B2.2 Commutative Algebra

Sheet 2 — HT26

Sections 1-7

Section A

1. Consider the ideals $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$ and $\mathfrak{m} = (x, y, z)$ of $K[x, y, z]$, where K is a field. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$. Determine the isolated and the embedded prime ideals of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$.

2. Let I be a radical ideal that is decomposible. Show that I has a minimal primary decomposition by prime ideals (so that in this case, the associated primes are the elements of the minimal primary decomposition itself).

   Furthermore, show that any two minimal primary decompositions by prime ideals of a radical ideal coincide.

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Section B

3. Let K be a field. Show that the ideal $(x^2, xy, y^2) \subseteq K[x, y]$ is a primary ideal, which is not irreducible.

4. (a) If I is a decomposable radical ideal, then all the associated primes of I are isolated.

   (b) if I is a decomposable ideal, there are only finitely many prime ideals, which contain I and are minimal among all the prime ideals containing I. These prime ideals are also the isolated ideals associated with I.

5. Let R be a ring. Let $I \subseteq R$ be an ideal. Then there are prime ideals that are minimal among all the prime ideals containing I.

   Furthermore, if $\mathfrak{p} \supseteq I$ is a prime ideal, then $\mathfrak{p}$ contains such a prime ideal.

6. Let R be a ring. Let $\mathcal{S}$ be the set of ideals in R that are not finitely generated; assume that $\mathcal{S} \neq \emptyset$.

   (a) Show that $\mathcal{S}$ has at least one maximal element.

   (b) Let I be maximal element of $\mathcal{S}$ (with respect to the relation of inclusion). Show that I is prime.

   (c) Suppose that all the prime ideals of R are finitely generated. Prove that R is noetherian.

   [Hint: exploit the fact that R/I is noetherian.]

7. Let R be a noetherian ring and $I \subseteq R$ an ideal. Then the quotient ring R/I is noetherian.

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Section C

8. Let R be a ring. Let $\mathcal{S}$ be the set of non-principal ideals in R; assume that $\mathcal{S} \neq \emptyset$. Prove that $\mathcal{S}$ admits maximal elements, and that every such element a prime ideal.

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