# **B2.2** Commutative Algebra

#### Sheet 4 — HT26

#### Sections 1-11

## Section A

1. Let R be a noetherian domain. Let  $\mathfrak{m}$  be a maximal ideal in R. Let  $r \in R \setminus \{0\}$  and suppose that (r) is an  $\mathfrak{m}$ -primary ideal. Show that height(r) = 1.

**Solution:** By assumption, the nilradical of (r) is  $\mathfrak{m}$ . Since the nilradical is the intersection of all the prime ideals containing (r), we see that every prime ideal containing (r) also contains  $\mathfrak{m}$ . On the other hand, a prime ideal containing  $\mathfrak{m}$  must be equal to  $\mathfrak{m}$ . We conclude that  $\mathfrak{m}$  is the only prime ideal containing (r). In particular,  $\mathfrak{m}$  is minimal among the prime ideals containing (r) and thus height(r) = height $(\mathfrak{m}) \leq 1$  by Krull's principal ideal theorem. On the other hand, height $(\mathfrak{m}) = 1$ , since we have the chain  $\mathfrak{m} \supset (0)$  (note that R is a domain).

2. Let R be a PID. Show that dim  $R \leq 1$ , and that dim R = 0 if and only if R is a field.

**Solution:** We have the prime ideal (0), since R is a domain. If R is a field, then we have no other prime ideals, and dim R = 0.

If R is not a field, then it has at least one non-trivial proper prime ideal. Every such ideal is maximal (see Sheet 0), and hence dim R = 1.

3. Let R be a noetherian ring. Let  $\mathfrak{p}, \mathfrak{p}'$  be prime ideals of R and suppose that  $\mathfrak{p} \subset \mathfrak{p}'$ . There exists a prime ideal  $\mathfrak{q}$  such that  $\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{p}'$  and  $\mathfrak{q}$  is maximal among prime ideals with this property.

**Solution:** Suppose that the conclusion does not hold. Let  $\mathfrak{q}_1$  be any prime ideal such that  $\mathfrak{p} \subseteq \mathfrak{q}_1 \subset \mathfrak{p}$  (we might eg take  $\mathfrak{q}_1 = \mathfrak{p}$ ). By assumption, there exists a prime ideal  $\mathfrak{q}_2$  such that  $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \mathfrak{p}$ . Applying the assumption again to  $\mathfrak{q}_2$ , we obtain a prime ideal  $\mathfrak{q}_3$  such that  $\mathfrak{q}_2 \subset \mathfrak{q}_3 \subset \mathfrak{p}$ . Continuing in this way we obtain an ascending sequence of ideals

$$\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \mathfrak{q}_3 \subset \dots$$

However, this sequence must stop since R is noetherian. This is a contradiction, so one of the prime ideals  $\mathfrak{q}_i$  must have the property mentioned in the lemma.

4. Let K be a field and let  $\mathfrak{p}$  be a non zero prime ideal of K[x]. Then height( $\mathfrak{p}$ ) = 1. In particular, we have  $\dim(K[x]) = 1$ .

**Solution:** This follows from the fact that non-zero prime ideals of K[x] are maximal and from the fact that the zero ideal in K[x] is prime, since K[x] is a domain.

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## Section B

- 5. Let R be a ring and let  $R_0$  be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated  $R_0$ -algebra. Suppose also that R is a field. Prove that R is a finite field.
- 6. Let R be an integrally closed domain. Let  $K = \operatorname{Frac}(R)$ . Let L|K be an algebraic field extension. Show that an element  $e \in L$  is integral over R if and only if the minimal polynomial of e over K has coefficients in R.
- 7. Let R be a PID. Let  $c_1, c_2 \in R$  be two distinct irreducible elements and let  $c = c_1 \cdot c_2$ . Show that  $(c) = (x, c_1)^2 \cdot (x, c_2)^2$  and that the ideals  $(x, c_i)$  are prime, as ideals in  $R[x]/(c-x^2)$ .
- 8. Let R be a ring (not necessarily noetherian). Suppose that  $\dim(R) < \infty$ . Show that  $\dim(R[x]) \leq 1 + 2\dim(R)$ .
- 9. Let A (resp. B) be a noetherian local ring with maximal ideal  $\mathfrak{m}_A$  (resp.  $\mathfrak{m}_B$ ). Let  $\phi \colon A \to B$  be a ring homomorphism and suppose that  $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$  (such a homomorphism is said to be 'local').

Suppose that

- (a) B is finite over A via  $\phi$ ;
- (b) the map  $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$  induced by  $\phi$  is surjective;
- (c) the map  $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$  induced by  $\phi$  is bijective.

