

B2.2 Commutative Algebra

Sheet 3 — HT26

Sections 1-10

Section A

1. Let R be a subring of a ring T . Suppose that T is integral over R . Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$. Show that if $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ then $\mathfrak{q}_1 = \mathfrak{q}_2$.

Solution: The ring $R/\mathfrak{p}$ can be viewed as a subring of $T/\mathfrak{q}_1$ and by assumption we have $(\mathfrak{q}_2 \bmod \mathfrak{q}_1) \cap R/\mathfrak{p} = (0)$. We may thus assume without loss of generality that R and T to be domains and that $\mathfrak{q}_1$ and $\mathfrak{p}$ are zero ideals.

Now let $e \in \mathfrak{q}_2 \setminus \{0\}$ and let $P(x) \in R[x]$ be a non-zero monic polynomial such that $P(e) = 0$. Since T is a domain, we may assume that the constant coefficient of $P(x)$ is non-zero (otherwise, replace $P(x)$ by $P(x)/x^k$ for a suitable $k \geq 1$). But then the constant term $P(0)$ is a linear combination of positive powers of e (since $P(e) = 0$), so $P(0) \in R \cap \mathfrak{q}_2 = (0)$. This is a contradiction.

2. Let R be a subring of a ring T and suppose that T is integral over R . Let $\mathfrak{p}$ be prime ideal of R and let $\mathfrak{q}$ be a prime ideal of T . Suppose that $\mathfrak{q} \cap R = \mathfrak{p}$. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k$ be primes ideal of R and suppose that $\mathfrak{p}_1 = \mathfrak{p}$. Show that there are prime ideals $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_k$ of T such that $\mathfrak{q}_1 = \mathfrak{q}$ and such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for all $i \in \{1, \dots, k\}$.

Solution: By induction on k , we only need to treat the case $k = 2$. Consider the extension of rings $R/\mathfrak{p} \subseteq T/\mathfrak{q}$. This is also an integral extension. Furthermore, there is a unique prime ideal $\mathfrak{p}'_2$ in $R/\mathfrak{p}$, which corresponds to $\mathfrak{p}_2$ via the quotient map. By the Going-up Theorem, there is a prime ideal $\mathfrak{q}'_2$ in $T/\mathfrak{q}$, which is such that $\mathfrak{q}'_2 \cap R/\mathfrak{p} = \mathfrak{p}'_2$. The prime ideal $\mathfrak{q}_2$ corresponding to $\mathfrak{q}'_2$ via the quotient map has the required properties.

Section B

3. Show that $\mathbb{Z}$ is integrally closed and that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$.
4. Let A be a ring and let $U(x) \in A[x]$ be a non zero monic polynomial. Then there exists a ring B containing A , which is integral over A and such that

$$U(x) = \prod_{i=1}^{\deg(U)} (x - b_i)$$

for some $b_i \in B$, where we set $\prod_{i=1}^{\deg(U)} (x - b_i) = 1$ if $\deg(U) = 0$.

5. Let S be a ring and let $R \subseteq S$ be a subring of S . Suppose that R is integrally closed in S . Let $P(x) \in R[x]$ and suppose that $P(x) = Q(x)J(x)$, where $Q(x), J(x) \in S[x]$ and $Q(x)$ and $J(x)$ are monic. Show that $Q(x), J(x) \in R[x]$. Use this to give a new proof of the fact that if $T(x) \in \mathbb{Z}[x]$ and $T(x) = T_1(x)T_2(x)$, where $T_1(x), T_2(x) \in \mathbb{Q}[x]$ are monic polynomials, then $T_1(x), T_2(x) \in \mathbb{Z}[x]$.
6. Let R be a ring. Show that the two following conditions are equivalent:
 - (a) R is a Jacobson ring.
 - (b) If $\mathfrak{p} \in \text{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \dots$.

7. Let k be field and let R be a finitely generated algebra over k . Show that the two following conditions are equivalent:
 - (a) $\text{Spec}(R)$ is finite.
 - (b) R is finite over k .
8. Let k be an algebraically closed field. Let $P_1, \dots, P_d \in k[x_1, \dots, x_d]$. Suppose that the set

$$\{(y_1, \dots, y_d) \in k^d \mid P_i(y_1, \dots, y_d) = 0 \forall i \in \{1, \dots, d\}\}$$

is finite. Show that

$$\text{Spec}(k[x_1, \dots, x_d]/(P_1, \dots, P_d))$$

is finite.

Section C

9. Let R be a noetherian ring and let T be a finitely generated R -algebra. Let G be a finite subgroup of the group of automorphisms of T as a R -algebra. Let T^G be the fixed point set of G (ie the subset of T , which is fixed by all the elements of G).

