_f \cong \tilde{M}_x \cong M_x / f$

Rmk could assume $t = \frac{m}{f}$ since can replace D_f with D_{fm} ($= D_f$).

could just ask $s(x) = t_x$ on a smaller open $p \in V \subseteq D_f$.

$\tilde{M} = \text{sheafification of } U \mapsto M \otimes_R \mathcal{O}_X(U)$

Call \tilde{M} the sheaf associated to M

UPSHOT \tilde{M} is \mathcal{O}_X -module on $X = \text{Spec } R$

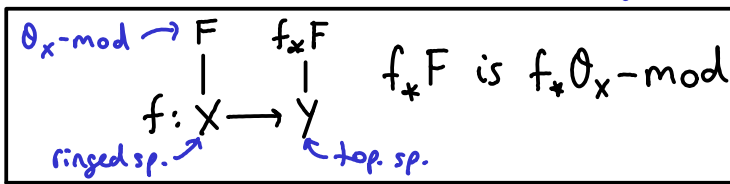
$\varphi : M \rightarrow N$ R -mod hom $\Rightarrow \tilde{M} \rightarrow \tilde{N}$ \mathcal{O}_X -mod morph by gluing $\tilde{M}(D_f) \rightarrow \tilde{N}(D_f)$
 (just need check stalks, then use sec. 3.0) \leftarrow for converse take global sections

\Rightarrow fully faithful exact functor $R\text{-Mods} \rightarrow \mathcal{O}_{\text{Spec}(R)}\text{-Mods}$

EXAMPLES. $\tilde{R} = \mathcal{O}_X$ ($X = \text{Spec } R$)
 $\bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} \tilde{M}_i$, so $\bigoplus_{i \in I} \tilde{R} \cong \bigoplus_{i \in I} \mathcal{O}_X$

$M \otimes R_f \xrightarrow{\varphi \otimes \text{id}} N \otimes R_f$
 $\begin{array}{ccc} \tilde{M}(D_f) & \rightarrow & \tilde{N}(D_f) \\ \parallel f & & \parallel f \\ M_f & \rightarrow & N_f \\ \parallel & & \parallel \\ M \otimes R_f & \xrightarrow{\varphi \otimes \text{id}} & N \otimes R_f \end{array}$

6.5 Direct image and inverse image

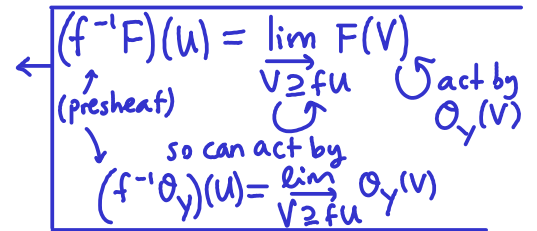
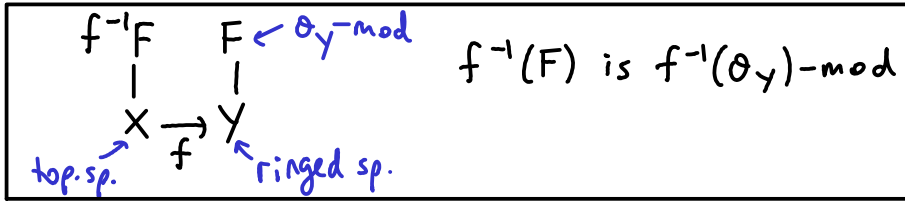


$(f_*F)(U) = F(f^{-1}(U))$ is $\mathcal{O}_X(f^{-1}(U))$ -mod $\cong f_*\mathcal{O}_X(U)$
 Example $\alpha: \text{Spec } S \rightarrow \text{Spec } R, \varphi = \alpha^\#: R \rightarrow S$
 N S -mod $\Rightarrow \alpha_*\tilde{N} = \tilde{R}N$ $\leftarrow R^N = N$ viewed as R -mod via φ
 Pf $(\alpha_*\tilde{N})(D_f) = \tilde{N}(D_{\varphi f}) = N_{\varphi f} = (R^N)_f$ compatible with restrictions \square

Algebra: Recall $R \xrightarrow{\varphi} S$ hom of rings, then S is R -mod via $r \cdot s = \varphi(r)s$.

$f: X \rightarrow Y$ morph of ringed spaces, then: \leftarrow (recall $\text{Mor}(\mathcal{O}_Y, f_*\mathcal{O}_X) \cong \text{Mor}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$)

$f^{-1}\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(U)$ makes \mathcal{O}_X an $f^{-1}\mathcal{O}_Y$ -mod on ringed space $(X, f^{-1}\mathcal{O}_Y)$



6.6 Operations on \mathcal{O}_X -mods

$\text{Hom}_{\mathcal{O}_X}(F, G): U \mapsto \text{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$ is a sheaf of \mathcal{O}_X -mods. \leftarrow left exact in F and in G

coproduct in \mathcal{O}_X -Mod: F_i \mathcal{O}_X -mods, $\bigoplus F_i = \text{sheafify}(U \rightarrow \bigoplus F_i(U))$

(Need sheafify: could get ∞ sums when globalise, e.g. $X = \mathbb{N}, F_i = \begin{cases} \mathbb{Z} & \text{on } \{i\} \\ 0 & \text{else} \end{cases}, s_n = (1, \dots, 1, 0, \dots)$ at $\{n\}$, try globalise)

Fact \exists canonical iso $\text{Mor}(\bigoplus F_i, G) \cong \prod \text{Mor}_{\mathcal{O}_X}(F_i, G)$ natural in F_i, G . \leftarrow right exact in F, G

product in \mathcal{O}_X -Mod: $F \otimes_{\mathcal{O}_X} G = \text{sheafify}(U \rightarrow F(U) \otimes_{\mathcal{O}_X(U)} G(U))$

Fact $\exists!$ \mathcal{O}_X -mod structure s.t. $F(U) \otimes_{\mathcal{O}_X(U)} G(U) \rightarrow (F \otimes_{\mathcal{O}_X} G)(U)$ hom of $\mathcal{O}_X(U)$ -mods

Universal property: $\text{Hom}_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G, H) = \text{Bilinear}_{\mathcal{O}_X}(F \times G, H)$

Rmk Stalks are $\text{Hom}_{\mathcal{O}_{X,x}}(F_x, G_x), \bigoplus (F_i)_x, F_x \otimes_{\mathcal{O}_{X,x}} G_x$.

for this require M finitely presented: \exists exact $\bigoplus_{\text{finite}} R \rightarrow \bigoplus_{\text{finite}} R \rightarrow M \rightarrow 0$

Examples on $X = \text{Spec } R: \widetilde{\bigoplus M_i} \cong \bigoplus \widetilde{M_i}, \widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}, \widetilde{\text{Hom}_R(M, N)} \cong \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$

Algebra $\text{Hom}_R(M \otimes_R N, P) \cong \text{Hom}_R(M, \text{Hom}_R(N, P))$ canonically, for R -mods M, N, P (so \otimes & Hom are adjoint)

Fact $\text{Hom}_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G, H) \cong \text{Hom}_{\mathcal{O}_X}(F, \text{Hom}_{\mathcal{O}_X}(G, H))$ canonically & functorial in F, G, H .

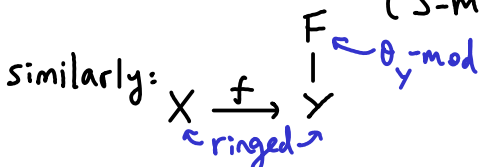
Cor $F \otimes_{\mathcal{O}_X}, \text{Hom}_{\mathcal{O}_X}(G, \cdot)$ adjoint, $F \otimes_{\mathcal{O}_X}$ right exact, $\text{Hom}_{\mathcal{O}_X}(G, \cdot)$ left exact.

Fact $f: X \rightarrow Y \Rightarrow f^{-1}(F \otimes_{\mathcal{O}_Y} G) \cong f^{-1}F \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}G$ canonically (F, G \mathcal{O}_Y -mod)

6.7 Pullback

Rmk $R \rightarrow S$ rings, M R -mod, N S -mod

$\Rightarrow M \otimes_R N$ is $\begin{cases} R\text{-mod since } N \text{ } R\text{-mod via } R \rightarrow S \\ S\text{-mod by } s \cdot (m \otimes n) = m \otimes sn \end{cases}$ ($r \cdot (m \otimes n) = (rm) \otimes n = m \otimes rn$)



$\Rightarrow f^*F = f^{-1}(F) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is an $f^{-1}\mathcal{O}_Y$ -mod but also an \mathcal{O}_X -mod!

can prove this using universal property, or by hand thinking about \otimes of presheaves.

Fact $\exists!$ \mathcal{O}_X -mod structure s.t. presheaf tensor product $f^{-1}(F)(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U) \rightarrow f^*F(U)$ is $\mathcal{O}_X(U)$ -mod hom

Example $f^*\mathcal{O}_Y = \mathcal{O}_X$ (since $f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$ canonically) $\mathcal{O}_X(U)$ -mod as by Rmk.

Exercise $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow f^* \circ g^* = (g \circ f)^*$ (use last Fact in 6.6, using Sec. 1.9)
 $f^*(F \otimes_{\mathcal{O}_Y} G) = f^*F \otimes_{\mathcal{O}_X} f^*G$ canonically & functorial

Upshot $f: X \rightarrow Y$ morph of ringed spaces $\Rightarrow \text{Mod}_{\mathcal{O}_X}(X) \xrightarrow{f^*} \text{Mod}_{\mathcal{O}_Y}(Y)$ and $\xleftarrow{f^*}$

Theorem f^*, f_* are adjoint functors: $\text{Mor}_{\mathcal{O}_X}(f^*F, G) \cong \text{Mor}_{\mathcal{O}_Y}(F, f_*G)$

(exercise) hence f_* left exact, f^* right exact

HWK 3 f_* commutes with limits \varprojlim for example \prod , f^* commutes with colimits \varinjlim for example \oplus

Example $f^*(\oplus \mathcal{O}_Y) = \oplus f^*\mathcal{O}_Y = \oplus \mathcal{O}_X$.
 (product in category of \mathcal{O}_X -Mods) (coproduct in cat. of \mathcal{O}_X -Mods)

Exercise Deduce from that $f^*(\text{Vect}(Y)) \subseteq \text{Vect}(X)$.

6.8 \tilde{M} on any scheme

M R -mod, $X \xrightarrow{\text{canonical}} \text{Spec } \Gamma(X, \mathcal{O}_X) \xrightarrow{\alpha} \text{Spec } R$ then get $F_M := \alpha^* \tilde{M}$
 ASSUME given a ring hom $R \rightarrow \Gamma(X)$

Easier: $(X, \mathcal{O}_X) \xrightarrow{\pi} \text{ringed space (point, } R)$ (on sheaves $\pi_* \mathcal{O}_X = \Gamma(X) \xleftarrow{\text{GIVEN}} R$)

$F_M := \pi^* M$
 = sheafify $(U \mapsto M \otimes_R \mathcal{O}_X(U))$ (since $\pi^{-1} M \otimes_{\pi^{-1} R} \mathcal{O}_X$ and $(\pi^{-1} R)(U) = R$, $(\pi^{-1} M)(U) = M$)

(get same answer since $X \xrightarrow{\alpha} \text{Spec } R \xrightarrow{\pi_1} \text{(point, } R)$, $\tilde{M} = \pi_1^* M$ by construction, $\pi^* = \alpha^* \pi_1^*$)

Claim $f: Y \rightarrow X$ (morph of ringed spaces) $\Rightarrow f^* F_M = F_N$ where $N = M \otimes_{\Gamma(X)} \Gamma(Y)$ is $\Gamma(Y)$ -module
 M $\Gamma(X)$ -module (case $R \xrightarrow{\text{id}} \Gamma(X)$)

Pf $Y \xrightarrow{f} X$
 $\pi_Y \downarrow \quad \downarrow \pi_X$
 $(\text{point, } \Gamma(Y)) \xrightarrow{\psi} (\text{point, } \Gamma(X))$
 using $f^\#: \Gamma(X) \rightarrow \Gamma(Y)$
 $f^* \pi_X^* M = \pi_Y^* \psi^* M$
 $\psi^* M = \psi^{-1} M \otimes_{\psi^{-1} \Gamma(X)} \Gamma(Y) = M \otimes_{\Gamma(X)} \Gamma(Y)$

Cor $\alpha: \text{Spec } S \rightarrow \text{Spec } R$ M R -mod $\Rightarrow \alpha^* \tilde{M} = \widetilde{M \otimes_R S}$
 (S is R-mod via the ring hom $R \rightarrow S$)

Example $D_f = \text{Spec } R_f \hookrightarrow \text{Spec } R \Rightarrow \tilde{M}|_{D_f} = \widetilde{M \otimes_R R_f} = \tilde{M}_f$
 stronger statement than saying $\tilde{M}(D_f) = M_f$

6.9 Classification of \mathcal{O}_X -homs $\tilde{M} \rightarrow F$

Lemma $X = \text{Spec } R \Rightarrow \text{Hom}_{\mathcal{O}_X}(\tilde{M}, F) \xleftarrow{!} \text{Hom}_R(M, \Gamma(X, F)) \forall \mathcal{O}_X\text{-mod } F$
 (compare Sec. 2.3) $\varphi \mapsto \varphi_x$ "F(X)

Pf $\pi: (X, \mathcal{O}_X) \rightarrow (\text{point, } R)$ morph of ringed spaces $(\pi^\#: R \xrightarrow{\text{id}} \pi_* \mathcal{O}_X = \mathcal{O}_X(X) = R)$
 $\tilde{M} = \pi^* M$, $\Gamma(X, F) = \pi_* F$
 $\Rightarrow \text{Hom}_{\mathcal{O}_X}(\tilde{M}, F) = \text{Hom}_{\mathcal{O}_X}(\pi^* M, F) \xleftarrow{\pi^*, \pi_* \text{ adjoint}} \text{Hom}_R(M, \pi_* F) = \text{Hom}_R(M, \Gamma(X, F)). \square$

Exercise Using 6.8: $\text{Hom}_{\mathcal{O}_X}(F_M, F) \xleftarrow{!} \text{Hom}_R(M, F(X))$ using $R \xrightarrow{\text{given}} \Gamma(X, \mathcal{O}_X)$ to make $F(X)$ an R -mod.

6.10 Flatness

Def F is flat \mathcal{O}_X -mod if $F \otimes_{\mathcal{O}_X} \cdot$ is exact
 so $\iff F_x$ flat $\mathcal{O}_{X,x}$ -mod $\forall x$.

