B2.2 Commutative Algebra

Sheet 2 — HT26

Sections 1-7

Section A

1. Consider the ideals $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$ and $\mathfrak{m} = (x, y, z)$ of $K[x, y, z]$, where K is a field. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$. Determine the isolated and the embedded prime ideals of $\mathfrak{p}_1 \cdot \mathfrak{p}_2$.

Solution: For future reference, note that we have

$$\mathfrak{m}^2 = ((x) + (y) + (z))^2 = (x^2, y^2, z^2, xy, xz, yz)$$

and

$$\mathfrak{p}_1 \cdot \mathfrak{p}_2 = ((x) + (y))((x) + (z)) = (x^2, xz, yx, yz).$$

We have $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ and we also clearly have $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{m}^2$ since $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq \mathfrak{m}$. Thus we have $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Note that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime since the rings $K[x, y, z]/\mathfrak{p}_1 \simeq K[z]$ and $K[x, y, z]/\mathfrak{p}_2 \simeq K[y]$ are domains. Note also that $\mathfrak{m}$ is a maximal ideal, since $K[x, y, z]/\mathfrak{m} \simeq K$ is a field. Thus $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}^2$ are primary (see after Lemma 6.4 for the latter). The radicals of the ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}^2$ are $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{m}$ (see again Lemma 6.4 for the latter). These three ideals are distinct. Finally, we have $\mathfrak{p}_1 \not\supseteq \mathfrak{p}_2 \cap \mathfrak{m}^2$ (because $z^2 \notin \mathfrak{p}_1$ but $z^2 \in \mathfrak{p}_2 \cap \mathfrak{m}^2$), $\mathfrak{p}_2 \not\supseteq \mathfrak{p}_1 \cap \mathfrak{m}^2$ (because $y^2 \notin \mathfrak{p}_2$ but $y^2 \in \mathfrak{p}_1 \cap \mathfrak{m}^2$) and $\mathfrak{m}^2 \not\supseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ (because $x \notin \mathfrak{m}^2$ but $x \in \mathfrak{p}_2 \cap \mathfrak{p}_2$). Hence if $\mathfrak{p}_1 \cdot \mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ then this decomposition is indeed primary and minimal. Thus we only have to show that $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \supseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$.

From the above, we have to show that

$$(x, y) \cap (x, z) \cap (x^2, y^2, z^2, xy, xz, yz) \subseteq (x^2, xz, yx, yz).$$

This is immediate, since all the ideals we are considering have the property that a polynomial lies in such an ideal if and only if all of the monomial summands of the polynomial lie in the ideal.

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2. Let I be a radical ideal that is decomposible. Show that I has a minimal primary decomposition by prime ideals (so that in this case, the associated primes are the elements of the minimal primary decomposition itself).

   Furthermore, show that any two minimal primary decompositions by prime ideals of a radical ideal coincide.

   Solution: Let $I = \bigcap_{i=1}^{k} J_i$ be a primary decomposition. Consider $I' = \mathfrak{r}(J_j) \cap \bigcap_{i \neq j} J_i$. We clearly have $I \subseteq I'$. If the inclusion is strict, then there exists $x \in I' \smallsetminus I$. There exists $n > 0$ such that $x^n \in J_j$, and hence $x^n \in I$. But I is radical, so $x \in I$. Thus $I' = I$. We may therefore replace J_j by the prime ideal $\mathfrak{r}(J_j)$; we still arrive at a primary decomposition this way (check this!).

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Section B

3. Let K be a field. Show that the ideal $(x^2, xy, y^2) \subseteq K[x, y]$ is a primary ideal, which is not irreducible.

4. (a) If I is a decomposable radical ideal, then all the associated primes of I are isolated.

   (b) if I is a decomposable ideal, there are only finitely many prime ideals, which contain I and are minimal among all the prime ideals containing I. These prime ideals are also the isolated ideals associated with I.

5. Let R be a ring. Let $I \subseteq R$ be an ideal. Then there are prime ideals that are minimal among all the prime ideals containing I.

   Furthermore, if $\mathfrak{p} \supseteq I$ is a prime ideal, then $\mathfrak{p}$ contains such a prime ideal.

6. Let R be a ring. Let $\mathcal{S}$ be the set of ideals in R that are not finitely generated; assume that $\mathcal{S} \neq \emptyset$.

   (a) Show that $\mathcal{S}$ has at least one maximal element.

   (b) Let I be maximal element of $\mathcal{S}$ (with respect to the relation of inclusion). Show that I is prime.

   (c) Suppose that all the prime ideals of R are finitely generated. Prove that R is noetherian.

   [Hint: exploit the fact that R/I is noetherian.]

7. Let R be a noetherian ring and $I \subseteq R$ an ideal. Then the quotient ring R/I is noetherian.

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Section C

8. Let R be a ring. Let $\mathcal{S}$ be the set of non-principal ideals in R; assume that $\mathcal{S} \neq \emptyset$. Prove that $\mathcal{S}$ admits maximal elements, and that every such element a prime ideal.

   Solution: The existence of maximal elements follows from Zorn's lemma. Let I be one such. Let $x, y \notin I$ and suppose for contradiction that $xy \in I$. Let $I_x = (x) + I$. By assumption, we have $I_x = (g_x)$ for some $g_x \in R$. Let $\phi\colon R \to I_x$ be the surjection of R-modules given by the formula $\phi(r) = rg_x$. We then have $I \subseteq \phi^{-1}(I)$.

   Suppose first that $I = \phi^{-1}(I)$. In other words, for all $r \in R$, we have $rg_x \in I$ if and only if $r \in I$. This contradicts the fact that $yg_x \in I$. So we conclude that $I \subset \phi^{-1}(I)$. From the definition of I, we then see that $\phi^{-1}(I)$ is a principal ideal of R, and hence so is $I = \phi(\phi^{-1}(I))$. This is a contradiction.

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