

B2.2 Commutative Algebra

Sheet 1 — HT26

Sections 1-5

Section A

1. Let R be a ring. Show that the Jacobson radical of R coincides with the set

$$\{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}.$$

2. Let R be a ring and let $S \subseteq R$ be a multiplicative subset. Let M be an R -module and suppose that for every $s \in S$ the map

$$[s]_M: M \rightarrow M, m \mapsto sm$$

is an isomorphism. Then there is a unique structure of an R_S -module on M such that $(r/1)m = rm$ for all $m \in M$ and $r \in R$.

Moreover, if $r/s \in R_S$ then we necessarily have $(r/s)(m) = [s]_M^{-1}(rm)$.

3. Let R be a ring and let N be the nilradical of R . Then the nilradical of $R[x]$ is $N[x]$.

Section B

4. Let R be a ring.
 - (a) Show that if $P(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geq 1$.
 - (b) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.
5. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:
 - (a) R has exactly one prime ideal.
 - (b) Every element of R is either a unit or is nilpotent.
 - (c) R/N is a field.
6. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S = \{1 + r \mid r \in I\}$.
 - (a) Show that S is a multiplicative set.
 - (b) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S .
 - (c) Prove the following generalisation of Nakayama's lemma:
Lemma. *Let M be a finitely generated R -module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and $rM = 0$.*
7. Let R be a ring and let M be a finitely generated R -module. Let $\phi: M \rightarrow M$ be a surjective homomorphism of R -modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$ -module on M and use the previous question.]
8. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ if and only if all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero-divisors of R is a union of prime ideals.

Section C

9. Let R be a ring. Consider the inclusion relation on the set $\text{Spec}(R)$. Show that there are minimal elements in $\text{Spec}(R)$.
10. Let R be a ring. The chain complex of R -modules

$$\cdots \rightarrow M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is exact if and only if the complex

$$\cdots \rightarrow M_{i,\mathfrak{m}} \xrightarrow{d_{i,\mathfrak{m}}} M_{i+1,\mathfrak{m}} \xrightarrow{d_{i+1,\mathfrak{m}}} \cdots$$

is exact for all the maximal ideals $\mathfrak{m}$ of R .