B2.2 Commutative Algebra

Sheet 3 — HT26

Sections 1-10

Section A

1. Let R be a subring of a ring T. Suppose that T is integral over R. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$. Show that if $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.

2. Let R be a subring of a ring T and suppose that T is integral over R. Let $\mathfrak{p}$ be prime ideal of R and let $\mathfrak{q}$ be a prime ideal of T. Suppose that $\mathfrak{q} \cap R = \mathfrak{p}$. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k$ be primes ideal of R and suppose that $\mathfrak{p}_1 = \mathfrak{p}$. Show that there are prime ideals $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_k$ of T such that $\mathfrak{q}_1 = \mathfrak{q}$ and such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all $i \in \{1, \ldots, k\}$.

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Section B

3. Show that $\mathbb{Z}$ is integrally closed and that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$.

4. Let A be a ring and let $U(x) \in A[x]$ be a non zero monic polynomial. Then there exists a ring B containing A, which is integral over A and such that
$$U(x) = \prod_{i=1}^{\deg(U)} (x - b_i)$$
for some $b_i \in B$, where we set $\prod_{i=1}^{\deg(U)}(x - b_i) = 1$ if $\deg(U) = 0$.

5. Let S be a ring and let $R \subseteq S$ be a subring of S. Suppose that R is integrally closed in S. Let $P(x) \in R[x]$ and suppose that $P(x) = Q(x)J(x)$, where $Q(x), J(x) \in S[x]$ and $Q(x)$ and $J(x)$ are monic. Show that $Q(x), J(x) \in R[x]$. Use this to give a new proof of the fact that if $T(x) \in \mathbb{Z}[x]$ and $T(x) = T_1(x)T_2(x)$, where $T_1(x), T_2(x) \in \mathbb{Q}[x]$ are monic polynomials, then $T_1(x), T_2(x) \in \mathbb{Z}[x]$.

6. Let R be a ring. Show that the two following conditions are equivalent:
   (a) R is a Jacobson ring.
   (b) If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

   Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \ldots$.

7. Let k be field and let R be a finitely generated algebra over k. Show that the two following conditions are equivalent:
   (a) $\mathrm{Spec}(R)$ is finite.
   (b) R is finite over k.

8. Let k be an algebraically closed field. Let $P_1, \ldots, P_d \in k[x_1, \ldots, x_d]$. Suppose that the set
$$\{(y_1, \ldots, y_d) \in k^d \mid P_i(y_1, \ldots, y_d) = 0 \, \forall i \in \{1, \ldots, d\}\}$$
is finite. Show that
$$\mathrm{Spec}(k[x_1, \ldots, x_d]/(P_1, \ldots, P_d))$$
is finite.

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Section C

9. Let R be a noetherian ring and let T be a finitely generated R-algebra. Let G be a finite subgroup of the group of automorphisms of T as a R-algebra. Let T^G be the fixed point set of G (ie the subset of T, which is fixed by all the elements of G).
   (a) Show that T is integral over T^G.
   (b) Show that T^G is a subring of T, which contains the image of R and that T^G is finitely generated over R.

10. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$ such that $\mathfrak{m} = (P_1, \ldots, P_d)$.

11. Let R be a domain. Show that $R[x]$ is integrally closed if R is integrally closed.

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