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REPRESENTATION THEORY OF SEMISIMPLE LIE ALGEBRAS

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1. The universal enveloping algebra of a Lie algebra

1.1. **Lie algebras.** Let k be a field.

Definition 1.1. *A Lie algebra $\mathfrak{g}$ over k is a k-vector space with a bilinear operation $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (called the Lie bracket) that satisfies the following identities:*

(1) *(alternating) $[x,x]=0$ for all $x\in\mathfrak{g}$. (When char $k\neq 2$, this is equivalent with "skew-symmetry": $[x,y]=-[y,x]$, $x,y\in\mathfrak{g}$.)*
(2) *(Jacobi identity): for all $x,y,z\in\mathfrak{g}$,*
$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0.$$

A Lie algebra is a non-associative algebra, and the Jacobi identity replaces the associativity condition. There are a few basic examples to keep in mind.

Example 1.2.

(1) Let A be an associative algebra. Then we may define a Lie algebra structure on $\mathfrak{g}=A$ by setting the bracket to equaled the commutator in A; $[x,y]=xy-yx$. One verifies immediately that the Jacobi identity is satisfied because of the associativity of the multiplication in A.
   (a) Let V be a k-vector space and $A=\mathrm{End}_k(V)$. Applying the construction above to this setting, we obtain the Lie algebra $\mathfrak{gl}(V)$ whose elements are the endomorphisms of V and the bracket is the commutator.
   (b) If V is finite dimensional and we fix a basis of V, we may identify $V\cong k^n$ and $\mathrm{End}_k(V)$ with $n\times n$ matrices with coefficients in k. Then the Lie algebra is $\mathfrak{gl}(n,k)$, the general linear Lie algebra of $n\times n$ matrices with the Lie bracket given by the commutator.
(2) Let V be a k-vector space, and consider $\mathfrak{sl}(V)=\{x:V\to V\mid \mathrm{tr}(x)=0\}$ with the Lie bracket given by the commutator in $\mathfrak{gl}(V)$ [1]. If V is finite dimensional and we fix a basis as before, we obtain the *special linear Lie algebra* $\mathfrak{sl}(n,k)$ of $n\times n$ matrices of trace 0. Notice that this algebra is not an example of (1): in order to define the bracket in $\mathfrak{sl}(V)$, we invoke the commutator in a larger algebra, $\mathfrak{gl}(V)$. In fact, one may prove that in general there is no associative algebra A such that $\mathfrak{sl}(2)$ is isomorphic with the Lie algebra obtained from A via the construction in (1).
(3) (Classical Lie algebras) Let V be a finite dimensional vector space over k and $B:V\times V\to k$ be a bilinear form. Define
$$\mathrm{Der}(B)=\{x\in\mathfrak{gl}(V)\mid B(xu,v)+B(u,xv)=0,\ \text{for all } u,v\in V\}. \tag{1.1.1}$$
   Thi is a Lie subalgebra of $\mathfrak{gl}(V)$, consisting of the linear maps that preserve B. Suppose that B is nondegenerate.
   (a) If B is symmetric, we obtain the *orthogonal Lie algebra with respect to B*, denoted by $\mathfrak{so}(V,B)$. When $k=\mathbb{C}$, all nondegenerate symmetric bilinear forms are equivalent, hence there is only one (up to isomorphism) orthogonal Lie algebra over $\mathbb{C}$. On the other hand, if $k=\mathbb{R}$, then the nondegenerate symmetric bilinear forms are classified by their signatures, and so are the orthogonal Lie algebras over $\mathbb{R}$.
   (b) If B is skew-symmetric, we obtain the *symplectic Lie algebra with respect to B*, denoted by $\mathfrak{sp}(V,B)$. Since B is nondegenerate, $\dim V$ must be even. Recall that, unlike the case of symmetric bilinear forms, the classification of skew-symmetric bilinear forms is independent of the field. In particular, there exists only one (up to equivalence) nondegenerate skew-symmetric bilinear form and thus, only one symplectic Lie algebra (up to isomorphism).

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[1] These notes are based on an earlier version of this course by Dan Ciubotaru.

[1] recall that the commutator of any two linear maps has trace 0

(4) Let $\mathfrak{g} = \mathfrak{gl}(n, k)$ be the general linear Lie algebra. We denote by $\mathfrak{n}^+$, $\mathfrak{h}$, $\mathfrak{n}^-$ the Lie subalgebras of strictly upper triangular matrices, diagonal matrices, and strictly lower triangular matrices, respectively. The vector space decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ will play an important role in the theory. We emphasize that this is not a Lie algebra decomposition.

1.2. Lie algebra representations.

Definition 1.3. *A* representation *of the Lie algebra $\mathfrak{g}$ over k is a k-vector space V together with a linear map $\rho : \mathfrak{g} \to \mathrm{End}_k(V)$ (the action) such that*

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x), \text{ for all } x, y \in \mathfrak{g}. \tag{1.2.1}$$

An equivalent way is to say that the map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a homomorphism of Lie algebras. We also say that V is a $\mathfrak{g}$-module*.*

If (ρ, V) is a representation of $\mathfrak{g}$, we will often write $x \cdot v$ in place of $\rho(x)(v)$ for the action, $x \in \mathfrak{g}$, $v \in V$.