← since exactness can be checked on stalks
 ← stalk is either 0 or $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{X,x}} \cdot = \text{id}$

Example $U \xrightarrow{i} X$ open subsch. $\implies i_* \mathcal{O}_U$ is flat \mathcal{O}_X -mod

Rmk Morph of schemes $f: X \rightarrow Y$ is flat $\iff \mathcal{O}_X$ flat $f^{-1} \mathcal{O}_Y$ -module \leftarrow (see $\textcircled{7}$ in Sec. 3.6)
 since recall $(f^{-1} \mathcal{O}_Y)_x = \mathcal{O}_{Y, f(x)}$

Claim $f: X \rightarrow Y$ flat $\implies f^*: \mathcal{O}_Y\text{-Mod} \rightarrow \mathcal{O}_X\text{-Mod}$ is exact (not just right exact)

Pf f^{-1} is exact $\implies \mathcal{O}_Y\text{-Mod} \xrightarrow{f^{-1}} f^{-1} \mathcal{O}_Y\text{-Mod}$ exact,
 $F \mapsto f^{-1} F$

$\otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$ exact by Rmk $\implies f^* F = f^{-1} F \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$ is composite of two exact functors \square

Facts (X, \mathcal{O}_X) ringed space
 • free \implies flat
 • Can take \oplus of flat mods

so kernels are flat

← Taking stalks, all follow from analogous statements for R -mods

Combine (break into SES's, show images $(F_n \rightarrow F_{n-1})$ flat) $\left\{ \begin{array}{l} \cdot \mathcal{O} \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow \mathcal{O} \text{ exact : outer two or last two flat } \implies \text{all flat} \\ \cdot \dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow F \rightarrow \mathcal{O} \text{ exact, all flat } \implies \text{"} \end{array} \right.$
 " , F_3 flat \implies sequence $\otimes_{\mathcal{O}_X}$ any \mathcal{O}_X -mod G is exact
 (so "flat resolution of flat \mathcal{O}_X -mod F ")

7. (QUASI-)COHERENT SHEAVES

7.1 QCoh(X)

Fact " \Leftarrow " holds also if just assume \mathcal{O}_X is coherent

Recall F coherent $\implies F$ locally finitely presented
 (Sec. 6.3) and " \Leftarrow " holds if X locally Noetherian scheme. $\left\{ \begin{array}{l} \text{now weaken this} \\ \text{condition by dropping finiteness} \end{array} \right.$

Def F quasi-coherent \iff F is locally presented, i.e. $\forall x, \exists$ open $x \in U \subseteq X$
 $\exists \bigoplus_{i \in I} \mathcal{O}_U \rightarrow \bigoplus_{j \in J} \mathcal{O}_U \rightarrow F|_U \rightarrow \mathcal{O} \text{ exact.}$
 where the maps are morphs of \mathcal{O}_U -mods \leftarrow where $\mathcal{O}_U = \mathcal{O}_X|_U$

SUMMARY: coherent \implies locally finitely presented \implies quasi-coherent (= locally presented)
 vector bundle \implies locally generated by finitely many sections \implies locally generated by sections

Lemma For $X = \text{Spec } R$: $\left(\exists \text{ exact sequence of } \mathcal{O}_X\text{-mods } \bigoplus_{i \in I} \mathcal{O}_X \rightarrow \bigoplus_{j \in J} \mathcal{O}_X \rightarrow F \rightarrow \mathcal{O} \right) \iff (F \cong \tilde{M} \text{ some } R\text{-module } M)$

Pf (\implies) Let $M = \bigoplus_{j \in J} R / \text{Im}(\bigoplus_{i \in I} R \rightarrow \bigoplus_{j \in J} R)$ (taking global sections)

by exact functor from 6.4: $\begin{array}{ccccccc} \bigoplus_{i \in I} \mathcal{O}_X & \rightarrow & \bigoplus_{j \in J} \mathcal{O}_X & \rightarrow & F & \rightarrow & \mathcal{O} \\ \parallel & & \parallel & & \parallel & & \\ \bigoplus_{i \in I} \tilde{R} & \rightarrow & \bigoplus_{j \in J} \tilde{R} & \rightarrow & \tilde{M} & \rightarrow & \mathcal{O} \end{array}$ exact $\left\{ \begin{array}{l} \text{by uniqueness of} \\ \text{cokernels up to iso:} \\ F \cong \tilde{M} \end{array} \right.$

(\impliedby) $F = \tilde{M}$: pick $J =$ set of generators m_j for R -mod M (e.g. $J = M$)
 pick $I =$ " " " " k_i " " $\text{Ker}(\bigoplus_{j \in J} R \rightarrow M)$
 apply \sim to $\bigoplus_{i \in I} R \rightarrow \bigoplus_{j \in J} R \rightarrow M \rightarrow \mathcal{O}$. \square
 send 1 in i -th copy of R to k_i \leftarrow send 1 in j -th copy of R to m_j

Cor
 \forall scheme X
 $FEQCoh(X) \iff \forall x \in X \exists$ affine open $x \in U \cong \text{Spec } R, F|_U \cong \tilde{M}$ some R -mod M
 $FCoh(X) \iff$ in addition require M is coherent R -mod

(Pf $\forall x$ pick U so that Lemma applies.)

(WLOG $M = F(U)$, $R = \mathcal{O}_x(U)$ as $F|_U(U) = \tilde{M}(U) = M$)

Idea: want \forall f.g. submodule of M to have finite presentation, indeed get exact sequence $R^m \rightarrow R^n \xrightarrow{\varphi} \text{Im } \varphi \rightarrow 0$ map to gens. of $\ker \varphi$

Rmk If R Noeth., coherent = f.g. (since R^n f.g., so its submods are f.g. as R Noeth.)

Example X loc. Noeth. scheme $\implies \mathcal{O}_x$ is coherent \implies ideal sheaf of any closed subsch. is coherent.

Rmk \forall scheme: $FEQCoh(X) \iff \exists$ affine open cover $X = \cup U_i$ s.t. $F|_{U_i} \cong \tilde{M}_i$ for R_i -mods M_i ; $U_i = \text{Spec } R_i$
 $FCoh(X) \iff$ " and M_i coherent. (WLOG: $R_i = \mathcal{O}_x(U_i), M_i = F(U_i)$)

(immediate from Cor)

Rmk restriction to open $V \subseteq X$: $QCoh(X) \rightarrow QCoh(V), Coh(X) \rightarrow Coh(V)$

Pf $x \in V \cap U = \cup D_{f_i}$ for $f_i \in R$ then $F|_U|_{D_{f_i}} \cong \tilde{M}|_{D_{f_i}} \cong \tilde{M}_{f_i}$ (and use fact that localization preserves "coherent" property) Example in 6.8

so again locally module. \square

Why is quasi-coherence a good notion?

Rings^{op} \rightarrow Aff, $R \mapsto (\text{Spec}(R), \mathcal{O}_{\text{Spec } R})$ equivalence of cats
 R -Mods $\rightarrow \mathcal{O}_{\text{Spec}(R)}$ -Mods, $M \mapsto \tilde{M}$ not equivalence of cats (notice $FEQ_{\mathcal{O}_x}$ -Mods)

Example $X = \text{Spec } k[x] = \mathbb{A}_k^1$, skyscraper sheaf at 0: $F(U) = \begin{cases} k[x] & \text{if } 0 \in U \\ 0 & \text{else} \end{cases}$
 \implies if the above were an equivalence of cats, then $F \cong \tilde{M}$ some $k[x]$ -mod M
 so $k[x] = F(X) \cong \tilde{M}(X) = M$. But $\widehat{k[x]} = \mathcal{O}_x$ is not isomorphic to F !

Solution restrict which \mathcal{O}_x -mods you allow: want them locally to look like \tilde{M} , just like when we studied sheaves of ideals that locally look like \tilde{I}

Will show later: For $X = \text{Spec } R$: R -Mods $\rightarrow QCoh(X)$ equivalence of categories $M \mapsto \tilde{M}$, $F(X) \leftarrow F$

7.2 Overview of general properties of $QCoh(X)$ and $Coh(X)$ for X scheme

1) $Coh(X)$ abelian category, and $Coh(X) \xrightarrow{\text{incl}} \mathcal{O}_X$ -Mod (for $Coh(X)$ properties enough if X ringed)
 $QCoh(X) \xrightarrow{\text{incl}} QCoh(X)$ are exact functors

In particular can take Ker, Coker, Image in both (not in $\text{Vect}(X)$) Easy for $QCoh$ since locally hom of mods $M_1 \rightarrow M_2$ so take \sim of Ker, Coker, Im

2) $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ exact in \mathcal{O}_X -Mods. Two of the $F_i \in QCoh(X) \implies$ all three are. Same holds for $Coh(X)$ (not for $\text{Vect}(X)$)
Trick $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3$ exact, and F_2, F_3 are, then F_1 is. (Pf. $F_1 \cong \text{Ker}(F_2 \rightarrow F_3)$, use (1). \square)

3) Can take finite $\oplus, \cdot \otimes_{\mathcal{O}_x}, \text{Hom}_{\mathcal{O}_x}(\cdot, \cdot)$ in $QCoh(X), Coh(X)$ and $\text{Vect}(X)$ (for $QCoh, \text{Hom}_{\mathcal{O}_x}(F, G)$ need assume F loc. finitely presented)

4) **Gabriel-Rosenberg thm**
 X quasi-compact & separated (e.g. variety) $\implies X$ is determined up to iso by $QCoh X$!

5) X loc. Noeth. scheme, $Z \hookrightarrow X$ closed subsch. $\implies 0 \rightarrow \mathcal{I}_{Z/X} \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Z \rightarrow 0$ exact in $Coh X$
 finite type subsheaf $F \subseteq G, G \in Coh(X) \implies F \in Coh(X)$ (combine to prove kernels exist in $Coh X$)

Hwk 4
 $\varphi: F \rightarrow G, G \in Coh X, F$ finite type $\implies \text{Ker } \varphi$ finite type
 $\varphi: F \rightarrow G, G \in Coh X, F$ finite type, $\varphi_x: F_x \rightarrow G_x$ injective $\implies \varphi|_U: F|_U \rightarrow G|_U$ inj. some $x \in U$

Hwk 4: Picard group $\text{Pic}(X) = \{\text{isomorphism classes of invertible sheaves}\}$ (we proved it in case $F=0$ in Pf. claim in sec. 6.3)
 group operation is $\cdot \otimes_{\mathcal{O}_x}$ (abelian group as $F \otimes_{\mathcal{O}_x} G \cong G \otimes_{\mathcal{O}_x} F$)

7.3 Pullback preserves quasi-coherence

$f: X \rightarrow Y$ morph ringed spaces

Claim $f^*: \text{QCoh}(Y) \rightarrow \text{QCoh}(X)$. If X loc. Noeth. scheme $\Rightarrow f^*: \text{Coh} Y \rightarrow \text{Coh} X$.

Pf If $\bigoplus_I \mathcal{O}_Y|_U \rightarrow \bigoplus_J \mathcal{O}_Y|_U \rightarrow G|_U \rightarrow 0$ exact ($f_x \in U \subseteq Y$ open)

without this can fail e.g. $f^* \mathcal{O}_Y = \mathcal{O}_X$
so if \mathcal{O}_Y coh, \mathcal{O}_X not coh, then fails

apply g^* where $g = f|_{f^{-1}U}: f^{-1}U \rightarrow U$, using g^* right exact & commutes with \bigoplus .

$\bigoplus_I \mathcal{O}_X|_{f^{-1}U} \rightarrow \bigoplus_J \mathcal{O}_X|_{f^{-1}U} \rightarrow f^* G|_{f^{-1}U} \rightarrow 0$ exact, and $x \in f^{-1}U$ open. using X loc. Noeth.

$F \in \text{Coh}(Y) \Rightarrow F$ locally finitely presented $\Rightarrow f^* F$ loc. finitely presented $\Rightarrow f^* F \in \text{Coh}(X)$ (above proof for I, J finite)

7.4 Push-forwards for X Noetherian

Claim $f: X \rightarrow Y$ morph of schemes, X Noetherian $\Rightarrow f_*: \text{QCoh} X \rightarrow \text{QCoh} Y$

Pf $0 \rightarrow F \rightarrow \prod F|_{U_i} \rightarrow \prod F|_{U_i \cap U_j} \rightarrow \dots$ exact by sheaf property, where $X = \cup U_i$ affine open cover

Recall f_* left-exact & commutes with limits e.g. with $\prod \Rightarrow 0 \rightarrow f_* F \rightarrow \prod f_*(F|_{U_i}) \rightarrow \prod f_*(F|_{U_i \cap U_j})$ exact

WLOG Y open affine = $\text{Spec} R$ (replace X by $f^{-1}(\text{Spec} R)$), WLOG $F|_{U_i} = \widetilde{F}(U_i)$, so $f_*(F|_{U_i}) = \widetilde{F}(U_i)_R$

similarly for U_{ijk} . If show $\prod f_*(F|_{U_i}), \prod f_*(F|_{U_{ijk}}) \in \text{QCoh}(Y)$ then $f_* F \in \text{QCoh}(Y)$ Trick (2) in 7.2

X Noeth $\Rightarrow U_{ij}$ quasi-compact \Rightarrow finite covers $\Rightarrow \prod$ is \bigoplus , but \sim commutes with \bigoplus so finally done! \square

Rmk X quasi-compact, separated $\Rightarrow f_*: \text{QCoh} X \rightarrow \text{QCoh} Y$ proof above but easier $U_{ijk} = U_i \cap U_j$ affine!

Non-examinable fact f proper, X, Y loc. Noeth. $\Rightarrow f_*: \text{Coh} X \rightarrow \text{Coh} Y$ otherwise in general f_* can ruin (quasi)-coherence

7.5 Gluing modules

Similar to Sec. 4.1: R ring $\ni f_1, \dots, f_n$ s.t. $1 \in \langle \text{all } f_i \rangle$

data: $M_i: R_{f_i}$ -mod (so have \widetilde{M}_i on $D_{f_i} \subseteq \text{Spec} R$)
 $\psi_{ij}: (M_i)_{f_j} \rightarrow (M_j)_{f_i}$ iso of $R_{f_i f_j}$ -mods cocycle condition
 $\psi_{ii} = \text{id}$ (so $\widetilde{M}_i \cong \widetilde{M}_j$ on $D_{f_i f_j} \subseteq \text{Spec} R$)

Define $M := \text{Ker} \left(\begin{array}{c} \bigoplus_i M_i \xrightarrow{\varphi} \bigoplus_{i,j} (M_i)_{f_j} \\ (m_i) \longmapsto \left(\frac{m_i}{1} - \psi_{ji} \left(\frac{m_j}{1} \right) \right) \end{array} \right)$

Idea: local data which agrees on overlaps

Call $\pi_i: M \rightarrow M_i$ the projections.