- (a) Show that T is integral over T^G .
 (b) Show that T^G is a subring of T , which contains the image of R and that T^G is finitely generated over R .

Solution: It is clear from the definitions that T^G is a subring which contains the image of R . Let $t \in T$. Then t satisfies the polynomial equation

$$\prod_{g \in G} (t - g(t)) = 0$$

The polynomial $M_t(x) = \prod_{g \in G} (x - g(t))$ has coefficients in T^G , because the coefficients are symmetric polynomials in the variables $g(t)$, and therefore are invariant under G . Hence t is integral over T^G . Since t was arbitrary, T is integral over T^G . Since T is also finitely generated as a T^G -algebra (because it is finitely generated as an R -algebra), we thus see that T is finite over T^G (see after Lemma 8.3). Hence T^G is finitely generated over R by the Artin–Tate Theorem.

10. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \dots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_d(x_1, \dots, x_d)$ such that $\mathfrak{m} = (P_1, \dots, P_d)$.

Solution: By induction on $d \geq 1$. If $d = 1$ then this follows from the fact that $k[x_1]$ is a PID. We suppose that the statement holds for $d - 1$. Let $K = k[x_1, \dots, x_d]/\mathfrak{m}$. By the weak Nullstellensatz, this is a finite field extension of k . Let $\phi: k[x_1, \dots, x_d] \rightarrow K$ be the natural surjective homomorphism of k -algebras. Let $L = \phi(k[x_1, \dots, x_{d-1}])$. This is a domain and by Lemma 8.9, L is a field, since it contains k and is contained inside an integral extension of k . Let $\psi: k[x_1, \dots, x_{d-1}] \rightarrow L$ be the surjective homomorphism of k -algebras arising by restricting ϕ . The map ψ induces a surjective homomorphism of k -algebras

$$\Psi: k[x_1, \dots, x_d] \simeq (k[x_1, \dots, x_{d-1}])[x_d] \rightarrow L[x_d]$$

and there is a surjective homomorphism of L -algebras

$$\Lambda: L[x_d] \rightarrow K$$

that sends x_d to $\phi(x_d)$. By construction, we have $\phi = \Lambda \circ \Psi$. In particular, we have $\mathfrak{m} = \Psi^{-1}(\Lambda^{-1}(0))$. Since $L[x_d]$ is a PID and $\phi(x_d)$ is algebraic over k , we have

$\Lambda^{-1}(0) = (P(x_d))$ for some non zero polynomial $P(x_d) \in L[x_d]$. Now let $P_d(x_1, \dots, x_d) \in (k[x_1, \dots, x_{d-1}])[x_d]$ be a preimage by Ψ of $P(x_d)$.

We claim that $\mathfrak{m} = (\ker(\Psi), P_d)$. To see this, note that $\Psi((\ker(\Psi), P_d)) = (P(x_d))$ and so we have $(\ker(\Psi), P_d) \subseteq \mathfrak{m}$. On the other hand, if $e \in \mathfrak{m}$ then $\Psi(e) \in (P(x_d))$ and thus there is an element $e' \in (P_d)$ such that $\Psi(e) = \Psi(e')$ (since Ψ is surjective). In particular, we have $e - e' \in \ker(\Psi)$, so that $e \in (\ker(\Psi), P_d)$.

Now by the inductive assumption, $\ker(\Psi)$ is generated by polynomials

$$P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_{d-1}(x_1, \dots, x_{d-1})$$

and so $\mathfrak{m}$ is generated by $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_d(x_1, \dots, x_d)$.

11. Let R be a domain. Show that $R[x]$ is integrally closed if R is integrally closed.

Solution: Suppose that R is integrally closed in its fraction field K . The fraction field of $R[x]$ is $K(x) = (K[x])(K[x] \setminus \{1\})^{-1}$. Let $Q(x) \in K(x)$ and suppose that $Q(x)$ is integral over $R[x]$. Suppose for a contradiction that $Q(x) \notin R[x]$, and take $Q(x)$ of smallest possible degree. Clearly $Q(x)$ is not the zero polynomial.

Then $Q(x)$ is in particular integral over $K[x]$ and we saw in the solution of Question 4, sheet 2, that $K[x]$ is integrally closed, since it is a PID. So we deduce that $Q(x) \in K[x]$.

Now let

$$Q^n + P_{n-1}Q^{n-1} + \dots + P_0 = 0$$

be a non trivial integral equation for Q with $P_i \in R[x]$ for all n . Evaluating at $x = 0$ shows that the constant term of $Q(x)$ is integral over R , and hence lies in R . Since the integral closure of $R[x]$ is a ring, we may subtract the constant term, and assume that the constant term of Q is zero. But then we can also divide by a power of x , and decrease the degree of Q . Contradiction.