Example 1.4.

(1) Let $\mathfrak{g} = \mathfrak{gl}(V)$ act on V in the usual way, i.e., the map ρ is the identity. In terms of matrices, if $\mathfrak{g} = \mathfrak{gl}(n, k)$ and $V = k^n$, then the action is just matrix multiplication: an $n \times n$ matrix times a column vector.

(2) If $\mathfrak{g}$ is any Lie algebra, the *adjoint representation* is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad}(x)(y) = [x, y]$, for all $x, y \in \mathfrak{g}$. The fact that this is a representation is equivalent with the Jacobi identity.

(3) If $\mathfrak{g}$ is any Lie algebra, the *trivial representation* is the one dimensional representation $\rho_0 : \mathfrak{g} \to \mathfrak{gl}(k)$, given by $\rho_0(x) = 0$ for all $x \in \mathfrak{g}$. More generally, if (ρ, V) is any $\mathfrak{g}$-representation, we write

$$V^{\mathfrak{g}} = \{v \in V \mid \rho(x)v = 0, \text{ for all } x \in \mathfrak{g}\}. \tag{1.2.2}$$

This the sum of all trivial subrepresentations of V.

(4) If V, W are $\mathfrak{g}$-modules, then so is $\mathrm{Hom}_k(V, W)$, via the rule

$$(x \cdot f)(v) = xf(v) - f(xv) \quad \text{for all} \quad f \in \mathrm{Hom}_k(V, W), x \in \mathfrak{g}, v \in V.$$

Later in the course we will specialize to certain types of representations.

1.3. **Tensor products.** Recall that the *tensor product* of two k-vector spaces U, V is a k-vector space $U \otimes V$. A typical element in $U \otimes V$ is $\sum_{i=1}^n u_i \otimes v_i$, where $u_i \in U$ and $v_i \in V$. This satisfies the following properties:

(1) $(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$, $u_1, u_2 \in U$, $v \in V$;
(2) $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$, $u \in U$, $v_1, v_2 \in V$;
(3) $(\lambda u) \otimes v = u \otimes (\lambda v) = \lambda(u \otimes v)$, $\lambda \in k$, $u \in U$, $v \in V$.

If $\{e_i \mid i \in I\}$ is a basis of U and $\{f_j \mid j \in J\}$ is a basis of V, then $\{e_i \otimes f_j \mid i \in I,\ j \in J\}$ is a basis of $U \otimes V$. In particular, $\dim(U \otimes V) = \dim U \cdot \dim V$.

More generally, if $V_1, \dots, V_n$ are k-vector spaces, we may define recursively the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_k$. This tensor product of vector spaces is associative, so we can ignore the order in which we construct this tensor product, e.g., $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$.

In particular, we can speak about the n-fold tensor product of a vector space V:

$$T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}.$$

If U and V are $\mathfrak{g}$-representations, then we define an action of $\mathfrak{g}$ on $U \otimes V$ by:

$$x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v), \quad x \in \mathfrak{g},\ u, v \in V. \tag{1.3.1}$$

One verifies easily that this is indeed a Lie algebra action:

$$\begin{aligned} x \cdot (y \cdot (u \otimes v)) &= x \cdot ((y \cdot u) \otimes v + u \otimes (y \cdot v)) \\ &= (x \cdot (y \cdot u)) \otimes v + (y \cdot u) \otimes (x \cdot v) + (x \cdot u) \otimes (y \cdot v) + u \otimes (x \cdot (y \cdot v)). \end{aligned}$$

Writing the similar equation for $y \cdot (x \cdot (u \otimes v))$ and subtracting, we see that the middle terms cancel, so:

$$[x, y] \cdot (u \otimes v) = [x, y] \cdot u \otimes v + u \otimes [x, y] \cdot v.$$

We can extend this definition to an action on tensor products $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, as the sum of actions on one component at a time.

Definition 1.5. *V be a k-vector space.*

(1) *Define $T^0(V) = k$, and set*
$$T(V) = \bigoplus_{n\geq 0} T^n(V). \tag{1.3.2}$$
Endow $T(V)$ with the multiplication given by the tensor product:
$$T^i(V) \times T^j(V) \to T^{i+j}(V),\ (x, y) \mapsto x \otimes y.$$
This makes $T(V)$ into an associative k-algebra with unity [2], called the tensor algebra *of V.*

(2) *The* symmetric algebra *$S(V)$ of V is defined as the quotient of $T(V)$ by the two-sided ideal generated by $\{x \otimes y - y \otimes x, x, y \in V\}$.*

(3) *The* exterior algebra *$\bigwedge V$ of V is defined as the quotient of $T(V)$ by the two-sided ideal generated by $\{x \otimes x, x \in V\}$.*

The image of $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in T^n(V)$ in $S^n(V)$ is denoted $x_1x_2 \cdots x_n$; its image in $\bigwedge^n(V)$ is denoted $x_1 \wedge x_2 \wedge \cdots \wedge x_n$.

