B2.2 Commutative Algebra

Sheet 4 — HT26

Sections 1-11

Section A

- 1. Let R be a noetherian domain. Let \mathfrak{m} be a maximal ideal in R. Let $r \in R \setminus \{0\}$ and suppose that (r) is an \mathfrak{m} -primary ideal. Show that height((r)) = 1.
- 2. Let R be a PID. Show that dim $R \leq 1$, and that dim R = 0 if and only if R is a field.
- 3. Let R be a noetherian ring. Let $\mathfrak{p}, \mathfrak{p}'$ be prime ideals of R and suppose that $\mathfrak{p} \subset \mathfrak{p}'$. There exists a prime ideal \mathfrak{q} such that $\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{p}'$ and \mathfrak{q} is maximal among prime ideals with this property.
- 4. Let K be a field and let \mathfrak{p} be a non zero prime ideal of K[x]. Then height(\mathfrak{p}) = 1. In particular, we have $\dim(K[x]) = 1$.

Mathematical Institute, University of Oxford Dawid Kielak: kielak@maths.ox.ac.uk

Section B

- 5. Let R be a ring and let R_0 be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated R_0 -algebra. Suppose also that R is a field. Prove that R is a finite field.
- 6. Let R be an integrally closed domain. Let $K = \operatorname{Frac}(R)$. Let L|K be an algebraic field extension. Show that an element $e \in L$ is integral over R if and only if the minimal polynomial of e over K has coefficients in R.
- 7. Let R be a PID. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c = c_1 \cdot c_2$. Show that $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ and that the ideals (x, c_i) are prime, as ideals in $R[x]/(c-x^2)$.
- 8. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$. Show that $\dim(R[x]) \leq 1 + 2\dim(R)$.
- 9. Let A (resp. B) be a noetherian local ring with maximal ideal \mathfrak{m}_A (resp. \mathfrak{m}_B). Let $\phi \colon A \to B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be 'local').

Suppose that

- (a) B is finite over A via ϕ ;
- (b) the map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;
- (c) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ induced by ϕ is bijective.

Prove that ϕ is surjective. [Hint: use Nakayama's lemma twice].

Mathematical Institute, University of Oxford Dawid Kielak: kielak@maths.ox.ac.uk

Section C

- 10. (a) Let R be a noetherian domain. Let I be a proper ideal of R. Then $\bigcap_{n\geq 0} I^n = 0$.
 - (b) Let R be a noetherian ring and let I be an ideal of R. Let M be a finitely generated R-module. Suppose that I is contained in the Jacobson radical of R. Then $\bigcap_{n\geq 0} I^n M = 0$.
- 11. Let $\phi: R \to T$ be a ring homomorphism. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let I be the ideal generated by $\phi(\mathfrak{p})$ in T.

Write $\psi \colon R/\mathfrak{p} \to T/I$ for the ring homomorphism induced by ϕ and let $S = (R/\mathfrak{p}) \setminus \{0\}$. Write $\psi_S \colon \operatorname{Frac}(R/\mathfrak{p}) \to (T/I)_{\psi(S)}$ for the induced ring homomorphism. Finally, write $\rho \colon T \to (T/I)_{\psi(S)}$ for the natural ring homomorphism.

- (a) Show that $\operatorname{Spec}(\rho)(\operatorname{Spec}((T/I)_{\psi(S)}))$ consists precisely of the prime ideals \mathfrak{q} of T, such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.
- (b) Show that the correspondence between
 - prime ideals \mathfrak{q} such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and
 - prime ideals of $(T/I)_{\psi(S)}$

respects inclusion in both directions.

(c) Deduce that when T = R[x] we have

$$(T/I)_{\psi(S)} = (R[x]/\mathfrak{p}[x])_{\psi(S)} \simeq (R/\mathfrak{p})[x]_{(R/\mathfrak{p})^*} = \operatorname{Frac}(R/\mathfrak{p})[x].$$

Mathematical Institute, University of Oxford Dawid Kielak: kielak@maths.ox.ac.uk