Gluing Lemma π_i induces isos $M_{f_i} \rightarrow M_i$ and $\psi_{ij} \circ \frac{\pi_i(m)}{1} = \frac{\pi_j(m)}{1} \forall m \in M$

Pf Enough to show π_ℓ iso after localising at every prime $q \in \text{Spec} R_{f_\ell}$

$\Rightarrow q = p R_{f_\ell}$ with $f_\ell \notin p \in \text{Spec} R$. By exactness of localisation

$$(M_{f_\ell})_q = M_p = \text{Ker} \left(\bigoplus (M_i)_p \xrightarrow{\varphi_p} \bigoplus ((M_i)_p)_{f_j} \right)$$

$f_\ell \in R_p$ is unit so WLOG replace: $R \rightarrow R_p, M \rightarrow M_p, M_i \rightarrow (M_i)_p, f_\ell \rightarrow 1$. "WLOG" in sense that localising at f_ℓ is like localising at 1 since f_ℓ is a unit in R_p

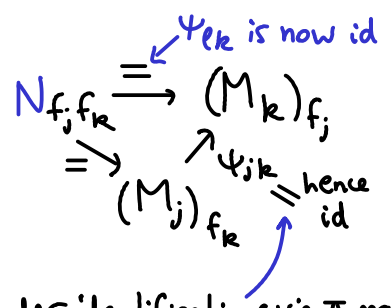
Abbreviate $N = M_p$ so: $\pi_\ell: M = \text{Ker} \varphi_p \subseteq (N \oplus \bigoplus_{i \neq \ell} M_i) \rightarrow N$

$$\psi_{\ell i}: N_{f_i} \xrightarrow{\cong} (M_i)_1 = M_i$$

WLOG $M_i = N_{f_i}$ (identify via ψ_{e_i}), so cocycle cond. becomes:

$$\Rightarrow 0 \rightarrow N \xrightarrow{\text{natural}} \bigoplus_i N_{f_i} \xrightarrow{\varphi_p} \bigoplus_{i,j} N_{f_i f_j}$$

$(N \rightarrow N \bigoplus_{i \neq l} N_{f_i}, n \mapsto n \bigoplus_{i \neq l} \frac{n}{f_i})$ $(x_i) \mapsto (\frac{x_i}{f_i} - \frac{x_j}{f_j})$



Sub-claim This is exact ($\Rightarrow N = \text{Ker } \varphi_p = M$, π_e iso, $\psi_{jk} = \text{id}$ under identifications via π maps)

Pf Enough to prove after localising at each max ideal m ← See 3.0

By $*$ not all $f_i \in m$ otherwise $1 \in \langle \text{all } f_i \rangle \subseteq m \nrightarrow$

Say $f_k \notin m$, so WLOG replace $N \rightsquigarrow N_m, R \rightsquigarrow R_m, f_k \rightsquigarrow 1$: ← $f_k \notin m$ so unit in localising R_m

$$\Rightarrow 0 \rightarrow N \rightarrow N \bigoplus_{i \neq k} N_{f_i} \rightarrow \bigoplus_{i,j} N_{f_i f_j}$$

clearly injective $n \bigoplus_{i \neq k} n_i \in \text{Ker}$ then $\frac{n}{f_i} = \frac{n_i}{f_i} \in N_{f_i f_k} = N_{f_i} \quad \forall i$ □
 hence $= n \bigoplus_{i \neq k} \frac{n}{f_i}$ so image of n via previous map

7.6 QCoh(X), Coh(X), Vect(X) for $X = \text{Spec } R$

Theorem For $X = \text{Spec } R$, \exists equivalence of categories

$$\begin{array}{ccc} R\text{-Mods} & \xrightarrow{\quad} & \text{QCoh}(X) \\ M & \xrightarrow{\quad} & \tilde{M} \\ F(X) = \Gamma(X, F) & \xleftarrow{\quad} & F \end{array}$$

← means: the two given functors compose to functors which are naturally iso to identity functors

Pf. Easy direction: $M \mapsto F = \tilde{M} \mapsto F(X) = \tilde{M}(X) = M$. Converse: given F want $F \cong \tilde{F}(X)$.

\Rightarrow locally $\forall p \in X, \exists p \in D_f$ s.t. $F|_{D_f} \xrightarrow{\varphi_f} \tilde{N}$ some R_f -mod N

By Cor in 7.1 using that D_f are basis of topology and $\text{Spec } R$ quasi-compact

cover X by finitely many such, say N_i on $D_{f_i}, i=1, \dots, n$, so $1 \in \langle \text{all } f_i \rangle$

\Rightarrow On overlaps: $\psi_{ij} : (\tilde{N}_i)_{f_j} \xrightarrow{\varphi_{f_i}^{-1}} F|_{D_{f_i f_j}} \xrightarrow{\varphi_{f_j}} (\tilde{N}_j)_{f_i}$ satisfy cocycle condition

← since $(\tilde{N}_i)_{f_j f_k}$ and other two are identified with $F|_{D_{f_i f_j f_k}}$

\Rightarrow by gluing them $\exists M$ with $M_{f_i} = N_i$ compatibly with the ψ_{ij}

But then \tilde{M}, F have isomorphic local gluing data for cover $X = D_{f_1} \cup \dots \cup D_{f_n}$ so $\tilde{M} \cong F$. □

(Explicitly: $m \in M \mapsto m_i = \frac{m}{f_i} \in M_{f_i} = N_i \xrightarrow{\varphi_{f_i}^{-1}} s_i \in F(D_{f_i})$ and $s_i|_{D_{f_i f_j}} = s_j|_{D_{f_i f_j}}$)
 so globalises to unique $s \in F(X)$. Recall $M \rightarrow F(X)$ determines $\tilde{M} \rightarrow F$ by Sec. 6.9)

Cor $X = \text{Spec } R$: $F \in \text{Coh } X \Leftrightarrow F = \tilde{M}$ for coherent module $M \cong F(X)$ and if R Noeth. get: $\Leftrightarrow F(X)$ f.g. R -mod

Pf $F = \tilde{F}(X)$ by Theorem. In definition of coherent take global sections $\Rightarrow F(X)$ coherent R -mod, and conversely if M coherent get \tilde{M} coherent since \sim is exact & fully faithful. □

Fact $X = \text{Spec } R$: $F \in \text{Vect } X \Leftrightarrow (F = \tilde{M} \text{ for finitely presented flat } R\text{-mod } M) \Leftrightarrow \text{f.g. projective } R\text{-mod}$

(see Hwk 4)

means in R -mods $\text{Hom}(M, \cdot)$ exact.

($\Leftrightarrow M$ is a direct summand of some free R -mod)

8. Čech Cohomology

8.1 Čech complex

Motivation for cohomology: assign group or ring of "invariants" to a space i.e. iso. spaces give isos of e.g. if $H^*(X) \cong H^*(Y)$ then $X \cong Y$ are not iso. spaces

notation:

$U_{ij} = U_i \cap U_j$
 $U_{ijk} = U_i \cap U_j \cap U_k$

X top. space, $X = \cup U_i$ open cover
 $U_I = U_{i_0} \cap \dots \cap U_{i_n}$ for $I = (i_0, \dots, i_n)$ multi-index, abbreviate $|I| = n$

ordered, allow repetitions

size is actually $n+1$

$$C^n = \prod_{\{U_i\}}^n \prod_{|I|=n} \Gamma(U_I, F)$$

$F \in \text{Ab}(X)$

so $s \in C^n$ is a collection $s_I \in F(U_I)$

called cochain

$$d = d^n: C^n \rightarrow C^{n+1}$$

$$(ds)_I = \sum_{j=0}^{n+1} (-1)^j s_{I_j} |_{U_I}$$

where $I_j = (i_0, \dots, \hat{i}_j, \dots, i_{n+1})$

omit

later also use notation $I_{jk\dots}$ if omit i_j, i_k, \dots

$\in F(U_I)$ so sum makes sense.

Example

$$C^0 = \prod_i \Gamma(U_i) \xrightarrow{d} \prod_{i,j} \Gamma(U_{ij}) = C^1$$

$$(s_i) \mapsto (s_j |_{U_{ij}} - s_i |_{U_{ij}})$$

$i_0 = i, i_1 = j$
 $I = (i_0, i_1)$
 $I_0 = (i_1) = j$

$$C^1 = \prod_{i,j} \Gamma(U_{ij}) \xrightarrow{d} \prod_{i,j,k} \Gamma(U_{ijk}) = C^2$$

$$(s_{ij}) \mapsto (s_{jk} |_{U_{ijk}} - s_{ik} |_{U_{ijk}} + s_{ij} |_{U_{ijk}})$$

if you took C3.1 Algebraic Top. notice similar to simplicial differential

Claim $d^2 = 0$, so (C^*, d) is a complex

Pf

$$(dds)_J = \sum_{k=0}^{n+2} (-1)^k (ds)_{J_k} |_{U_J} = \sum_{k=0}^{n+2} \left(\sum_{j < k} (-1)^{k+j} s_{J_{kj}} |_{U_J} + \sum_{j > k} (-1)^{k+j-1} s_{J_{kj}} |_{U_J} \right)$$

$$= 0. \square$$

anti-symmetry if swap j, k (notice full sum is over all $j \neq k$)

$$\text{Def } H^n(X, F) = \prod_{\{U_i\}}^n H^n(X, F) = \text{Ker } d^n / \text{Im } d^{n-1}$$

$(H^n(X, F))$ depend on choice of U_i
Rmk $d^n \circ d^{n-1} = 0$ so $\text{Im } d^{n-1} \subseteq \text{Ker } d^n$

Lemma $H^0(X, F) = \Gamma(X, F)$

Pf $s_j |_{U_{ij}} = s_i |_{U_{ij}}$ says s glues to global section. \square

called coboundaries called cocycles

"Co" sometimes omitted. Emphasizes doing Cohomology

Terminology 1) hom of complexes $f: C^n \rightarrow C^n$ is chain map if $f \circ d = d \circ f$

2) $h: C^n \rightarrow C^{n-1}$ is chain homotopy between chain maps f, g if $f - g = d \circ h + h \circ d$

Consequences: 1) $f: H^n \rightarrow H^n$ via $f[c] = [fc]$ well-defined

2) $f = g: H^n \rightarrow H^n \leftarrow (dc = 0 \Rightarrow [fc - gc] = [dhc] = 0)$

$$\left[\begin{array}{l} [c] = [c + db] \\ \text{but} \\ [fdb] = [dfb] = 0 \end{array} \right.$$

Key trick To show $H^* = 0$ can find chain homotopy between $\text{id}, 0$.
 i.e. C^* is exact, also called acyclic

Rmk If a homomorphism $d_n: C_n \rightarrow C_{n-1}$ decreases the degree by 1, and $d_{n-1} \circ d_n = 0$ then $H_n = \text{Ker } d_n / \text{Im } d_{n+1}$ is called the homology of (C_*, d_*) . In this case a chain homotopy is degree increasing: $h: C_n \rightarrow C_{n+1}$ with $f_n - g_n = d_{n+1} \circ h_n - h_{n-1} \circ d_n$.

8.2 Čech complex with ordering

e.g. if X quasi-compact

Repetitions of indices are annoying since $C^n \neq 0$ all $n \geq 0$ even if finite # U_i

Trick pick total ordering on indices

C_+^n : as C^n but only allow $I = (i_0, \dots, i_n)$ if $i_0 < i_1 < \dots < i_n$, d as before
 $\Rightarrow C_+^n \subseteq C^n$ subcomplex

so if finite cover with N sets,
 $C_+^n = 0$ for $n \geq N$
 $H_+^n = 0$ "

Claim $H_+^n \cong H^n$

I'm doing a hands-on proof based on
 Serre "FAC" 1955 sec. 20, p. 214
 Godement "Théorie des faisceaux" 1958 p. 60
 Eilenberg & Steenrod "Foundations of Alg. Top." 1952, VI. 6

Non-examinable Proof ("Serre's Trick")

Let $S_* =$ free abelian group generated by all index sets I , so $S_n = \langle I : |I| = n \rangle$

Differential: $\partial I = \sum (-1)^j I_j$ so $\partial: S_n \rightarrow S_{n-1}$.

(I is really a function $\{0, 1, \dots, n\} \rightarrow \{\text{indices}\}$)

S_*^+ = subgroup generated by strictly ordered index sets I

(so strictly increasing function for chosen total order on set)

Step 1 S_*, S_*^+ are acyclic

Pf $h: S_*^+ \rightarrow S_{*+1}^+$, $h(I) = \begin{cases} (\ell, I) & \text{if } \ell \neq i_0 \\ 0 & \text{if } \ell = i_0 \end{cases} \Rightarrow$ if $\ell \neq i_0$: $\partial h I = \partial(\ell, I) = I + \sum (-1)^{j+1} (\ell, I_j)$
 $h \partial I = h \sum (-1)^j I_j = \sum (-1)^j (\ell, I_j)$

$\ell :=$ minimal index

$\Rightarrow \text{id} - 0 = \partial h + h \partial$ ✓ For S_* it is even easier: $h(I) = (\ell, I)$ works. \square

Step 2 $f(I) := \begin{cases} 0 & \text{if } \exists \text{ repeated indices in } I \\ \text{sign}(\sigma) \cdot \sigma(I) & \text{otherwise, where } \sigma \text{ unique permutation s.t. } \sigma I \text{ ordered} \end{cases}$

$\sigma_{j_0 < j_1 < \dots}$

$\Rightarrow f$ chain map, $f = \text{id}$ on S_0 , $f(S_*) \subseteq S_*^+$, $f \circ f = f$ (i.e. f is id on S_*^+ , f is a projection to S_*^+)

Pf $\sigma(I) \in S_*^+$ and if I is ordered then $\sigma = \text{id}$. On S_0 : $f((i_0)) = (i_0)$.

$\partial f I = \sum (-1)^j \text{sign}(\sigma) \sigma(I)_j \iff$ for $k = \sigma^{-1}(j)$ get same set, $\text{sign}(\sigma) = \text{sign}(\tau) \cdot (-1)^{k-j}$ since $f \partial I = \sum (-1)^k \text{sign}(\tau) \tau(I_k)$ σ does an extra $k-j$ transpositions to move i_j to position k

Step 3 General trick: C_* free acyclic complex, a chain map $f: C_* \rightarrow C_*$ has $f_0 = \text{id}: C_0 \rightarrow C_0$ then f, id are chain homotopic: $\exists k: C_* \rightarrow C_{*+1}$ with $f - \text{id} = \partial k + k \partial$

Pf Build k inductively by equation $\partial_{n+1} \circ k_n = f_n - \text{id} - k_{n-1} \circ \partial_n$

$n=0$: $\partial_1 \circ k_0 = \underbrace{f_0 - \text{id}}_{=0} - \underbrace{k_{-1} \circ \partial_0}_{=0}$ so just define $k_0 = 0$.

Warning the k_i do not make diagram commute

Assume true for $n-1$: $\partial_n k_{n-1} = f_{n-1} - \text{id} - k_{n-2} \partial_{n-1}$ \otimes

$\partial_n (f_n - \text{id} - k_{n-1} \circ \partial_n) = f_{n-1} \partial_n - \partial_n - (\partial_n \circ k_{n-1}) \partial_n$
 $\otimes \rightarrow = f_{n-1} \partial_n - \partial_n - (f_{n-1} - \text{id} - k_{n-2} \partial_{n-1}) \partial_n$
 $= 0$ since $\partial \circ \partial = 0$ and f is chain map

$\Rightarrow \forall c_n \in C_n, (f_n - \text{id} - k_{n-1} \partial_n) c_n = \partial_{n+1} c_{n+1}$ some $c_{n+1} \in C_{n+1}$. As C_n is free we can pick basis elts c_n of C_n and pick such c_{n+1} , then define $k_n(c_n) := c_{n+1}$ and extend k_n linearly to get $k_n: C_n \rightarrow C_{n+1} \Rightarrow$ get required equation for n .

Step 4 chain maps/homotopies on S_*, S_*^+ induce corresponding chain maps/homopies on C^*, C_*^+

Pf If $\varphi(I) = \sum n_{II'} \cdot I'$, $n_{II'} \in \mathbb{Z}$ then define $(\check{\varphi}(s))_I = \sum n_{II'} \cdot s_{I'} |_{U_I}$
 ($\check{\varphi}$ hom on S_* or S_*^+) ($\check{\varphi}$ hom on C^* or C_*^+ respectively)

Example $d = \check{\partial}$, and for f of Step 2: $(\check{f}(s))_I = \begin{cases} 0 & \text{if } \exists \text{ repeated indices in } I \\ \text{sign}(\sigma) \cdot s_{\sigma(I)} |_{U_I} & \text{else} \end{cases}$

Conclusion: $\check{f}: C^* \rightarrow C^*$ chain hpic to id and surjects onto $C_*^+ \Rightarrow [\check{f}] = \text{id}: H^* \rightarrow H^*$ hence equal. \square

Cor H_+^* is independent of choice of total ordering on set of indices (since $H_+^* \cong H^*$)

$\check{H}_{\{U_i\}}^m(X, F) = 0$ for $m \geq N$ if $X = \cup U_i$ if finite cover with N sets (since $U_i = \emptyset$ in C_*^+ if $|I| \geq N$)

Example $X = \mathbb{P}_k^n$ with cover by $N = n+1$ affine sets $U_i \cong \mathbb{A}_k^n$ (HWK 2)

8.3 Affines have no cohomology except H^0

← (compare $H^*(\mathbb{C}^n) = 0$ for $* \geq 1$ in algebraic topology)

Theorem $X = \text{Spec } R$

$F \in \text{QCoh}(X)$

$$\Rightarrow \check{H}^n(X, F) = 0 \text{ for } n \geq 1$$

$X = \cup U_i$: finite affine open cover

Pf X separated (since affine) $\Rightarrow U_I$ all affine (Sec. 5.3, ⑧)

Easy case: minimal index l satisfies $U_l = X$

use ordered Čech cohomology. $s \in \mathbb{C}^n, h_s \in \mathbb{C}^{n-1}$
 $I = (i_0, \dots, i_{n-1})$
 $i_0 < i_1 < \dots < i_{n-1}$

$$U_{l,I} = U_l \cap U_I = X \cap U_I = U_I$$

← Exercise check case $I = (l, i_1, \dots)$ also works.

chain homotopy: $(h_s)_I = \begin{cases} 0 & \text{if } i_0 = l \\ s_{l,I} & \text{if } i_0 \neq l \text{ (so } l < i_0) \end{cases}$

for I with $i_0 \neq l$:

$$\left. \begin{aligned} (d(hs))_I &= \sum (-1)^j (hs)_{I_j} = \sum (-1)^j s_{l,I_j} \\ (h(ds))_I &= (ds)_{l,I} = s_I + \sum (-1)^{j+1} s_{l,I_j} \end{aligned} \right\} \Rightarrow \text{id} = dh + hd \Rightarrow \text{Key Trick } \checkmark \text{ (Sec. 8.1)}$$

General case $X = \text{Spec } R = \cup U_i, U_i = \text{Spec } R_i$

By easy case, know result for space U_l with covering $U(U_l \cap U_i)$, for minimal l .

