B2.2 Commutative Algebra

Sheet 4 — HT26

Sections 1-11

Section A

1. Let R be a noetherian domain. Let $\mathfrak{m}$ be a maximal ideal in R. Let $r \in R \smallsetminus \{0\}$ and suppose that (r) is an $\mathfrak{m}$-primary ideal. Show that $\text{height}((r)) = 1$.

   Solution: By assumption, the nilradical of (r) is $\mathfrak{m}$. Since the nilradical is the intersection of all the prime ideals containing (r), we see that every prime ideal containing (r) also contains $\mathfrak{m}$. On the other hand, a prime ideal containing $\mathfrak{m}$ must be equal to $\mathfrak{m}$. We conclude that $\mathfrak{m}$ is the only prime ideal containing (r). In particular, $\mathfrak{m}$ is minimal among the prime ideals containing (r) and thus $\text{height}((r)) = \text{height}(\mathfrak{m}) \leqslant 1$ by Krull's principal ideal theorem. On the other hand, $\text{height}(\mathfrak{m}) = 1$, since we have the chain $\mathfrak{m} \supset (0)$ (note that R is a domain).

2. Let R be a PID. Show that $\dim R \leqslant 1$, and that $\dim R = 0$ if and only if R is a field.

   Solution: We have the prime ideal (0), since R is a domain. If R is a field, then we have no other prime ideals, and $\dim R = 0$.

   If R is not a field, then it has at least one non-trivial proper prime ideal. Every such ideal is maximal (see Sheet 0), and hence $\dim R = 1$.

3. Let R be a noetherian ring. Let $\mathfrak{p}, \mathfrak{p}'$ be prime ideals of R and suppose that $\mathfrak{p} \subset \mathfrak{p}'$. There exists a prime ideal $\mathfrak{q}$ such that $\mathfrak{p} \subseteq \mathfrak{q} \subset \mathfrak{p}'$ and $\mathfrak{q}$ is maximal among prime ideals with this property.

   Solution: Suppose that the conclusion does not hold. Let $\mathfrak{q}_1$ be any prime ideal such that $\mathfrak{p} \subseteq \mathfrak{q}_1 \subset \mathfrak{p}$ (we might eg take $\mathfrak{q}_1 = \mathfrak{p}$). By assumption, there exists a prime ideal $\mathfrak{q}_2$ such that $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \mathfrak{p}$. Applying the assumption again to $\mathfrak{q}_2$, we obtain a prime ideal $\mathfrak{q}_3$ such that $\mathfrak{q}_2 \subset \mathfrak{q}_3 \subset \mathfrak{p}$. Continuing in this way we obtain an ascending sequence of ideals

   $$\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \mathfrak{q}_3 \subset \dots$$

   However, this sequence must stop since R is noetherian. This is a contradiction, so one of the prime ideals $\mathfrak{q}_i$ must have the property mentioned in the lemma.

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4. Let K be a field and let $\mathfrak{p}$ be a non zero prime ideal of $K[x]$. Then $\text{height}(\mathfrak{p}) = 1$. In particular, we have $\dim(K[x]) = 1$.

   Solution: This follows from the fact that non-zero prime ideals of $K[x]$ are maximal and from the fact that the zero ideal in $K[x]$ is prime, since $K[x]$ is a domain.

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Section B

5. Let R be a ring and let R_0 be the prime ring of R (the image of $\mathbb{Z}$ under the unique ring homomorphism $\mathbb{Z} \to R$). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

6. Let R be an integrally closed domain. Let $K = \mathrm{Frac}(R)$. Let $L|K$ be an algebraic field extension. Show that an element $e \in L$ is integral over R if and only if the minimal polynomial of e over K has coefficients in R.

7. Let R be a PID. Let $c_1, c_2 \in R$ be two distinct irreducible elements and let $c = c_1 \cdot c_2$. Show that $(c) = (x, c_1)^2 \cdot (x, c_2)^2$ and that the ideals (x, c_i) are prime, as ideals in $R[x]/(c - x^2)$.

8. Let R be a ring (not necessarily noetherian). Suppose that $\dim(R) < \infty$. Show that $\dim(R[x]) \leqslant 1 + 2\dim(R)$.

9. Let A (resp. B) be a noetherian local ring with maximal ideal $\mathfrak{m}_A$ (resp. $\mathfrak{m}_B$). Let $\phi\colon A \to B$ be a ring homomorphism and suppose that $\phi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ (such a homomorphism is said to be 'local').

   Suppose that

   (a) B is finite over A via ϕ;

   (b) the map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ induced by ϕ is surjective;

   (c) the map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ induced by ϕ is bijective.

   Prove that ϕ is surjective. [Hint: use Nakayama's lemma twice].

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Section C

10. (a) Let R be a noetherian domain. Let I be a proper ideal of R. Then $\bigcap_{n \geqslant 0} I^n = 0$.

    (b) Let R be a noetherian ring and let I be an ideal of R. Let M be a finitely generated R-module. Suppose that I is contained in the Jacobson radical of R. Then $\bigcap_{n \geqslant 0} I^n M = 0$.

    Solution: Part (a) is clear.

    If $r \in 1+I$ then r is a unit (a similar reasoning was made during the proof of Nakayama's lemma). Indeed, if r is not a unit, then r is contained in some maximal ideal $\mathfrak{m}$. But then 1 is also contained in $\mathfrak{m}$, since $I \subseteq \mathfrak{m}$, which is a contradiction. Hence $\ker(r_M) = 0$ and the result follows from Krull's theorem.

