

B2.2 Commutative Algebra

Sheet 4 — HT26

Sections 1-11

Section A

1. Let R be a noetherian domain. Let \mathfrak{m} be a maximal ideal in R . Let $r \in R \setminus \{0\}$ and suppose that (r) is an \mathfrak{m} -primary ideal. Show that $\text{height}((r)) = 1$.
2. Let R be a PID. Show that $\dim R \leq 1$, and that $\dim R = 0$ if and only if R is a field.
3. Let R be a noetherian ring. Let $\mathfrak{p}, \mathfrak{p}'$ be prime ideals of R and suppose that $\mathfrak{p} \subset \mathfrak{p}'$. There exists a prime ideal \mathfrak{q} such that $\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{p}'$ and \mathfrak{q} is maximal among prime ideals with this property.
4. Let K be a field and let \mathfrak{p} be a non zero prime ideal of $K[x]$. Then $\text{height}(\mathfrak{p}) = 1$. In particular, we have $\dim(K[x]) = 1$.

Section B

5. Let R be a ring and let R_0 be the prime ring of R (the image of \mathbb{Z} under the unique ring homomorphism $\mathbb{Z} \rightarrow R$). Suppose that R is a finitely generated R_0 -algebra. Suppose also that R is a field. Prove that R is a finite field.
6. Let R be an integrally closed domain. Let $K = \text{Frac}(R)$. Let $L|K$ be an algebraic field extension. Show that an element $e \in L$ is integral over R if and only if the minimal polynomial of e over K has coefficients in R .
7. Let R be a PID. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c = c_1 \cdot c_2$. Show that $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ and that the ideals (x, c_i) are prime, as ideals in $R[x]/(c - x^2)$.
8. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$. Show that $\dim(R[x]) \leq 1 + 2 \dim(R)$.
9. Let A (resp. B) be a noetherian local ring with maximal ideal \mathfrak{m}_A (resp. \mathfrak{m}_B). Let $\phi: A \rightarrow B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be ‘local’).

Suppose that

- (a) B is finite over A via ϕ ;
- (b) the map $\mathfrak{m}_A \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;
- (c) the map $A/\mathfrak{m}_A \rightarrow B/\mathfrak{m}_B$ induced by ϕ is bijective.

Prove that ϕ is surjective. [Hint: use Nakayama’s lemma twice].

Section C

10. (a) Let R be a noetherian domain. Let I be a proper ideal of R . Then $\bigcap_{n \geq 0} I^n = 0$.
- (b) Let R be a noetherian ring and let I be an ideal of R . Let M be a finitely generated R -module. Suppose that I is contained in the Jacobson radical of R . Then $\bigcap_{n \geq 0} I^n M = 0$.
11. Let $\phi: R \rightarrow T$ be a ring homomorphism. Let $\mathfrak{p} \in \text{Spec}(R)$ and let I be the ideal generated by $\phi(\mathfrak{p})$ in T .

Write $\psi: R/\mathfrak{p} \rightarrow T/I$ for the ring homomorphism induced by ϕ and let $S = (R/\mathfrak{p}) \setminus \{0\}$. Write $\psi_S: \text{Frac}(R/\mathfrak{p}) \rightarrow (T/I)_{\psi(S)}$ for the induced ring homomorphism. Finally, write $\rho: T \rightarrow (T/I)_{\psi(S)}$ for the natural ring homomorphism.

- (a) Show that $\text{Spec}(\rho)$ is injective and that its image consists precisely of the prime ideals \mathfrak{q} of T such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.
- (b) Show that the correspondence induced by $\text{Spec}(\rho)$ between
- prime ideals \mathfrak{q} such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and
 - prime ideals of $(T/I)_{\psi(S)}$

respects inclusion in both directions.

- (c) Now suppose that $T = R[x]$ and that $\phi: R \rightarrow R[x]$ is the obvious map. Show that we have

$$(T/I)_{\psi(S)} = (R[x]/\mathfrak{p}[x])_{\psi(S)} \simeq \text{Frac}(R/\mathfrak{p})[x]$$

and deduce that if $\mathfrak{q}_0 \supset \cdots \supset \mathfrak{q}_k$ is a proper chain of prime ideals in $\text{Spec}(R[x])$ such that $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}$ for all i , then $k \leq 1$.