Ordering of indices does not affect H^* , so know result for \exists any l by Cor of 8.2

\Rightarrow **Reduce to claim**: if C^* exact when restrict to $U_i \forall i$, then C^* exact

$F \in \text{QCoh}(X), U_I$ affine say $\text{Spec } R_I \xrightarrow{7.6} F|_{U_I} \cong \widetilde{M}_I$ some R_I -module M_I

$$C^n = \prod_{|I|=n} F(U_I, F) = \prod_{|I|=n} M_I \text{ finite product so } = \bigoplus \text{ (in particular, an } R\text{-mod) (since } R \rightarrow R_I \text{ from } U_I \rightarrow X)$$

$\Rightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \rightarrow \dots$ is a complex of R -mods

and by assumption of exactness on U_i have:

$$C^0 \otimes_R R_i \rightarrow C^1 \otimes_R R_i \rightarrow \dots \text{ exact } \forall i$$

\Rightarrow localising further by $\cdot \otimes_{R_i} (R_i)_P$ get exactness of localisation of C^* at each $P \in \text{Spec } R$.

\Rightarrow by Sec. 3.0 deduce exactness of C^* . \square

← using $F|_{U_i/U_i} = \widetilde{M}_I|_{U_i} \cong \widetilde{M}_I \otimes R_i$ by 6.8 and $\bigoplus \widetilde{N}_i = \widetilde{\bigoplus N_i}$. U_i cover X so $P \in U_i$: some i

Rmk Chain homotopy trick above can be used to show $\check{H}^*(X, \underline{A}) = 0$ for $* \geq 1$ if X irreducible scheme and \underline{A} is constant sheaf with values in abelian group A . (e.g. $\check{H}^1 = 0$: given cocycle g_{ij} fix index i_0 define $h \in \check{C}^0$ by $h_i = g_{i_0 i} \in \Gamma(U_{i_0}) = A = \Gamma(U_i)$ Cocycle $\Rightarrow g_{i_0 j} = g_{i_0 i} + g_{i j}$ so $(dh)_{i_0 j} = h_j - h_{i_0} = g_{ij}$)

8.4 Independence of cover

Theorem X separated, quasi-compact $\Rightarrow \check{H}^*(X, F)$ independent of choice of

Pf Will use ordered Čech cohomology. $F \in \text{Coh}(X)$ finite affine open cover

X separated $\Rightarrow \bigcap_{\text{finite}} \text{affines}$ is affine (Sec. 5.3, ⑧)

$X = \cup U_i, X = \cup V_j$ take mixed intersections: $C^{n,m} = \prod_{\substack{|I|=n \\ |J|=m}} \Gamma(U_I \cap V_J, F)$

$$C^{n,0} \cong \prod_{|I|=n} \check{C}^n(\{V_j \cap U_I\}, (F|_{U_I}))$$

$$C^{0,m} \cong \prod_{|J|=m} \check{C}^0(\{U_i \cap V_J\}, (F|_{V_J}))$$

finite affine cover of the affine U_I so by 8.3 $H^* = 0$

similar

"bi-complex"

$$\begin{array}{cccc} \dots & \dots & \dots & \dots \\ \uparrow & \uparrow & \uparrow & \uparrow \\ C^{0,2} & \rightarrow & C^{1,2} & \rightarrow & C^{2,2} & \rightarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ C^{0,1} & \rightarrow & C^{1,1} & \rightarrow & C^{2,1} & \rightarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ C^{0,0} & \rightarrow & C^{1,0} & \rightarrow & C^{2,0} & \rightarrow & \dots \end{array}$$

\Rightarrow rows & columns are exact except for degree 0:

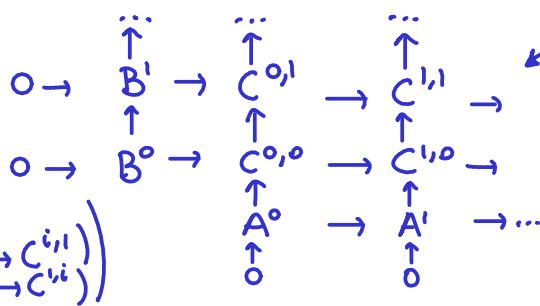
$$H^0(C^{n,0}) = \prod_{|I|=n} \Gamma(U_I, F) = \check{C}^n(\{U_i\}, (F))$$

$$H^0(C^{0,m}) = \prod_{|J|=m} \Gamma(V_J, F) = \check{C}^0(\{V_j\}, (F))$$

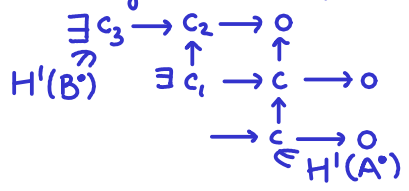
General fact from homological algebra

$C^{i,j}$ bi-complex, $H^i(C^{n,\bullet}) = 0 \forall i > 0, \forall n$
 $H^i(C^{\bullet,m}) = 0 \forall i > 0, \forall m \Rightarrow H^0(C^{n,\bullet}) \text{ complex in } n$
 $H^0(C^{\bullet,m}) \text{ " " } m$ } with iso cohomology $H^*(A^0) \cong H^*(B^0)$

Sketch Pf



Now rows & cols are exact, so can diagram chase, and get a "zig-zag":



so $H^1(A^0) \rightarrow H^1(B^0)$
 $c \mapsto c_3$
 via the iso \square

(Note \rightarrow
 $A^i = \ker(C^{i,0} \rightarrow C^{i,1})$
 $B^i = \ker(C^{0,i} \rightarrow C^{1,i})$)

8.5 Induced Long Exact Sequence on \check{H}^*

Recall $\Gamma(X, \cdot): \text{Ab}(X) \rightarrow \text{Ab}$ is always left exact (Sec. 1.9)

Lemma U open affine \subseteq scheme $X \Rightarrow \Gamma(U, \cdot): \text{QCoh } X \rightarrow \text{Ab}$ is exact

Recall Sec. 6.4
 $R\text{-mod} \rightarrow \text{QCoh}(\text{Spec } R)$
 $M \mapsto \tilde{M}$
 is exact and fully faithful

Pf Given $F_1 \rightarrow F_2 \rightarrow F_3$ exact. Exactness is local condition (indeed stalks)
 \Rightarrow WLOG $F_i|_U = \tilde{M}_i$. $\tilde{M}_1 \rightarrow \tilde{M}_2 \rightarrow \tilde{M}_3$ exact $\Leftrightarrow M_1 \rightarrow M_2 \rightarrow M_3$ exact \square

Claim X separated, $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ SES in $\text{QCoh}(X)$

SES = short exact sequence
 LES = long " " " "

\Rightarrow get LES $0 \rightarrow H^0(X, F_1) \rightarrow H^0(X, F_2) \rightarrow H^0(X, F_3) \rightarrow H^1(X, F_1) \rightarrow H^1(X, F_2) \rightarrow \dots$
 (using affine cover) $\Gamma(X, F_1) \quad \Gamma(X, F_2) \quad \Gamma(X, F_3)$
 (e.g. Ker measures failure of $\Gamma(X, \cdot)$ being right-exact)

Pf $0 \rightarrow F_1(U_I) \rightarrow F_2(U_I) \rightarrow F_3(U_I) \rightarrow 0$ exact by Lemma.
 (U_I affine since X separated)
 $\Rightarrow 0 \rightarrow \check{C}^*(F_1) \rightarrow \check{C}^*(F_2) \rightarrow \check{C}^*(F_3) \rightarrow 0$ exact, claim follows \square

homological algebra:
 SES of chain complexes induces LES on cohomology (e.g. see my C3.1 notes)

8.6 Dealing with infinite covers

A refinement of an open cover $X = \cup U_i$ is an open cover $X = \cup V_j$ s.t. $\forall j, V_j \subseteq U_i$ some i
 Make choices \Rightarrow restrictions $F(U_{i(j)}) \rightarrow F(V_j) \Rightarrow \check{C}_{\{U_i\}}^*(X, F) \rightarrow \check{C}_{\{V_j\}}^*(X, F)$ chain map.
 (top. space \rightarrow sheaf \rightarrow on V_j get restriction from $F(U_{i(j_0) \dots i(j_n)})$)

Fact $\check{H}_{\{U_i\}}^*(X, F) \rightarrow \check{H}_{\{V_j\}}^*(X, F)$ does not depend on choices made (Serre "FAC", Sec. 21)

Def $\check{H}^*(X, F) = \varinjlim \check{H}_{\{U_i\}}^*(X, F)$
 (so each class is represented by a Čech cocycle for some cover, and identify cocycles if they differ by a boundary after passing to some common refinement)

Non-examinable Rmk For any topological space homotopy equivalent to a CW complex (e.g. any manifold)

$\check{H}^*(X, \underline{A}) \cong H^*(X, A) =$ singular cohomology of X with coefficients in A
 (\underline{A} is "constant sheaf" with values in A : actually means sheafify, so $\underline{A}(U) = \{\text{locally constant } U \rightarrow A\}$)

Rmk X quasi-compact scheme \Rightarrow can use finite covers by affine opens, and can refine any cover by such a cover
 \Rightarrow can calculate $\check{H}^*(X, F)$ by only using finite affine covers

so for smooth manifolds, and $A = \mathbb{R}$, get de Rham cohomology
 $U_i = \cup_j A_{ij}$
 $X = \cup_{i,j} A_{ij}$
 pick finite subcover

Cor Theorem in 8.3 holds \forall cover (using definition \star)
Cor X separated quasi-compact sch. \Rightarrow can calculate $\check{H}^*(X, F)$ with one cover!
 (by Theorem 8.4 \Rightarrow maps in \varinjlim for such covers are isos so $\check{H}_{\{U_i\}}^*(X, F) \rightarrow \varinjlim \dots$ is iso.)

8.7 Application: line bundles and $\check{H}^1(X, \mathcal{O}_X^*)$

X scheme, $F \in \text{Vect}(X)$

$\Rightarrow \exists$ open cover $X = \cup U_i$ with $F|_{U_i} \xrightarrow{\cong \varphi_i} \mathcal{O}_{U_i}^{\oplus n_i}$ some $n_i \in \mathbb{N}$

called a trivialization over U_i

and can compare trivializations on overlaps:

$$F|_{U_{ij}} \xrightarrow{\cong \varphi_i} \mathcal{O}_{U_{ij}}^{\oplus n_i}$$

$$\parallel \cong \downarrow \alpha_{ij}$$

$$F|_{U_{ji}} \xrightarrow{\cong \varphi_j} \mathcal{O}_{U_{ji}}^{\oplus n_j} = \mathcal{O}_{U_{ij}}^{\oplus n_j}$$

α_{ij} called transition maps

$\mathcal{O}_{U_{ij}}$ -module iso described by an invertible $n_j \times n_i$ matrix with entries in $\mathcal{O}_{U_{ij}}(U_{ij})$

(see sec. 6.2: $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)^{\oplus n}$)

$\Rightarrow n_i = n_j$ if $U_{ij} \neq \emptyset$, so the rank of F is locally constant.

Conversely, given such data φ_i, α_{ij} satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on U_{ijk} determines by giving a vector bundle.

Rmk $\alpha_{ji} = \alpha_{ij}^{-1}$

This is the actual definition of vector bundle in terms of compatible local trivializations.

Def $\mathcal{O}_X^* \subseteq \mathcal{O}_X$ sheaf of invertible functions. So $\mathcal{O}_X^*(U) = \{f \in \mathcal{O}_X(U) : \exists g \in \mathcal{O}_X(U) \text{ s.t. } f \cdot g = 1\}$

Note that $\mathcal{O}_X^*(U)$ is an abelian group under multiplication.

Theorem {isomorphism classes of line bundles that admit a trivialization over U_i } $\xleftrightarrow{1:1} \check{H}_{\{U_i\}}^1(X, \mathcal{O}_X^*)$

and $\text{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*)$ as groups.

\leftarrow (Pic X defined in 7.2)

Pf $\alpha_{ij} : \mathcal{O}_{U_{ij}} \rightarrow \mathcal{O}_{U_{ij}}$ given by multiplication by element $\in \mathcal{O}_{U_{ij}}^*$

• tensoring line bundles that admit a trivialization on U_{ij} : $\mathcal{O}_{U_{ij}} \cong \mathcal{O}_{U_{ij}} \otimes_{\mathcal{O}_{U_{ij}}} \mathcal{O}_{U_{ij}} \xrightarrow{\alpha_{ij} \otimes \tilde{\alpha}_{ij}} \mathcal{O}_{U_{ij}} \otimes_{\mathcal{O}_{U_{ij}}} \mathcal{O}_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$

• Cocycle condition can be rewritten: $\alpha_{jk} \cdot \alpha_{ik}^{-1} \cdot \alpha_{ij} = 1$

(which is the statement $s_{jk} - s_{ik} + s_{ij} = 0$ in multiplicative notation)

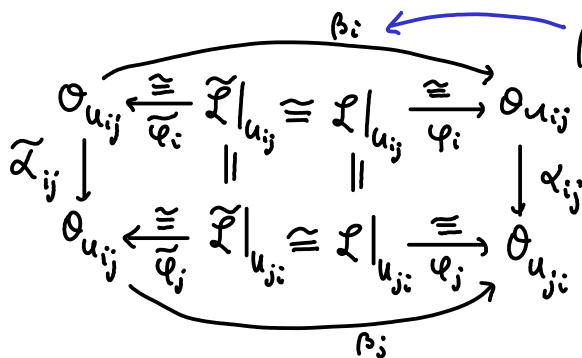
multiplication by $\alpha_{ij} \cdot \tilde{\alpha}_{ij} \in \mathcal{O}_{U_{ij}}^*$

$\Rightarrow (\alpha_{ij}) \in \check{H}_{\{U_i\}}^1(X, \mathcal{O}_X^*)$

\leftarrow ($(s_i) \in \check{C}^0, d(s_i) = s_j - s_i$ on U_{ij}) in additive notation

In \check{H}^1 we identify $[(\tilde{\alpha}_{ij})] = [(\alpha_{ij})] \iff \alpha_{ij} = \tilde{\alpha}_{ij} \beta_j \beta_i^{-1}$ some $\beta_i \in \mathcal{O}_{U_i}^*$

This corresponds precisely to how the \check{C}^1 class changes under an iso of line bundles $\mathcal{L} \cong \tilde{\mathcal{L}}$ as in claim:



$\beta_i := \text{composite } (\mathcal{O}_{U_i} \xrightarrow{\cong \tilde{\varphi}_i} \tilde{\mathcal{L}}|_{U_i} \cong \mathcal{L}|_{U_i} \xrightarrow{\cong \varphi_i} \mathcal{O}_{U_i}) \in \mathcal{O}_{U_i}^*$

\leftarrow in the case $\mathcal{L} = \tilde{\mathcal{L}}$ the diagram shows that the \check{C}^1 class changes by a boundary chain if we change the choice of trivialization on each U_i . Hence the \check{H}^1 class does not depend on the choices of the φ_i .