Prove that  $\phi$  is surjective. [Hint: use Nakayama's lemma twice].

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# Section C

- 10. (a) Let R be a noetherian domain. Let I be a proper ideal of R. Then  $\bigcap_{n\geq 0} I^n = 0$ .
  - (b) Let R be a noetherian ring and let I be an ideal of R. Let M be a finitely generated R-module. Suppose that I is contained in the Jacobson radical of R. Then  $\bigcap_{n\geqslant 0} I^n M=0$ .

Solution: Part (a) is clear.

If  $r \in 1+I$  then r is a unit (a similar reasoning was made during the proof of Nakayama's lemma). Indeed, if r is not a unit, then r is contained in some maximal ideal  $\mathfrak{m}$ . But then 1 is also contained in  $\mathfrak{m}$ , since  $I \subseteq \mathfrak{m}$ , which is a contradiction. Hence  $\ker(r_M) = 0$  and the result follows from Krull's theorem.

11. Let  $\phi: R \to T$  be a ring homomorphism. Let  $\mathfrak{p} \in \operatorname{Spec}(R)$  and let I be the ideal generated by  $\phi(\mathfrak{p})$  in T.

Write  $\psi \colon R/\mathfrak{p} \to T/I$  for the ring homomorphism induced by  $\phi$  and let  $S = (R/\mathfrak{p}) \setminus \{0\}$ . Write  $\psi_S \colon \operatorname{Frac}(R/\mathfrak{p}) \to (T/I)_{\psi(S)}$  for the induced ring homomorphism. Finally, write  $\rho \colon T \to (T/I)_{\psi(S)}$  for the natural ring homomorphism.

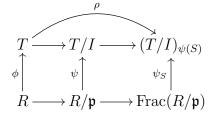
- (a) Show that  $\operatorname{Spec}(\rho)(\operatorname{Spec}((T/I)_{\psi(S)}))$  consists precisely of the prime ideals  $\mathfrak{q}$  of T, such that  $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ .
- (b) Show that the correspondence between
  - prime ideals  $\mathfrak{q}$  such that  $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ , and
  - prime ideals of  $(T/I)_{\psi(S)}$

respects inclusion in both directions.

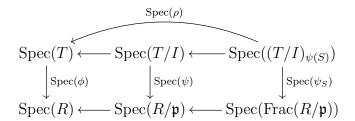
(c) Deduce that when T = R[x] we have

$$(T/I)_{\psi(S)} = (R[x]/\mathfrak{p}[x])_{\psi(S)} \simeq (R/\mathfrak{p})[x]_{(R/\mathfrak{p})^*} = \operatorname{Frac}(R/\mathfrak{p})[x].$$

**Solution:** We have a commutative diagram of rings



leading to a commutative diagram of spectra



The lemma is saying that the fibre of  $\operatorname{Spec}(\phi)$  above  $\mathfrak{p}$  is precisely the image of  $\operatorname{Spec}(\rho)$ .

Note first that  $\operatorname{Spec}(\operatorname{Frac}(R/\mathfrak{p}))$  consists of one point, since  $\operatorname{Frac}(R/\mathfrak{p})$  is a field. The image of  $\operatorname{Spec}(\operatorname{Frac}(R/\mathfrak{p}))$  in  $\operatorname{Spec}(R/\mathfrak{p})$  is the ideal  $(0) \subseteq R/\mathfrak{p}$  and the preimage of the ideal  $(0) \subseteq R/\mathfrak{p}$  in R is  $\mathfrak{p}$ . Thus the image of  $\operatorname{Spec}(\rho)$  is contained in the fibre of  $\operatorname{Spec}(\phi)$  above  $\mathfrak{p}$ , since the diagram is commutative.

Now suppose that  $\mathfrak{q} \in \operatorname{Spec}(T)$  and that  $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$  (i.e.,  $\mathfrak{q}$  lies inside the fibre of  $\operatorname{Spec}(\phi)$  above  $\mathfrak{p}$ ).

Then  $\mathfrak{q} \supseteq I$  and there is thus an ideal  $\mathfrak{q}' \in \operatorname{Spec}(T/I)$ , such that  $\mathfrak{q}$  is the image of  $\mathfrak{q}'$  in  $\operatorname{Spec}(T)$ . On the other hand, we know that  $\psi^{-1}(\mathfrak{q}')$  is the 0 ideal, since  $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$  and the diagram of rings is commutative. In other words, we have  $\mathfrak{q}' \cap \psi(S) = \emptyset$ . We conclude that  $\mathfrak{q}'$  lies in the image of the map  $\operatorname{Spec}((T/I)_{\psi(S)}) \to \operatorname{Spec}(T/I)$ .

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