B2.2 Commutative Algebra

Sheet 4 — HT26

Sections 1-11

Section A

1. Let R be a noetherian domain. Let $\mathfrak{m}$ be a maximal ideal in R. Let $r \in R \setminus \{0\}$ and suppose that (r) is an $\mathfrak{m}$-primary ideal. Show that $\operatorname{height}((r)) = 1$.

2. Let R be a PID. Show that $\dim R \leqslant 1$, and that $\dim R = 0$ if and only if R is a field.

3. Let R be a noetherian ring. Let $\mathfrak{p}, \mathfrak{p}'$ be prime ideals of R and suppose that $\mathfrak{p} \subset \mathfrak{p}'$. There exists a prime ideal $\mathfrak{q}$ such that $\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{p}'$ and $\mathfrak{q}$ is maximal among prime ideals with this property.

4. Let K be a field and let $\mathfrak{p}$ be a non zero prime ideal of $K[x]$. Then $\operatorname{height}(\mathfrak{p}) = 1$. In particular, we have $\dim(K[x]) = 1$.

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Section B

5. Let R be a ring and let R_0 be the prime ring of R (the image of $\mathbb{Z}$ under the unique ring homomorphism $\mathbb{Z} \to R$). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

6. Let R be an integrally closed domain. Let $K = \mathrm{Frac}(R)$. Let $L|K$ be an algebraic field extension. Show that an element $e \in L$ is integral over R if and only if the minimal polynomial of e over K has coefficients in R.

7. Let R be a PID. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c = c_1 \cdot c_2$. Show that $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ and that the ideals (x, c_i) are prime, as ideals in $R[x]/(c - x^2)$.

8. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$. Show that $\dim(R[x]) \leqslant 1 + 2\dim(R)$.

9. Let A (resp. B) be a noetherian local ring with maximal ideal $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$). Let $\phi\colon A \to B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be 'local').

   Suppose that

   (a) B is finite over A via ϕ;

   (b) the map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;

   (c) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ induced by ϕ is bijective.

   Prove that ϕ is surjective. [Hint: use Nakayama's lemma twice].

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Section C

10. (a) Let R be a noetherian domain. Let I be a proper ideal of R. Then $\bigcap_{n\geqslant 0} I^n = 0$.

    (b) Let R be a noetherian ring and let I be an ideal of R. Let M be a finitely generated R-module. Suppose that I is contained in the Jacobson radical of R. Then $\bigcap_{n\geqslant 0} I^n M = 0$.

11. Let $\phi\colon R \to T$ be a ring homomorphism. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let I be the ideal generated by $\phi(\mathfrak{p})$ in T.

    Write $\psi\colon R/\mathfrak{p} \to T/I$ for the ring homomorphism induced by ϕ and let $S = (R/\mathfrak{p}) \smallsetminus \{0\}$. Write $\psi_S\colon \operatorname{Frac}(R/\mathfrak{p}) \to (T/I)_{\psi(S)}$ for the induced ring homomorphism. Finally, write $\rho\colon T \to (T/I)_{\psi(S)}$ for the natural ring homomorphism.

    (a) Show that $\operatorname{Spec}(\rho)(\operatorname{Spec}((T/I)_{\psi(S)}))$ consists precisely of the prime ideals $\mathfrak{q}$ of T, such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

    (b) Show that the correspondence between

    - prime ideals $\mathfrak{q}$ such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and
    - prime ideals of $(T/I)_{\psi(S)}$

    respects inclusion in both directions.

    (c) Deduce that when $T = R[x]$ we have

$$(T/I)_{\psi(S)} = (R[x]/\mathfrak{p}[x])_{\psi(S)} \simeq (R/\mathfrak{p})[x]_{(R/\mathfrak{p})^*} = \operatorname{Frac}(R/\mathfrak{p})[x].$$

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