□

Rmk \mathcal{L} line bundle with transition maps α_{ij} } and $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X =$ trivial line bundle
 $\Rightarrow \mathcal{L}^{-1}$ " " " " α_{ij}^{-1}

FACT line bundles on \mathbb{A}^n are always trivial
 indeed vector bundles on \mathbb{A}^n are always trivial \leftarrow (Serre's Conjecture 1955, Quillen-Suslin Theorem 1976)

EXAMPLE $\text{Pic}(\mathbb{P}^1) \quad \mathbb{P}^1_k = A_0 \cup A_1$
 $\quad \quad \quad \cong \text{Spec } k[t] \quad \cong \text{Spec } k[t^{-1}]$
 In C3.4 course: view $\mathbb{P}^1_k = k^2 \setminus \{0\} / k^*$ -rescaling
 Have homogeneous coordinates $[x_0 : x_1]$
 and A_0 corresponds to $\{[1:t] : t \in \mathbb{A}^1\}$ where $t = x_1/x_0$

\mathcal{L} line bundle on $\mathbb{P}^1_k \Rightarrow \mathcal{L}|_{A_i}$ trivial since $A_i \cong \mathbb{A}^1$.

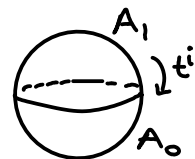
$(\alpha_{10} : \mathcal{L}|_{A_1} \rightarrow \mathcal{L}|_{A_0}) \in k[t, t^{-1}]^* = \{a t^i : a \in k^*, i \in \mathbb{Z}\}$ \leftarrow note: $A_0 \cap A_1 = \text{Spec } k[t, t^{-1}]$

$\beta_0 \in k[t]^* = k^*$, $\beta_1 \in k[t^{-1}]^* = k^*$ \leftarrow exercise

$$\Rightarrow \text{Pic}(\mathbb{P}^1_k) \cong \check{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^*) \cong \mathbb{Z}$$

$$\mathcal{O}(i) \leftrightarrow (\alpha_{10} = t^i) \leftrightarrow i$$

so define $\mathcal{O}(i)$ by using
 $\alpha_{10} = t^i$
 $\alpha_{01} = t^{-i}$



Rmk $\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^1}$ trivial line bundle.

HWK 4 Ideal sheaf of a closed point in \mathbb{P}^1_k is $\cong \mathcal{O}(-1)$, for disjoint union of n closed pts get $\cong \mathcal{O}(-n)$
 for order n point $(t^n) \subseteq k[t]$ (i.e. closed subscheme $\text{Spec } k[t]/(t^n) \subseteq A_0 \subseteq \mathbb{P}^1_k$) get $\mathcal{O}(-n)$.

Non-examinable Rmk (for differential geometers): i determines the Chern class $c_1(\mathcal{L}) : i = \int c_1(\mathcal{L})$

$T\mathbb{P}^1_{\mathbb{C}}$ is $\mathcal{O}(2)$ since $2 = \chi(\mathbb{P}^1_{\mathbb{C}}) = \chi(S^2)$ and $c_1(T\mathbb{P}^1_{\mathbb{C}}) =$ Euler class of $\mathbb{P}^1_{\mathbb{C}}$, and $T^*\mathbb{P}^1_{\mathbb{C}} = \mathcal{O}(-2)$.

$\mathcal{O}(-1) \rightarrow \mathbb{P}^1_{\mathbb{C}}$ is blow-up of \mathbb{C}^2 at 0 : the lines through 0 in \mathbb{C}^2 are the fibres.

Theorem

Cultural Rmk
 Symmetry is "Serre duality".
 For \mathbb{P}^1 :
 $\check{H}^1(\mathcal{O}(i)) \cong \check{H}^0(\mathcal{O}(-i-2))$
 $= \check{H}^0(\mathcal{O}(-i-2))$

- $\check{H}^0(\mathbb{P}^1_k, \mathcal{O}(i)) = \begin{cases} 0 & i < 0 \\ \{f \in k[t] : \deg f \leq i\} \cong k[x_0, x_1]_i & i \geq 0 \end{cases}$ $\leftarrow t = x_1/x_0$
 \leftarrow i -th graded part, so homogeneous polys in x_0, x_1 of degree i
- $\check{H}^1(\mathbb{P}^1_k, \mathcal{O}(i)) = \begin{cases} 0 & i \geq -1 \\ k[t^{-1}]/k + t^i k[t^{-1}] \cong k[x_0, x_1]_{-i-2} & i < -1 \end{cases}$
 \leftarrow exercise
- $\check{H}^n(\mathbb{P}^1_k, \mathcal{O}(i)) = 0$ for $n \geq 2$

Pf By 8.6, since \mathbb{P}^1_k separated & quasi-compact, enough to calculate $\check{H}^i_{\{A_0, A_1\}}(\mathbb{P}^1_k, \mathcal{O}(i))$.

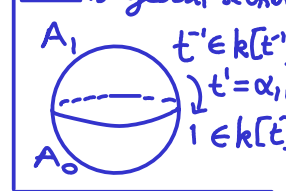
3) no triple ordered overlaps or higher

1) $\check{H}^0 = \Gamma : g(t^{-1}) \in k[t^{-1}]$ on A_1 , $f(t) \in k[t]$ on A_0 , on overlap: $t^i g(t^{-1}) = f(t) \in k[t, t^{-1}]$
 $\Rightarrow \deg f \leq i$ and g is determined by f from equation

example $\mathcal{O}(1)$
 $s=1$ on A_0
 $s=t^{-1}$ on A_1
 is global section

$$\Gamma(A_0, \mathcal{L}) \oplus \Gamma(A_1, \mathcal{L}) \xrightarrow{d} \Gamma(A_0 \cap A_1, \mathcal{L}) \xrightarrow{d} 0$$

$$\begin{matrix} \parallel & \parallel & \parallel \\ k[t] & k[t^{-1}] & k[t, t^{-1}] \\ (f, g) & \xrightarrow{\quad} & t^i g(t^{-1}) - f(t) \end{matrix}$$



$$\check{H}^1 = k[t, t^{-1}] / k[t] + t^i k[t^{-1}]$$

\leftarrow need to transition from $g(t^{-1}) \in \mathcal{O}_{A_1}(A_{10})$ to $\mathcal{O}_{A_0}(A_{01})$ via $\alpha_{10} : \mathcal{O}_{A_{10}} \cong \mathcal{L}|_{A_{10}} = \mathcal{L}|_{A_{01}} \cong \mathcal{O}_{A_{01}}$
 • is all of $k[t, t^{-1}]$ if $i \geq -1$
 • does not contain $t^{-1}, t^{-2}, \dots, t^{i+1}$ if $i < -1$

EXAMPLE: \mathbb{P}^n

called hyperplane bundle or Serre's twisting sheaf
 $X = \mathbb{P}_k^n = A_0 \cup A_1 \cup \dots \cup A_n$ $A_i = \text{Spec } k \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right]$ *omit $\frac{x_i}{x_i}$*
 $\mathcal{O}(1) =$ line bundle with $\alpha_{ij} = \left(\frac{x_i}{x_j} \right)$: $k \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] \rightarrow k \left[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j} \right]$ *\mathbb{P}_k^1 case: $t = x_1/x_0$
 $\alpha_{01}: k[t] \rightarrow k[t^{-1}]$ is multiplication by $\frac{x_0}{x_1} = t^{-1}$ ✓*
 $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ so $\alpha_{ij} = \left(\frac{x_i}{x_j} \right)^m$ *both equal to $\Gamma(A_i \cap A_j, \mathcal{O}_X)$*
 $m \in \mathbb{Z}$ tensor m times

Rmk $\mathcal{O}(-1)$ called tautological line bundle because in C3.4 course each (closed) point of \mathbb{P}_k^n is a 1-dim vector subspace $V \subseteq k^{n+1}$ ($\mathbb{P}_k^n = k^{n+1} \setminus \{0\} / \sim$ k^* -rescaling)
 so get obvious line bundle: over the point $[V] \in \mathbb{P}^n$ have the line V .

Hwk 4 $\text{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$ generated by the $\mathcal{O}(1)$

$\Gamma(\mathbb{P}_k^n, \mathcal{O}(m)) = \begin{cases} k[x_0, \dots, x_n]_m & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$ *So homogeneous polys of deg = m. So on A_i get polys of deg $\leq m$ in the variables $\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}$*

8.8 Divisors

Let $(X, \mathcal{O}_X = \mathcal{O})$ be an integral scheme (i.e. reduced & irreducible) *← see Sec. 3.5*

Recall from Sec. 3.5 that \forall open $\emptyset \neq U \subseteq X$ can view $\mathcal{O}(U) \xrightarrow{\text{injective}} K(X) =$ function field.

Abbreviate: $K = K(X)$, $K^* = K \setminus \{0\}$ (non-zero rational functions are invertible) *← rational functions on X*

$\mathcal{O}^* \subseteq \mathcal{O}$ subsheaf of sections of \mathcal{O} admitting inverses in \mathcal{O} (so can view $\mathcal{O}^* \subseteq K^*$)

$X = \cup U_i$ open cover

$f_i \in K^*$ s.t. $\frac{f_i}{f_j} \Big|_{U_i \cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$ } this data is called a Cartier divisor *(see ⊗ below)*

\Rightarrow get line bundle $\mathcal{L} \subseteq K$ via $\mathcal{L}(U) := \mathcal{O}(U) \cdot \frac{1}{f_i} \subseteq K$

(so $\mathcal{O}^(U) \subseteq K^*$ or can also view K, K^* as constant sheaves on X with values in K, K^*)*
(so free $\mathcal{O}(U)$ -mod with basis $1/f_i$)

Exercise

① Obvious trivialisations $\varphi_i: \mathcal{L}(U_i) \rightarrow \mathcal{O}(U_i)$, $g \cdot \frac{1}{f_i} \mapsto g$

Yields transition maps $\alpha_{ij} = \varphi_j \circ \varphi_i^{-1} \Big|_{U_i \cap U_j} = \frac{f_j}{f_i}$ (from U_i to U_j)

② If $D_1 = (U_i, f_i), D_2 = (V_k, g_k)$ are two Cartier divisors on X yielding line bundles $\mathcal{L}_1, \mathcal{L}_2$ then
 $D_1 + D_2 = (U_i \cap V_k, \frac{f_i g_k}{g_k})$ is a Cartier divisor yielding the line bundle $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ [in particular $-D_1$ is $(U_i, \frac{1}{f_i})$ yields \mathcal{L}_1^{-1}]
 $D_1 - D_2 = (U_i \cap V_k, \frac{f_i}{g_k})$ " " " " $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$

Key Example Recall $\mathbb{P}^n = \cup U_i$ for $U_i = \text{Spec } \mathbb{Z} \left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right] \cong \mathbb{A}^n$

Let $m \in \mathbb{Z}$, $f_0 = 1, f_i = \left(\frac{x_0}{x_i} \right)^m \in K(\mathbb{P}^n) = \left\{ \frac{p}{q} \in \mathbb{Q}(x_0, \dots, x_n) : p, q \in \mathbb{Z}[x_0, \dots, x_n] \text{ homogeneous of same degree} \right\}$

$\mathcal{L}(U_0) = \mathcal{O}_{\mathbb{P}^n}(U_0) \cdot 1 \subseteq K(\mathbb{P}^n)$ *(side remark: $K(\mathbb{P}^n) \cong k(U_i) \cong k(\mathbb{A}^n) \cong \mathbb{Q}(z_1, \dots, z_n)$)*

$\mathcal{L}(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i) \cdot \left(\frac{x_0}{x_i} \right)^m \subseteq K(\mathbb{P}^n)$ transition $\alpha_{ij} = \left(\frac{x_0}{x_j} \cdot \frac{x_i}{x_0} \right)^m = \left(\frac{x_i}{x_j} \right)^m$ (from U_i to U_j) so $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(m)$.

Rmk This does not look very "symmetric" in the x_i . One can define an $\mathcal{O}_{\mathbb{P}^n}$ -module F

by $F(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i) \cdot x_i^m$ which is a line bundle with the same transitions $\alpha_{ij} = \left(\frac{x_i}{x_j} \right)^m$.

So $F \cong \mathcal{L}$ above, but we cannot pick $f_i = x_i^m$ for the Cartier divisor since $x_i^m \notin K(\mathbb{P}^n)$.

* Actually want to identify Cartier divisors related by refining the cover, so if $X = \cup U_i = \cup V_j$ and $V_j \subseteq U_{i(j)}$ *← compare Sec. 8.6* then identify (U_i, f_i) and $(V_j, f_{i(j)})$.
 Also identify (U_i, f_i) with $(U_i, f_i g_i)$ if $g_i \in \mathcal{O}^*(U_i)$ *← (i.e. rescaling f_i by invertible regular fns)*

Viewing K, K^* as constant sheaves, have an exact sequence

$$0 \rightarrow \mathcal{O}^* \rightarrow K^* \rightarrow K^*/\mathcal{O}^* \rightarrow 0$$

Because of \otimes , a Cartier divisor is just a global section of K^*/\mathcal{O}^* so $\in \check{H}^0(X, K^*/\mathcal{O}^*)$

Take LES: $0 \rightarrow \check{H}^0(X, \mathcal{O}^*) \rightarrow \check{H}^0(X, K^*) \rightarrow \check{H}^0(X, K^*/\mathcal{O}^*) \rightarrow \check{H}^1(X, \mathcal{O}^*) = \text{Pic}(X) \rightarrow \check{H}^1(X, K^*)$

A Cartier divisor in image of $\check{H}^0(X, K^*/\mathcal{O}^*)$ is called principal (i.e. use cover X and one $f \in K^*$ or (U_i, f_i) and $f_i \in \mathcal{O}^*(U_i) \cdot f, \forall i$)

Two Cartier divisors D, D' are linearly equivalent if $D - D'$ is principal. Write $D \sim D'$.

Get abelian group $\text{CaCl}(X)$ of Cartier divisors modulo linear equivalence.

$\Rightarrow \text{CaCl}(X) \cong \text{Pic}(X)$ by the LES, in particular $(D, D' \text{ yield iso. line bundles } \mathcal{L}(D) \cong \mathcal{L}(D')) \Leftrightarrow D \sim D'$.

because K^* is constant sheaf and X is irreducible (end of Sec 8.3)

Cultural Rmks (Non-examinable) There is another notion of divisor: Weil divisor.

$\dim \mathcal{O}_{X, z} = 1$ at generic point $z \in Z$ (think hypersurfaces)

This means a formal sum $\sum_{z \in Z} n_z \cdot Z$ of integral closed subschemes Z_i of codim = 1

Example rational function $f \in K(X) \Rightarrow \exists$ an "order of vanishing" $\text{ord}_z(f)$ of f along such subschemes Z .