11. Let $\phi\colon R \to T$ be a ring homomorphism. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let I be the ideal generated by $\phi(\mathfrak{p})$ in T.

    Write $\psi\colon R/\mathfrak{p} \to T/I$ for the ring homomorphism induced by ϕ and let $S = (R/\mathfrak{p}) \setminus \{0\}$. Write $\psi_S\colon \operatorname{Frac}(R/\mathfrak{p}) \to (T/I)_{\psi(S)}$ for the induced ring homomorphism. Finally, write $\rho\colon T \to (T/I)_{\psi(S)}$ for the natural ring homomorphism.

    (a) Show that $\operatorname{Spec}(\rho)$ is injective and that its image consists precisely of the prime ideals $\mathfrak{q}$ of T such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

    (b) Show that the correspondence induced by $\operatorname{Spec}(\rho)$ between

    - prime ideals $\mathfrak{q}$ such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and
    - prime ideals of $(T/I)_{\psi(S)}$

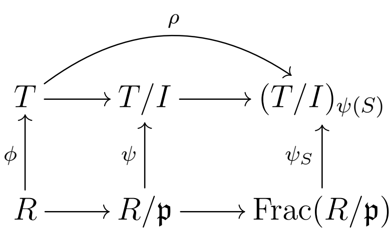
    respects inclusion in both directions.

    (c) Now suppose that $T = R[x]$ and that $\phi\colon R \to R[x]$ is the obvious map. Show that we have

    $$(T/I)_{\psi(S)} = (R[x]/\mathfrak{p}[x])_{\psi(S)} \simeq \operatorname{Frac}(R/\mathfrak{p})[x]$$

    and deduce that if $\mathfrak{q}_0 \supset \cdots \supset \mathfrak{q}_k$ is a proper chain of prime ideals in $\operatorname{Spec}(R[x])$ such that $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}$ for all i, then $k \leqslant 1$.

    Solution: We have a commutative diagram of rings



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leading to a commutative diagram of spectra

$$\begin{array}{ccccc}
\mathrm{Spec}(T) & \longleftarrow & \mathrm{Spec}(T/I) & \longleftarrow & \mathrm{Spec}((T/I)_{\psi(S)}) \quad (\text{with } \mathrm{Spec}(\rho): \mathrm{Spec}((T/I)_{\psi(S)}) \to \mathrm{Spec}(T)) \\
\downarrow{\scriptstyle \mathrm{Spec}(\phi)} & & \downarrow{\scriptstyle \mathrm{Spec}(\psi)} & & \downarrow{\scriptstyle \mathrm{Spec}(\psi_S)} \\
\mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R/\mathfrak{p}) & \longleftarrow & \mathrm{Spec}(\mathrm{Frac}(R/\mathfrak{p}))
\end{array}$$

(a) Note first that $\mathrm{Spec}(\mathrm{Frac}(R/\mathfrak{p}))$ consists of one point, since $\mathrm{Frac}(R/\mathfrak{p})$ is a field. The image of $\mathrm{Spec}(\mathrm{Frac}(R/\mathfrak{p}))$ in $\mathrm{Spec}(R/\mathfrak{p})$ is the ideal $(0) \subseteq R/\mathfrak{p}$ and the preimage of the ideal $(0) \subseteq R/\mathfrak{p}$ in R is $\mathfrak{p}$. Thus the image of $\mathrm{Spec}(\rho)$ is contained in the fibre of $\mathrm{Spec}(\phi)$ above $\mathfrak{p}$, since the diagram is commutative.

Now suppose that $\mathfrak{q} \in \mathrm{Spec}(T)$ and that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ (i.e., $\mathfrak{q}$ lies inside the fibre of $\mathrm{Spec}(\phi)$ above $\mathfrak{p}$).

Then $\mathfrak{q} \supseteq I$ and there is thus an ideal $\mathfrak{q}' \in \mathrm{Spec}(T/I)$, such that $\mathfrak{q}$ is the image of $\mathfrak{q}'$ in $\mathrm{Spec}(T)$. On the other hand, we know that $\psi^{-1}(\mathfrak{q}')$ is the 0 ideal, since $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and the diagram of rings is commutative. In other words, we have $\mathfrak{q}' \cap \psi(S) = \emptyset$. We conclude that $\mathfrak{q}'$ lies in the image of the map $\mathrm{Spec}((T/I)_{\psi(S)}) \to \mathrm{Spec}(T/I)$.

It is easy to check that $\mathrm{Spec}(\rho)$ is injective.

(b) This is a trivial check.

(c) We have

$$(T/I)_{\psi(S)} = (R[x]/\mathfrak{p}[x])_{\psi(S)} \simeq \big((R/\mathfrak{p})[x]\big)_{\psi(S)} \simeq \mathrm{Frac}(R/\mathfrak{p})[x]$$

where the last isomorphism is $(\sum_i^N r_i x^i)/s \mapsto \sum_i^N (r_i/s)x^i$ with inverse $\sum_i^N (r_i/s_i)x^i \mapsto (\sum_i^N r_i' x^i)/s$ with $s = \prod s_i$ and $r_i' = s_1 s_2 \cdots s_{i-1} r_i s_{i+1} \ldots s_N$.

By the correspondence above, the chain of primes can only be as long as the longest chain of primes in $\mathrm{Frac}(R/\mathfrak{p})[x]$. But here every non-zero prime is maximal, so the longest chain has length 1.

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