B2.2 Commutative Algebra

Sheet 1 — HT26

Sections 1-5

Section A

1. Let R be a ring. Show that the Jacobson radical of R coincides with the set

$$\{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}.$$

Solution: Suppose that x lies in the Jacobson radical of R. Suppose for contradiction that $1 - xy$ is not a unit for some $y \in R$. Let $\mathfrak{m}$ be a maximal ideal containing $1 - xy$. We know that $xy \in \mathfrak{m}$ since $x \in \mathfrak{m}$ and thus we conclude that $1 \in \mathfrak{m}$, a contradiction.

Suppose now that $x \in R$ and that there is a maximal ideal $\mathfrak{m}$ not containing x. Then $x + \mathfrak{m}$ is non-trivial in the field $R/\mathfrak{m}$ and hence it is a unit; thus, there is a $y \in R$ such that $xy + \mathfrak{m} = 1 + \mathfrak{m}$. In other words, $1 - xy \in \mathfrak{m}$ and so $1 - xy$ is not a unit.

2. Let R be a ring and let $S \subseteq R$ be a multiplicative subset. Let M be an R-module and suppose that for every $s \in S$ the map

$$[s]_M \colon M \to M,\ m \mapsto sm$$

is an isomorphism. Then there is a unique structure of an R_S-module on M such that $(r/1)m = rm$ for all $m \in M$ and $r \in R$.

Moreover, if $r/s \in R_S$ then we necessarily have $(r/s)(m) = [s]_M^{-1}(rm)$.

Solution: The proof is essentially identical to that of Lemma 5.1.

3. Let R be a ring and let N be the nilradical of R. Then the nilradical of $R[x]$ is $N[x]$.

Solution: Any element of $N[x]$ is a polynomial with nilpotent coefficients and is thus clearly nilpotent (check). On the other hand, let $P(x) = a_0 + a_1 x + \cdots + a_d x^d \in R[x]$ be an element of the nilradical of $R[x]$ (i.e., a nilpotent polynomial). Suppose for contradiction that $P(x)$ has a coefficient a_i, which is not nilpotent. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a prime ideal, such that $a_i \notin \mathfrak{p}$. Then $P(x) + \mathfrak{p}[x] \in (R/\mathfrak{p})[x]$ is a non-zero nilpotent polynomial. This is contradiction, since $(R/\mathfrak{p})[x]$ is a domain.

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Section B

4. Let R be a ring.
   (a) Show that if $P(x) = a_0 + a_1 x + \cdots + a_k x^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geqslant 1$.
   (b) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.

5. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:
   (a) R has exactly one prime ideal.
   (b) Every element of R is either a unit or is nilpotent.
   (c) R/N is a field.

6. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S = \{1 + r \mid r \in I\}$.
   (a) Show that S is a multiplicative set.
   (b) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S.
   (c) Prove the following generalisation of Nakayama's lemma:

   Lemma. *Let M be a finitely generated R-module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and $rM = 0$.*

7. Let R be a ring and let M be a finitely generated R-module. Let $\phi\colon M \to M$ be a surjective homomorphism of R-modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$-module on M and use the previous question.]

8. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ if and only if all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero-divisors of R is a union of prime ideals.

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Section C

9. Let R be a ring. Consider the inclusion relation on the set $\mathrm{Spec}(R)$. Show that there are minimal elements in $\mathrm{Spec}(R)$.

   Solution: Let $\mathcal{T}$ be a totally ordered subset of $\mathrm{Spec}(R)$ for the relation $\supseteq$. Note that the maximal elements for the relation $\supseteq$ are the minimal elements for the inclusion relation (which is $\subseteq$). Let $I = \cap_{\mathfrak{p}\in\mathcal{T}}\mathfrak{p}$. Then I is an ideal. We claim that I is prime.

   To see this, let $x, y \in R$ and suppose for contradiction that $x, y \in R \smallsetminus I$ and that $xy \in I$. By assumption there are prime ideals $\mathfrak{p}_x, \mathfrak{p}_y \in \mathcal{T}$ such that $x \notin \mathfrak{p}_x$ and $y \notin \mathfrak{p}_y$. Suppose without restriction of generality that $\mathfrak{p}_x \supseteq \mathfrak{p}_y$ (recall that $\mathcal{T}$ is totally ordered). We have $xy \in \mathfrak{p}_y$ and thus either x or y lies in $\mathfrak{p}_y$. This contradicts the fact that $x, y \notin \mathfrak{p}_y$. The ideal I thus lies in $\mathrm{Spec}(R)$ and it is a lower bound for $\mathcal{T}$. We may thus apply Zorn's lemma to conclude that there are minimal elements in $\mathrm{Spec}(R)$.

10. Let R be a ring. The chan complex of R-modules

$$\cdots \to M_i \overset{d_i}{\to} M_{i-1} \overset{d_{i-1}}{\to} \cdots$$

   is exact if and only if the complex

$$\cdots \to M_{i,\mathfrak{m}} \overset{d_{i,\mathfrak{m}}}{\to} M_{i+1,\mathfrak{m}} \overset{d_{i+1,\mathfrak{m}}}{\to} \cdots$$

   is exact for all the maximal ideals $\mathfrak{m}$ of R.

   Solution: "$\Rightarrow$": By Lemma 5.8.

   "$\Leftarrow$": We will prove the contrapositive. Suppose that the first chain complex is not exact. Then

$$\ker(d_i)/\mathrm{im}(d_{i+1}) \neq 0$$

   for some $i \in \mathbb{Z}$. Take $a \in \ker(d_i)/\mathrm{im} d_{i+1} \smallsetminus 0$, and let $\mathfrak{m}$ be a maximal ideal containing $\mathrm{Ann}(a)$ (which exists as $1 \notin \mathrm{Ann}(a)$). Then $(\ker(d_{i+1})/\mathrm{im}(d_i))_{\mathfrak{m}} \neq 0$ for otherwise there would be an element $u \in R \smallsetminus \mathfrak{m} \subseteq R \smallsetminus \mathrm{Ann}(a)$ such that

$$u \cdot a = 0,$$

   which is a contradiction.

   By Lemma 5.8, there is a natural isomorphism

$$\ker(d_i)_{\mathfrak{m}}/\mathrm{im}(d_{i+1})_{\mathfrak{m}} \simeq \big(\ker(d_i)/\mathrm{im}(d_{i+1})\big)_{\mathfrak{m}} \not\simeq 0.$$

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