\Rightarrow Weil divisor $\text{div}(f) := \sum_Z \text{ord}_z(f) \cdot Z$ called principal Weil divisor ($f = \frac{a}{b} \in K(X)$ then $\text{ord}_z(f)$ is $\text{length}_{\mathcal{O}_z}(R/a) - \text{length}_{\mathcal{O}_z}(R/b)$ for $R = \mathcal{O}_{X, z}$)

Example Cartier divisor (U_i, f_i) yields Weil divisor $W = \sum_i \sum_{z \in U_i} \text{ord}_z(f_i) \cdot Z$ (notice compute the order of $f_i|_{U_i} \in K(U_i)$ along $Z \cap U_i \subseteq U_i$)

On \mathbb{P}^1 : Cartier divisor $(U_0, 1), (U_1, \frac{x_0}{x_1})$ yields $W = + \text{point } [0:1]$ (since $f_i = \frac{x_0}{x_1}$ has a simple zero at $x_0=0$ but ignore pole $x_1=0$)

Cartier divisor $(U_0, 1), (U_1, (\frac{x_0}{x_1})^m)$ yields $W = m \cdot p$ where $m \in \mathbb{Z}, p = [0:1]$ since $[1:0] \notin U_1$

On \mathbb{P}^n : $(U_0, 1), (U_i, \frac{x_0}{x_i})$ yields $W = H$ where $H \cong \mathbb{P}^{n-1}$ is the hyperplane $x_0 = 0$ (case $m < 0$ is when f_i has a pole at $p = (x_0=0)$)

The lack of "symmetry" mentioned in Rmk above is because it involves a choice of Weil divisor H .

We could have picked any hyperplane to get $\mathcal{L} \cong \mathcal{O}(+1)$. More complicated choices are possible

e.g. Cartier divisor D on \mathbb{P}^1 with $W = \sum n_i \cdot p_i$ any points p_i , and $n_i \in \mathbb{Z}$, yields $\mathcal{L}(D) \cong \mathcal{O}(\sum n_i)$ (compare Hwk 4)

Weil divisors $\text{Div}(X)$ modulo principal Weil divisors define the class group $\text{Cl}(X)$ (abelian group).

Weil divisor D defines an \mathcal{O}_X -module $\mathcal{O}_X(D)$ by $\Gamma(U, \mathcal{O}_X(D)) = \{f \in K : \text{div}(f) + D \geq 0 \text{ on } U\}$

But $\mathcal{O}_X(D)$ need not be a line bundle (i.e. invertible sheaf). When it is a line bundle the Weil divisor

is Cartier since on some cover $X = \cup U_i$ have trivialisations $\mathcal{O}(U_i) \cong \Gamma(\mathcal{O}_X(D), U_i)$

means the " n_z " above are ≥ 0 so if D has $m_z \cdot Z$ and $m_z > 0$ then f is allowed to have a pole of order $\leq m_z$ along $Z \cap U$ if $m_z \leq 0$ then f must vanish with order $\geq -m_z$ along $Z \cap U$

\Rightarrow Cartier divisor (U_i, f_i) and $\mathcal{L}(U_i) = \mathcal{O}(U_i) \cdot \frac{1}{f_i} = \Gamma(U_i, \mathcal{O}_X(D))$ $1 \mapsto f_i \in K$

Weil divisor is Cartier if locally principal: so locally looks like $\text{div}(\text{rational fn})$ (e.g. $D = \text{div}(f)$ gives $\mathcal{O}_X(D) \cong \mathcal{O}$ via $g \mapsto g \cdot f$)

(also need mild condition: X is "normal")

X non-singular variety $\Rightarrow \text{CaCl}(X) \cong \text{Cl}(X) \cong \text{Pic}(X)$ e.g. get \mathbb{Z} for \mathbb{P}^n (more generally, if local rings are UFD.)

For X singular it can fail: $X = \text{Spec } k[x, y, z]/(xy - z^2) \subseteq \mathbb{A}_k^3$ has $\text{CaCl}(X) = 0$ but $\text{Cl}(X) = \mathbb{Z}/2$ generated by the hypersurface $Z = (y = z = 0)$. (At $0 \in Z$ we really need 2 equations to cut out Z , one is not enough, so not locally principal.)

Rmk in a UFD, height 1 prime ideals are principal, so asking that local rings are UFD ensures Weil divisors are locally cut out by one equation, hence Cartier.

Cultural Remark: Riemann-Roch Theorem (non-examinable)

C projective non-singular algebraic curve / alg. closed field k
 $F = \mathcal{O}_C(D)$ for divisor D of degree d (dim(global sections) often written $l(D)$)

$$\chi(C, F) := \sum (-1)^m \dim \check{H}^m(C, F) = h^0(F) - h^1(F) = \deg D + \chi(C, \mathcal{O}_C)$$

$= d + 1 - g$ $\leftarrow h^m = \dim_k \check{H}^m(C, \cdot)$ $\leftarrow \sum n_i$ if $D = \sum n_i p_i$ $\leftarrow \chi(C, \mathcal{O}_C) = 1 - \text{genus}(C)$ (usual genus if $k = \mathbb{C}$ so Riemann surface)

8.9 Čech cohomology computations on \mathbb{P}^n

Recall the key example in Sec. 8.8:

$$\mathbb{P}^n = \bigcup_{i=0}^n U_i \quad \text{where } U_i = D_{x_i} = \text{Spec } \mathbb{Z} \left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right] \cong \mathbb{A}^n$$

Line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ for $d \in \mathbb{Z}$ has:

$$\Gamma(U_i, \mathcal{L}) = \left(\mathbb{Z} [x_0, \dots, x_n] \left[\frac{1}{x_i} \right] \right)_d \leftarrow \text{so poly. in } x\text{'s of degree } N+d \text{ any } N \geq 0.$$

example: $d=0$ gives $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}$ and $\Gamma(U_i, \mathcal{L}) = \left\{ \frac{p(x)}{x_i^N} : p \in \mathbb{Z}[x_0, \dots, x_n], \deg p = N, N \geq 0 \right\}$ ← the classical functions on U_i , well-defined when rescale homogeneous coords.

Use ordered Čech cohomology using obvious ordering of $i \in \{0, 1, \dots, n\}$.

$$\Gamma(U_{i_0 \dots i_k}, \mathcal{L}) = \left(\mathbb{Z} [x_0, \dots, x_n] \left[\frac{1}{x_{i_0} \dots x_{i_k}} \right] \right)_d$$

← $(U_{i_0 \dots i_k} = U_{i_0} \cap \dots \cap U_{i_k})$
 $0 \leq i_0 < \dots < i_k \leq n$

Warm-up example $\check{H}^1(\mathbb{P}^2, \mathcal{L}) = 0$

Proof $c_{ij} \in \check{C}^1$ is \mathbb{Z} -combo of terms $\frac{x_0^{m_0} x_1^{m_1} x_2^{m_2}}{(x_i x_j)^N}$ where total degree $\sum m_i - 2N = d$

c cocycle $\Rightarrow (dc)_{012} = 0 \stackrel{\star}{=} c_{12} - c_{02} + c_{01} \in \Gamma(U_{012}, \mathcal{L})$ ← (e.g. $c_{12} \in \Gamma(U_{12}, \mathcal{L})$ and we restrict to U_{012})

Want to show cocycle c is a coboundary i.e. $\exists b_i \in \Gamma(U_i, \mathcal{L}), (db)_{ij} = b_j - b_i = c_{ij}$.

Want $b_i \in \Gamma(U_i, \mathcal{L})$ so only x_i denominators allowed.

(e.g. $\frac{x_1^2 x_3}{x_1 x_2} = \frac{x_1 x_3}{x_2}$)

Key observation: c_{12} cannot have both x_1, x_2 arising as a denominator (after simplify) because c_{02} has no x_2 's at denom, c_{01} has no x_1 's at denom.

Expand terms depending on denominators: e.g. $c_{12, x_1} =$ terms of c_{12} which have x_1 denominators

$$\begin{aligned} c_{12} &= c_{12, x_2} + c_{12, x_1} + p_{12} \\ -c_{02} &= -c_{02, x_2} - c_{02, x_0} - p_{02} \\ c_{01} &= c_{01, x_1} + c_{01, x_0} + p_{01} \end{aligned}$$

p_{ij} are leftover terms, so no denoms (so polys of degree d if $d \geq 0$, otherwise zero)

\Rightarrow calling $b_2 = c_{12, x_2}, b_1 = -c_{12, x_1}, b_0 = -c_{02, x_0}$ get $\begin{cases} c_{12} = b_2 - b_1 + p_{12} \\ -c_{02} = -b_2 + b_0 - p_{02} \\ c_{01} = b_1 - b_0 + p_{01} \end{cases}$

\Rightarrow replacing c by $c - db$ remains to consider the case $c_{ij} = p_{ij}$ (no denominators)

Trick 1 Let $q_2 = \alpha, q_0 = q_1 = 0$ then $(dq)_{ij} = q_j - q_i = \begin{cases} \alpha & \text{if } (i,j) = (0,2) \\ \alpha & \text{if } (i,j) = (1,2) \\ 0 & \text{else} \end{cases}$

Taking $\alpha = p_{12}$ we can replace c by $c - dq$ and assume $p_{12} = 0$ (and redefine p_{02} due to)

Whereas for $q_2 = q_1 = 0, q_0 = -\beta$ get $(dq)_{ij} = \begin{cases} \beta & \text{if } (0,2) \\ \beta & \text{if } (0,1) \\ 0 & \text{else} \end{cases}$

Taking $\beta = p_{02}$ replacing c by $c - dq$ we can assume $p_{02} = 0$, so also $p_{01} = 0$, so $c = 0$.

Lemma $\check{H}^1(\mathbb{P}^n, \mathcal{L}) = 0 \quad \forall n \geq 2$ ← $(n=1$ fails because don't have triple overlaps we computed the $n=1$ case in Sec. 8.7)

Proof ← TRY ON YOUR OWN FIRST!
 The first part of proof of $n=2$ case is same: replace $0, 1, 2$ by indices i_0, i_1, i_2 .
 So reduce to case of cocycle $c \in \check{C}^1$ with c_{ij} having no denominators (so polys of degree d when $d \geq 0$)
 Doing Trick 1 now is messy I think, so I'll use another trick first.

Trick 2 $\frac{c_{ij}}{x_0^d} \in (\mathbb{Z}[x_0, \dots, x_n][\frac{1}{x_0}])_0 = \mathbb{Z}[z_1, \dots, z_n] = \text{global sections on } U_0 \cong \mathbb{A}^n$

This is a 1-cocycle on \mathbb{A}^n and we know $\check{H}^1(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = 0$ (by Sec. 8.3 since \mathbb{A}^n affine)
 So $\exists \beta_i \in \mathbb{Z}[z_1, \dots, z_n][\frac{1}{z_i}]$ with $(d\beta)_{ij} = \frac{c_{ij}}{x_0^d}$ for $1 \leq i < j$

Since c_{ij} has no denominators, β_i cannot have any z_i denominator. (drop deg $\neq d$ terms from β_i won't affect)
 Since c_{ij} is homog. of deg = d in the x 's, WLOG β_i is homogeneous of deg = d in z 's
 \Rightarrow Take $b_i = x_0^d \beta_i = \text{homog. deg } d \text{ poly in } x$'s with $(db)_{ij} = c_{ij}$ for $1 \leq i < j$.
 \Rightarrow Replace c by $c - db$, can assume $c_{ij} = 0$ for $i \neq 0$.

Final trick $(dc)_{0ij} = 0 = 0 - c_{0j} + c_{0i}$ so all c_{0i} are the same say = β , so use
 Trick 1 with $q_i = 0$ for $i \neq 0$, $q_0 = -\beta$ then $(dq)_{ij} = \begin{cases} \beta & \text{if } i=0 \\ 0 & \text{if } i \neq 0 \end{cases}$ so $c = dq$. \square

Theorem For $\mathcal{L} = \mathcal{O}(d)$, $d \in \mathbb{Z}$, $n \geq 2$ degree d homog. polys (so $\{0\}$ if $d < 0$)

$$\check{H}^*(\mathbb{P}^n, \mathcal{L}) = \begin{cases} \mathbb{Z}[x_0, \dots, x_n]_d & \text{for } * = 0 \leftarrow \text{Hwk 4, global sections of } \mathcal{O}_{\mathbb{P}^n}(d) \\ 0 & \text{for } 0 < * < n \\ \mathbb{Z} \left\{ \frac{1}{x_0 x_1 \dots x_n} \cdot \frac{1}{x^m} \text{ of total degree } d \text{ and all } m_i \geq 0 \right\} & \text{for } * = n \leftarrow (\mathbb{Z}\{\dots\} \text{ means free } \mathbb{Z}\text{-module with that basis}) \\ 0 & \text{for } * > n \leftarrow \text{no } n+2 \text{ overlaps or higher since } n+1 \text{ sets } U_i \text{ cover} \end{cases}$$

(same for $\mathbb{P}^n_{\mathbb{R}}$ if replace \mathbb{Z} by a ring R)

Proof $0 < * = k < n$ is same as for \check{H}^1 : exercise for you.

(Hint: $\pm b_{i_0 \dots \hat{i}_a \dots \hat{i}_b \dots i_k} = \text{terms in } c_{i_0 \dots \hat{i}_a \dots \hat{i}_b \dots i_k$ with no x_{i_b} at denominator
 notice those must cancel with similar terms in $c_{i_0 \dots \hat{i}_a \dots \hat{i}_b \dots i_k$
 Pick sign it has as a term in $(db)_{i_0 \dots \hat{i}_a \dots \hat{i}_b \dots i_k} \leftarrow \text{since want this to give } c_{i_0 \dots \hat{i}_a \dots \hat{i}_b \dots i_k$)

Case $* = n$: only one possible overlap: $U_{012\dots n}$, any chain $c \in \check{C}^n$ is cocycle since no higher overlaps. Question becomes what are possible $(db)_{01\dots n}$ for $b_i \in \Gamma(U_{0\dots \hat{i} \dots n}, \mathcal{L})$.

$$(db)_{01\dots n} = \underbrace{b_{12\dots n}}_{\text{no } x_0 \text{ at denom}} - \underbrace{b_{02\dots n}}_{\text{no } x_1 \text{ at denom}} + b_{013\dots n} - \dots$$

so can get all x^m with some $m_i \geq 0$ (i.e. some x_i not in denom.)

$$\begin{aligned} \Rightarrow \check{H}^n &= \mathbb{Z}\{x^m : \sum m_i = d\} / \mathbb{Z}\{x^m : \sum m_i = d, \text{ some } m_i \geq 0\} \\ &\cong \mathbb{Z}\{x^m : \sum m_i = d, \text{ all } m_i < 0\} \\ &= \frac{1}{x_0 \dots x_n} \cdot \mathbb{Z}\left\{ \frac{1}{x^m} : \sum m_i = -d - n - 1, \text{ all } m_i \geq 0 \right\}. \quad \square \end{aligned}$$

Rmk (non-examinable)
 Serre duality thm for \mathbb{P}^n_k :
 $\check{H}^i(\mathbb{P}^n_k, \mathcal{L}) \cong \check{H}^{n-i}(\mathbb{P}^n_k, K \otimes \mathcal{L}^{-1})^*$
 Where $K = \mathcal{O}(-n-1)$, \mathcal{L} line bble
 So explains this "symmetry" (means 0 for $d < 0$)

Exercise deduce the ranks $h^i = \text{rank}_{\mathbb{Z}} \check{H}^i$ are $h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \binom{n+d}{n} & \text{if } i=0 \\ \binom{-d-1}{n} & \text{if } i=n \\ 0 & \text{else} \end{cases}$
 (means 0 for $d \geq -n$)

Motivation for chapter 9: Now that we know $\check{H}^*(\mathbb{P}^n, \mathcal{O}(d))$, one might hope to compute $\check{H}^*(\mathbb{P}^n, F)$ for other $F \in \text{Coh}(\mathbb{P}^n)$ by first finding a resolution $\dots \rightarrow \mathcal{L}_2 \rightarrow \mathcal{L}_1 \rightarrow F \rightarrow 0$ with $\mathcal{L}_i = \bigoplus \mathcal{O}(d_{ij})$ and exploiting LES's.

8.10 Product on Čech cohomology (Non-examinable section)

(X, \mathcal{O}_X) any ringed space

$$\begin{aligned} \check{H}^p_{\{U_i\}}(X, F) \times \check{H}^q_{\{U_i\}}(X, G) &\longrightarrow \check{H}^{p+q}_{\{U_i\}}(X, F \otimes_{\mathcal{O}_X} G) \\ ((s_I), (t_I)) &\longmapsto (s_I \otimes t_I) \end{aligned}$$

[using $F = G = \mathbb{R}$ for $\mathcal{O}_X = \text{smooth real functions}$ so $\mathbb{R} \otimes_{\mathcal{O}_X} \mathbb{R} \cong \mathbb{R}$]

Rmk In 8.6 where we took constant coefficients $F = G = \mathbb{Z}$ (note: $\mathbb{Z} \otimes_{\mathcal{O}_X} \mathbb{Z} \cong \mathbb{Z}$) we recover the cup product on singular cohomology (respectively on de Rham cohomology)

9. Sheaf cohomology

9.1 Resolutions

(Reference for more details: Lang, Algebra, Chapter XX §4-6)

Motivation: "represent" an object in an abelian category A by "nicer objects" at the cost of using a chain cx (sec. 1.8)

right resolution of $M \in A$ means an exact sequence $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$ in A

left resolution $\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$, or $P_0 \rightarrow M$

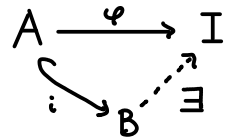
abbreviated as $M \rightarrow I^\bullet$

Def I injective if $\text{Hom}(\cdot, I)$ exact

P projective if $\text{Hom}(P, \cdot)$ exact

\Rightarrow (both always left exact)

Exercise I injective is equivalent to: $\forall \text{inj } A \xrightarrow{i} B$ can "extend" $\varphi: A \rightarrow I$



Fact Injective resolution $M \rightarrow I^\bullet$ means I^n are injective

projective resolution $P_\bullet \rightarrow M$ " P_n " projective

$f, g: A \rightarrow B$ additive functors of abelian cats (see 1.7)

f left exact \Rightarrow right-derived functor

$$R^n f(M) = H^n(f(I^\bullet))$$

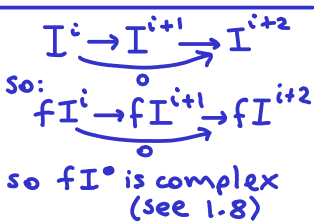
$M \rightarrow I^\bullet$ inj. res.

see 1.8

g right exact \Rightarrow left-derived functor

$$L_n g(M) = H_n(g(P_\bullet))$$

$P_\bullet \rightarrow M$ proj. res.



Later will see why choice of I^\bullet, P_\bullet does not matter.

$$\ker(fI^i \rightarrow fI^{i+1}) \cong \text{Im}(fM \rightarrow fI^i)$$

Warning f left exact only implies $0 \rightarrow fM \rightarrow fI^0 \rightarrow f(\text{Im}(I^0 \rightarrow I^1)) \rightarrow 0$ exact. Deduce: $R^0 f(M) \cong fM$
Similarly $L_0 g \cong g$, so $R^0 f, L_0 g$ remember the functors f, g .

Classical Examples

$A = S\text{-Mods}$, $f = \text{Hom}(M, \cdot)$

$N \rightarrow I^\bullet$ inj. res.

$$\Rightarrow \text{Ext}_S^n(M, N) = (R^n f)(N) = H^n(\text{Hom}_S(M, I^\bullet)) \quad (\text{Ext}_S^0(M, N) \cong \text{Hom}_S(M, N))$$

(Similarly: $f = \text{Hom}(\cdot, N): S\text{-Mods}^{\text{op}} \rightarrow \text{Ab}$, $\text{Ext}_S^n(M, N) = (R^n f)(M) = H_n(\text{Hom}(P_\bullet, N))$
 $P_\bullet \rightarrow M$ proj. res.

$$g = M \otimes_S \cdot \text{ right exact} \Rightarrow \text{Tor}_S^n(M, N) = (L_n g)(N) = H_n(M \otimes_S P_\bullet) \quad (\text{Tor}_S^0(M, N) \cong M \otimes_S N)$$

(Similarly: $g = \cdot \otimes_S N$, $\text{Tor}_S^n(M, N) = (L_n g)(M) = H_n(P_\bullet \otimes_S N)$ for $P_\bullet \rightarrow M$ proj. res.)

For R -mods: I injective \Leftrightarrow if $I \subseteq$ any mod M then \exists mod $J: I \oplus J = M$ (compare linear algebra "extending a basis")
 P projective $\Leftrightarrow P$ is a direct summand of a free R -mod

Fact $M \rightarrow I^\bullet$ inj. res., $N \rightarrow J^\bullet$ inj. res., \downarrow morph \Rightarrow can extend $M \rightarrow I^\bullet \rightarrow N \rightarrow J^\bullet$ and any 2 choices $\Rightarrow f(M) \rightarrow H^*(f(I^\bullet))$ and $f(N) \rightarrow H^*(f(J^\bullet))$ are chain homotopic $\Rightarrow \exists!$

Key idea I inj $\Rightarrow \text{Hom}(\cdot, I)$ right exact \Rightarrow if $A \xrightarrow{\text{mono}} B$ then any $A \rightarrow I$ can be extended to $B \rightarrow I$. E.g. $M \hookrightarrow I^0 \Rightarrow M \hookrightarrow I^0$
then consider $\text{Coker}(M \hookrightarrow I^0) \hookrightarrow I^1$ and continue inductively. Try proving the rest.

Cor 1) $R^n f(M) = H^n(fI^\bullet)$ independent of choice of inj. res. $M \rightarrow I^\bullet$
2) $M \rightarrow N$ induces $R^n f(M) \rightarrow R^n f(N)$, indeed $R^n f: A \rightarrow A$ is functor.

Pf 1) Apply fact to $M=N$, get $H^*(fI^\bullet) \rightarrow H^*(fJ^\bullet) \rightarrow H^*(fI^\bullet)$ composite is id by uniqueness.
2) By Fact, $R^n f(M) = H^n(fI^\bullet) \rightarrow H^n(fJ^\bullet) = R^n f(N)$. Exercise: check functor. \square

Lemma f left exact, $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ SES $\Rightarrow \exists$ canonical & functorial LES

$$0 \rightarrow R^0 f(M_1) \rightarrow R^0 f(M_2) \rightarrow R^0 f(M_3) \rightarrow R^1 f(M_1) \rightarrow R^1 f(M_2) \rightarrow R^1 f(M_3) \rightarrow R^2 f(M_1) \rightarrow \dots$$

$\begin{matrix} \parallel & \parallel & \parallel & & & & \\ fM_1 & fM_2 & fM_3 & & & & \end{matrix}$

Sketch Pf

$$0 \rightarrow I_1^\circ \rightarrow I_2^\circ = I_1^\circ \oplus I_3^\circ \rightarrow I_3^\circ \rightarrow 0$$

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

where these triples are just $R^n f$ applied to the SES
 first pick inj. res. I_1°, I_3° then define I_2° that way so get obvious SES.

use obvious map $M_2 \rightarrow M_3 \rightarrow I_3^\circ$ and $M_1 \hookrightarrow I_1^\circ$ extends via $M_1 \rightarrow M_2$ to $M_2 \rightarrow I_1^\circ$
 Exercise: $M_2 \hookrightarrow I_2^\circ = I_1^\circ \oplus I_3^\circ$ is injective.
 Then take cokernels $M_i' = \text{Coker}(M_i \rightarrow I_i^\circ)$, check that $0 \rightarrow M_1' \rightarrow M_2' \rightarrow M_3' \rightarrow 0$ exact, and repeat construction.

(Fact additive functors preserve \oplus)

$$\Rightarrow 0 \rightarrow fI_1^\circ \rightarrow fI_2^\circ = fI_1^\circ \oplus fI_3^\circ \rightarrow fI_3^\circ \rightarrow 0$$

$$0 \rightarrow fM_1 \rightarrow fM_2 \rightarrow fM_3 \rightarrow 0$$

f may only be left exact, but here clearly fI_2° surjects onto fI_3° since have projection onto fI_3° summand.

Finally take the LES associated to the SES of complexes $0 \rightarrow fI_1^\circ \rightarrow fI_2^\circ \rightarrow fI_3^\circ \rightarrow 0$. \square

Rmk Indeed $R^* f$ satisfies universal property that " $R^* f = f$ and Lemma holds," then it follows that $R^* f(M) = H^*(f(I^\circ))$ for any inj. res. $M \rightarrow I^\circ$ (see end of next section)

Hwk 4 $\text{Ab}(X)$ has enough injectives i.e. can build inj. resolutions of any object $F \in \text{Ab}(X)$.

$\Gamma(X, \cdot): \text{Ab}(X) \rightarrow \text{Ab}$ left exact \Rightarrow can define sheaf cohomology $H^n(X, F) = R^n \Gamma(X, F)$ (Sec. 1.9)

We now ask how this relates to $\check{H}^n(X, F)$ for $F \in \text{Coh}(X) \subseteq \text{Ab}(X)$ and X scheme.

9.2 Acyclic resolutions (in an abelian cat.)

Rmk If I inj. object \Rightarrow resolution $0 \rightarrow I \xrightarrow{\text{id}} I^\circ = I \rightarrow 0 \rightarrow 0 \rightarrow \dots \Rightarrow R^n f(I) = 0 \quad \forall n \geq 1$
 So for sheaf cohomology: $H^n(X, I) = 0 \quad \forall n \geq 1$ if I injective sheaf.

Def An acyclic resolution of F is an exact sequence $0 \rightarrow F \rightarrow J^0 \rightarrow J^1 \rightarrow \dots$ with $H^n(X, J^k) = 0 \quad \forall n \geq 1$ (so we weakened the condition of being an inj. resolution)

Claim Any acyclic resolution can be used to compute sheaf cohomology, i.e.

$$H^n(X, F) = \text{cohomology of chain complex } \Gamma(X, J^0) \rightarrow \Gamma(X, J^1) \rightarrow \dots$$

Pf Trick "break down into SES and take LES":

Let $C_1 = \text{Coker}(F \rightarrow J_0) \cong \text{Im}(J_0 \rightarrow J_1)$ so \exists natural monomorph. $C_1 \hookrightarrow J_1$
 $C_{n+1} = \text{Coker}(C_n \rightarrow J_n) \cong \text{Im}(J_n \rightarrow J_{n+1})$ " " $C_{n+1} \hookrightarrow J_{n+1}$

$$\left. \begin{matrix} 0 \rightarrow F \rightarrow J_0 \rightarrow C_1 \rightarrow 0 \\ 0 \rightarrow C_1 \rightarrow J_1 \rightarrow C_2 \rightarrow 0 \\ 0 \rightarrow C_n \rightarrow J_n \rightarrow C_{n+1} \rightarrow 0 \end{matrix} \right\} \text{exact, and } 0 \rightarrow F \rightarrow J_0 \rightarrow J_1 \rightarrow J_2 \rightarrow J_3 \rightarrow \dots$$

$\begin{matrix} \searrow & \nearrow & \searrow & \nearrow & \dots \\ & C_1 & & C_2 & \dots \end{matrix}$

Trick If $G_1, G_2 \in \text{QCoh } X$, $H^n(X, G_i) = 0 \ \forall n \geq 1 \Rightarrow G_1 \oplus G_2$ also, since:

$$0 \rightarrow G_1 \rightarrow G_1 \oplus G_2 \rightarrow G_2 \rightarrow 0 \text{ SES} \stackrel{(iii)}{\Rightarrow} \text{take LES get } H^n(X, G_1 \oplus G_2) = 0, \ n \geq 1 \checkmark$$

$0 \rightarrow F \rightarrow J^\bullet$ exact \Leftrightarrow exact on stalks $\Leftrightarrow 0 \rightarrow \Gamma(U, F) \rightarrow \Gamma(U, J^\bullet)$ exact \forall affine open U

$$0 \rightarrow \Gamma(U, F) \rightarrow \Gamma(U, J_0) \rightarrow \Gamma(U, J_1) \rightarrow \dots$$

exact since $\Gamma(U, \cdot)$ left exact (Sec. 1.9)

exact since $\check{H}^n(U, F) = 0$ for $n \geq 1$
 \uparrow for cover $U = \cup U_i$
 since U affine, using sec. 8.3 \square

stronger than quasi-compact

Cor X separated, Noetherian \Rightarrow sheaf cohomology $H^n(X, F) \cong \check{H}^n(X, F) \ \forall F \in \text{QCoh}(X)$

Non-examinable

Pf Sheaf cohomology $H(X, F) =$ cohomology of $\Gamma(X, I^0) \rightarrow \Gamma(X, I^1) \rightarrow \dots$ for $F \rightarrow I^\bullet$ any injective resolution.

i) $\Gamma(X, \cdot)$ left exact $\Rightarrow H^0(X, F) \cong \Gamma(X, F)$

iii) Lemma in 9.1 proves \exists LES

ii) by the Theorem below. \square

general consequence see 9.1, or explicitly:
 $0 \rightarrow \Gamma(X, F) \rightarrow \Gamma(X, I^0) \rightarrow \Gamma(X, I^1)$
 exact, so $\text{im } \Gamma(X, F) \rightarrow \Gamma(X, I^0)$ is ker of $\Gamma(X, I^0) \rightarrow \Gamma(X, I^1)$ which is H^0

Theorem R Noeth., $F \in \text{QCoh}(\text{Spec } R) \Rightarrow H^n(\text{Spec } R, F) = 0 \ \forall n \geq 1$

Cultural Rmk
Serre's Theorem:
 X Noeth. scheme then:
 X affine $\Leftrightarrow H^n(X, F) = 0 \ \forall n \geq 1 \ \forall F \in \text{QCoh}(X)$

Non-examinable proof ideas The cleanest proof is to build machinery:

- 1) A sheaf F is flasque if all restrictions $F(U) \rightarrow F(V)$ are surjective.
- 2) \forall flasque F on a top. space X , have $H^n(X, F) = 0 \ \forall n \geq 1$ (Hartshorne III.2.5)
- 3) \forall injective R -module I , and R Noeth. $\Rightarrow \tilde{I}$ on $\text{Spec } R$ is flasque (Hartshorne III.3.4)

Cor Flasque resolutions are acyclic by (2), so can be used to compute $H^n(X, F)$ by 9.2

Pf Thm $F \cong \tilde{M}$ for $M = \Gamma(X, F)$ by 7.6. Pick injective resolution of the R -module $M: 0 \rightarrow M \rightarrow I^\bullet$

$\Rightarrow 0 \rightarrow \tilde{M} \rightarrow \tilde{I}^\bullet$ exact, each \tilde{I}^n flasque, so can use this to compute $H^n(X, F)$ by Cor

$\Rightarrow H^n(X, \tilde{M}) = H^n(\Gamma(X, \tilde{I}^\bullet)) = H^n(I^\bullet) \stackrel{\leftarrow n \geq 1}{=} 0$ since I^\bullet exact sequence except in degree 0. \square
 (in deg=0 get M , and $H^0(X, \tilde{M}) = \tilde{M}(X) = M$)

Rmk Injective \mathcal{O}_X -mods are flasque (Hartshorne III.2.4)

Cultural Rmk For any scheme X and sheaf F of abelian groups have $\check{H}^0(X, F) \cong H^0(X, F) = \Gamma(X, F)$ but also in degree 1: $\check{H}^1(X, F) \cong H^1(X, F)$. So for example $\text{Pic}(X) \cong \check{H}^1(X, \mathcal{O}^*) \cong H^1(X, \mathcal{O}^*)$ in 8.7.

9.4 Product on sheaf cohomology

(Non-examinable section) (X, \mathcal{O}_X) any ringed space

Fact \exists product $H^p(X, F) \times H^q(X, G) \rightarrow H^{p+q}(X, F \otimes_{\mathcal{O}_X} G)$

idea $0 \rightarrow F \rightarrow I^\bullet \Rightarrow 0 \rightarrow F \otimes G \rightarrow I^\bullet \otimes J^\bullet$ unfortunately not a resolution
 $0 \rightarrow G \rightarrow J^\bullet$ \leftarrow bi-complex (compare 8.4) with maps $d \otimes \text{id}, \text{id} \otimes d$
 then take total complex: total degree is sum of degrees

rows & cols not exact

need I^\bullet, J^\bullet to be "pure acyclic resolutions" to ensure this \rightarrow

is resolution. Then given any inj. res. $F \otimes G \rightarrow K^\bullet$, the identity $F \otimes G \xrightarrow{\text{id}} F \otimes G$ extends to $I^\bullet \otimes J^\bullet \rightarrow K^\bullet$.

(e.g. degree 2 part is $(I^2 \otimes J^0) \oplus (I^1 \otimes J^1) \oplus (I^0 \otimes J^2)$)

Taking $\Gamma(X, \cdot)$ yields the result. (see key idea under the Fact in 9.1)

10. Qcoh(P^n), GRADED MODULES, PROJ(R)

(Non-examinable chapter)

10.1 Graded modules and Qcoh(P^n)

Def graded ring means a ring R s.t.

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \dots \text{ as abelian groups (so a graded abelian gp graded by } \mathbb{N})$$

$$R_i \cdot R_j \subseteq R_{i+j} \quad \leftarrow \text{Rmk } R_0 \subseteq R \text{ subring since } R_0 \cdot R_0 \subseteq R_0$$

The elements of R_n are called homogeneous elements of degree n

Graded module means R-mod M s.t.

$$M = \dots \oplus M_{-2} \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus M_2 \oplus \dots \text{ as abelian groups (so graded by } \mathbb{Z})$$

$$R_i \cdot M_j \subseteq M_{i+j} \quad \leftarrow \text{(often write } M_0 \text{ to emphasize } \exists \text{ grading)}$$

A morphism of graded R-mods is R-mod hom $M \xrightarrow{\varphi} N$, with $\varphi(M_n) \subseteq N_n \quad \forall n$

From now on: $R = k[x_0, \dots, x_n]$ $R_m =$ homogeneous polys of $\text{deg} = m$ (so $R_0 = k$)

$$X = \mathbb{P}_k^n = A_0 \cup A_1 \cup \dots \cup A_n \text{ for}$$

$$A_i = \text{Spec } k \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] = \text{Spec} \left((k[x_0, \dots, x_n]_{x_i})_0 \right)$$

↑
omit $\frac{x_i}{x_i}$

$$A_i \cap A_j = \text{Spec } k \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j} \right] = \text{Spec} \left((k[x_0, \dots, x_n]_{x_i x_j})_0 \right)$$

Claim \exists exact, full & faithful functor

$$\begin{aligned} \{ \text{graded } R\text{-mods} \} &\longrightarrow \text{Qcoh}(\mathbb{P}^n) \\ M &\longmapsto \tilde{M} \end{aligned}$$

Pf Let $M_i = (M_{x_i})_0$ \leftarrow 0-th graded piece and $M_{ij} = (M_{x_i x_j})_0$

Define $\tilde{M}|_{A_i} = \tilde{M}_i$ these glue since $\tilde{M}_i|_{A_i \cap A_j} \cong \tilde{M}_{ij} \cong \tilde{M}_j|_{A_i \cap A_j}$ \leftarrow using $((M_{x_i})_0)_{\frac{x_i}{x_j}} \cong (M_{x_i x_j})_0$

Exactness is a local condition, so it holds since it holds in affine case.

Full & faithful: $\text{Hom}(\tilde{M}|_{A_i}, \tilde{N}|_{A_i}) = \text{Hom}(\tilde{M}_i, \tilde{N}_i) = \text{Hom}_{(R_{x_i})_0\text{-mods}}((M_{x_i})_0, (N_{x_i})_0)$

this reduces the problem to an exercise in graded R-mods. (omitted here) \square

Warning Not an equivalence of categories because:

Hwk 4 if $M_n = N_n$ for $n > N$ then $\tilde{M} \cong \tilde{N}$

$$\begin{aligned} \text{unlike case from 7.6:} \\ R\text{-Mods} &\simeq \text{Qcoh}(\text{Spec } R) \\ M &\longmapsto \tilde{M} \\ F(M) &\longleftarrow F \end{aligned}$$

Fact If work with graded R-mods "modulo" identifying those which would give rise to "same" \tilde{M} , then get equivalence of categories. So work with $\{R\text{-mods } M\} / \{R\text{-mods } M : \tilde{M} = 0\}$. \star

For $X = \mathbb{P}^n$, $\tilde{M} = 0 \Leftrightarrow M$ is locally nilpotent, i.e. $\forall m \in M, \exists d \text{ s.t. } x_i^d \cdot m = 0 \quad \forall i$.

If M is f.g., then $\tilde{M} = 0 \Leftrightarrow M$ is finite dim v.s./k

In reverse direction: $\{ \text{graded } R\text{-mods} \} \longleftarrow \text{Qcoh}(\mathbb{P}^n)$

\leftarrow (o stands for grading $d \geq 0$) $\Gamma_*(F) := \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, F(d)) \longleftarrow F$ where $F(d) = F \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(d)$ \leftarrow called twisting

Fact $F \cong \Gamma_*(F)$

When we mod out by the M with $\tilde{M} = 0$ as in \star , this functor together with the functor of claim define an equivalence of cats.

$\text{Coh}(\mathbb{P}^n)$ corresponds to the f.g. graded modules under the equivalence.

Rmk The preferred representative of M in the quotient \star is the saturation $\Gamma_*(\tilde{M})$ of M .
 Call M a saturated module if $M \cong \Gamma_*(\tilde{M})$. \leftarrow (think of this like a sheafification)

Def $M[d]$ new graded R -mod with $M[d]_i = M_{d+i}$

so shift the module down by m :

	-1	0	1	2	...
M	...	M_0	M_1	M_2	...
$M[1]$...	M_0	M_1

Example $\mathcal{L} := \widetilde{R[d]}$ on \mathbb{P}^n \leftarrow (so $k[x_0, \dots, x_n][d]$)

$\mathcal{L}(A_i) = (R[d]_{x_i})_0 = x_i^d k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] = x_i^d \cdot (R_{x_i})_0$

so line bundle, since on each A_i have $(R_{x_i})_0 \cong \mathcal{L}(A_i)$, $1 \mapsto x_i^d$.
 Note $\mathcal{O}_{\mathbb{P}^n}(A_i) = (R_{x_i})_0$ (see box at top of page) and $\mathcal{L}|_{A_i} = \widetilde{\mathcal{L}(A_i)}$, $\mathcal{O}_{\mathbb{P}^n}|_{A_i} = \widetilde{\mathcal{O}_{\mathbb{P}^n}(A_i)} \Rightarrow \mathcal{O}_{\mathbb{P}^n}|_{A_i} \cong \mathcal{L}|_{A_i}$

line bundle with $\alpha_{ij} = (x_i/x_j)^d$. Hence $\mathcal{L} = \mathcal{O}(d)$.

$\mathcal{O}_{\mathbb{P}^n}|_{A_{ij}} \cong \mathcal{L}|_{A_{ij}} = \mathcal{L}|_{A_{ji}} \cong \mathcal{O}_{\mathbb{P}^n}|_{A_{ji}}$, $f \mapsto x_i^d f \mapsto x_j^{-d} x_i^d f$

Exercise $\widetilde{M[d]} \cong \tilde{M}(d) (= \tilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(d))$ \leftarrow (e.g. $\widetilde{R[d]} = \tilde{R}(d) = \mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}(d) = \mathcal{O}(d)$)

Rmk $k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ (but this does not generalise due to above issue about cats)

The construction of \tilde{M} is so similar to the $\text{Spec } R$ case of \tilde{M} , because \exists analogue of $\text{Spec } R$: Proj R .

10.2 Proj(R) and Qcoh(Proj R)

$\text{Proj}(R) = \{ \text{graded prime ideals } I \subseteq R \text{ not containing the irrelevant ideal} \}$

R any graded ring

means $I = \bigoplus_{n \geq 0} (I \cap R_n)$

(\Leftrightarrow generated by homogeneous elts)

$R_+ := \bigoplus_{n > 0} R_n$

in \mathbb{P}^n we remove the max ideal (x_0, \dots, x_n) (irrelevant ideal) because don't allow the closed point $[0: \dots : 0]$

$\mathbb{V}(I) = \{ p \in \text{Proj } R : p \supseteq I \}$ define closed sets of Zariski topology

f homogeneous of degree $> 0 \Rightarrow D_f = \text{Proj } R \setminus \mathbb{V}(f) = \{ p \in \text{Proj } R : f \notin p \}$ basis of open sets

Warning $\text{Proj } R = \bigcup D_f \Leftrightarrow R_+ \subseteq \sqrt{\langle \text{all } f_i \rangle}$ \leftarrow (example: $\mathbb{P}^n = D_{x_0} \cup \dots \cup D_{x_n}$ and $(x_0, \dots, x_n) = k[x_0, \dots, x_n]_+$)

Fact $D_f \cong \text{Spec}((R_f)_0)$ as topological spaces

$p \mapsto p R_f \cap (R_f)_0$ (inverse map: $p_0 \mapsto \bigoplus_{k \geq 0} \{ a_k \in R_k : \frac{a_k}{f^k} \in p_0 \}$)

Sheaf $\mathcal{O} := \mathcal{O}_{\text{Proj}(R)}$:

$\mathcal{O}|_{D_f} = \mathcal{O}_{\text{Spec}((R_f)_0)}$ then glue. \leftarrow (on $D_{fg} = D_f \cap D_g$ get $\mathcal{O}_{\text{Spec}((R_{fg})_0)}$)

Warning Proj is not functorial like Spec

If $\varphi: R \rightarrow S$ graded hom of rings, $\varphi(R_+) \supseteq S_+$ then get morph $\varphi^\#: \text{Proj } S \rightarrow \text{Proj } R$
 but not all morphs arise in this way. $I \mapsto \varphi^{-1}(I)$

more generally, suffices $\sqrt{\varphi(R_+) \cdot S} = S_+$

Examples

- 1) $S = R[x_0, \dots, x_n]$ with usual grading $\Rightarrow \text{Proj } R = \mathbb{P}_R^n$ (or $\mathbb{P}_{\text{Spec } R}^n$)
- 2) $R^{(d)} := \bigoplus_{n \geq 0} R_{d-n}$ then the inclusion $R^{(d)} \rightarrow R$ induces an iso $\text{Proj } R \cong \text{Proj } R^{(d)}$

- 3) S graded ring generated as an S_0 -algebra by $n+1$ elements $s_0, \dots, s_n \in S_1$
 $\Rightarrow S_0[x_0, \dots, x_n] \xrightarrow{\varphi} S \Rightarrow S \cong S_0[x_0, \dots, x_n] / \underbrace{\text{Ker } \varphi}_{\text{call this } I} \Rightarrow \text{Proj } S \cong \mathbb{V}(I) \subseteq \mathbb{P}_{S_0}^n$
 closed subscheme

Example $k[x, y]^{(2)} = k[x^2, xy, y^2]$

$k[X, Y, Z] \twoheadrightarrow k[x^2, xy, y^2], X \mapsto x^2, Y \mapsto xy, Z \mapsto y^2$

$\Rightarrow \mathbb{P}^1 = \text{Proj } k[x, y] \cong \text{Proj } k[x, y]^{(2)} \cong \text{Proj } k[X, Y, Z] / (XZ - Y^2)$ closed subscheme of \mathbb{P}^2

is the Veronese embedding $v_2: \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. Similarly get $v_d: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$

$N = \#$ degree d monomials in x_0, \dots, x_n
 so $N = \binom{n+d}{d}$

- 4) every closed subscheme of $\text{Proj } R$ arises as $\text{Proj } (R/I)$ some graded ideal I .

Fact $R = \bigoplus_{n \geq 0} R_n$ graded ring \Rightarrow get line bundles $\mathcal{O}(d) = \widetilde{R}_d$ on $\text{Proj } R$, and

\exists exact, full & faithful functor

$$\begin{aligned} \{\text{graded } R\text{-mods}\} &\longrightarrow \text{QCoh}(\text{Proj } R) \\ M &\longmapsto \widetilde{M} \\ \Gamma_0(F) &\longleftarrow F \end{aligned}$$

Note: this tells us $\text{QCoh}(\cdot) \forall$ proj. variety!

\widetilde{M} built by gluing as in 10.1 namely
 $\widetilde{M}(D_f) = M_{(f)}$ is homogeneous localization at f
 (so localize at f and take 0-th graded part)
 stalk $\widetilde{M}_I = M_{(I)}$ = homogeneous localization at the homog. prime ideal I
 = 0-th graded part of M_I

where $\Gamma_d(F) := \Gamma(\text{Proj } R, F(d)) \leftarrow (F(d) = F \otimes_{\mathcal{O}_X} \mathcal{O}(d) \text{ and } \mathcal{O}_X = \widetilde{R} \text{ on } X = \text{Proj } R)$

again, not an equivalence of cats, but $\Gamma_0(F) \cong F$ and the two functors define an equivalence of cats if we work with saturated graded R -mods ($M \cong \Gamma_0(\widetilde{M})$)

Fact If R_0 Noetherian, R generated as R_0 -algebra by finitely many elts $\in \underline{R}_1$

Example:
 $R = k[x_0, \dots, x_n] / I$
 then $x_0, \dots, x_n \in R_1$ generate.

then $\otimes \{ \text{f.g. } R\text{-mods} \} / \{ \text{f.g. "torsion" } R\text{-mods} \} \longrightarrow \text{Coh}(\text{Proj } R)$ is equiv. of cats.

$M \longmapsto \widetilde{M}$ and quasi-inverse $\Gamma_0(F) \longleftarrow F$

Here "torsion" means $\forall m \in M, \exists N \in \mathbb{N}: (R_+)^N \cdot m = 0$. For M f.g. A -mod: this holds $\iff M_k = 0$ for large k
 So \otimes same as working with f.g. R -mods modulo identifying those that "agree" in large degrees.

Exercise M "torsion" $\implies M_f = 0 \forall$ homogeneous $f \in R_+ \implies \widetilde{M}(D_f) = M_{(f)} = 0 \implies \widetilde{M} = 0$.

(homogeneous localisation at f)

Now assume only R Noeth graded ring.

Exercise Show R_0 Noeth, and R generated as R_0 -alg. by finitely many $f_1, \dots, f_a \in R$.

Let $d := \text{lcm}(\text{deg } f_i)$. Call homogeneous $m \in M$ irrelevant if $(R_+ \cdot m)_{N \cdot d} = 0$ for all large N .

M called irrelevant if all are irrelevant. Fact \otimes holds if replace "torsion" by "irrelevant".