Remark 1.6. *Let $\{x_i \mid i \in I\}$ be a basis of V where $(I, \leq)$ is an ordered set.*

(1) *A basis of $S^n(V)$ is given by $\{x_{i_1}x_{i_2} \ldots x_{i_n} \mid i_1 \leq i_2 \leq \cdots \leq i_n,\ i_j \in I\}$.*

(2) *A basis of $\bigwedge^n V$ is given by $\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} \mid i_1 < i_2 < \cdots < i_n,\ i_j \in I\}$.*

In particular, if $n > \dim V$, then $\bigwedge^n V = 0$.

Lemma 1.7. *Let V be a $\mathfrak{g}$-module; then $T(V)$, $S(V)$ and $\bigwedge V$ are graded algebras, and each inherits a $\mathfrak{g}$-action from V.*

Example 1.8. Suppose the characteristic of the field k is not 2. We can decompose $V \otimes V$ as a direct sum:
$$V \otimes V = S^2(V) \oplus \bigwedge^2 V,$$
where we embed $S^2(V)$ into $V \otimes V$ via:
$$xy \mapsto \frac{1}{2}(x \otimes y + y \otimes x),$$
and we embed $\bigwedge^2 V$ into $V \otimes V$ via:
$$x \wedge y \mapsto \frac{1}{2}(x \otimes y - y \otimes x).$$
If V is a $\mathfrak{g}$-representation, then this decomposition of $\mathfrak{g}$-invariant, in other words, it is a decomposition as $\mathfrak{g}$-representations.

1.4. **The universal enveloping algebra: definition.** Let $\mathfrak{g}$ be a Lie algebra. The goal is to assign to $\mathfrak{g}$ an associative k-algebra with 1 such that the representation theory of $\mathfrak{g}$ is equivalent with the representation theory of this associative algebra.

Definition 1.9. *Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. Let J be the two-sided ideal of $T(\mathfrak{g})$ generated by all the elements $x \otimes y - y \otimes x - [x, y]$, with $x, y \in \mathfrak{g}$. The* universal enveloping algebra of $\mathfrak{g}$ *is the associative unital k-algebra:*
$$U(\mathfrak{g}) := T(\mathfrak{g})/J. \tag{1.4.1}$$
There is a canonical linear map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ obtained by composing the identity map $\mathfrak{g} \to T^1(\mathfrak{g})$ with the quotient map $T(\mathfrak{g}) \to U(\mathfrak{g})$.

For example, If $\mathfrak{g}$ is a commutative Lie algebra (so the bracket is identically zero), then $U(\mathfrak{g}) = S(\mathfrak{g})$, the symmetric algebra generated by (the vector space) $\mathfrak{g}$.

The adjective "universal" is motivated by the following universal property whose proof is straightforward.

Lemma 1.10. *Let A be an associative unital algebra together with a linear map $\tau : \mathfrak{g} \to A$ such that*
$$\tau(x)\tau(y) - \tau(y)\tau(x) = \tau([x, y]) \quad \text{for all} \quad x, y \in \mathfrak{g}.$$
There exists a unique unital algebra homomorphism $\tau' : U(\mathfrak{g}) \to A$ such that
$$\tau' \circ \iota = \tau.$$

[2] The element 1 comes from $k = T^0(V)$

In particular, notice that the lemma says that every Lie algebra representation of $\mathfrak{g}$ can be lifted to a representation of the associative algebra $U(\mathfrak{g})$. Indeed, if we have $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a Lie algebra representation, take $A = \mathrm{End}_k(V)$ and $\tau = \rho$ and the claim follows from the universal property. Conversely, given any representation of $U(\mathfrak{g})$, we obtain a Lie algebra representation of $\mathfrak{g}$ by composing with the canonical map ι. Therefore, Lie algebra representations of $\mathfrak{g}$ are the same thing as representations of $U(\mathfrak{g})$.

1.5. **Filtration by degree and the associated graded algebra.** We assume from now on that k has characteristic 0 and that $\mathfrak{g}$ is a finite dimensional Lie algebra over k.

The tensor algebra $T(\mathfrak{g})$ has a natural filtration by degree via the subspaces

$$T_n(\mathfrak{g}) := \bigoplus_{i=0}^{n} T^i(\mathfrak{g}) \quad \subset \quad T(\mathfrak{g}).$$

Let $U_n(\mathfrak{g})$ denote the image of $T_n(\mathfrak{g})$ in $U(\mathfrak{g})$. Then $\{U_n(\mathfrak{g})\}$ is a filtration by subspaces of $U(\mathfrak{g})$:

$$U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \cdots \subset U_n(\mathfrak{g}) \subset \cdots, \quad U(\mathfrak{g}) = \bigcup_{n \geq 0} U_n(\mathfrak{g}). \tag{1.5.1}$$

Definition 1.11. *The* associated graded algebra $\mathrm{gr}\, U(\mathfrak{g})$ *is defined as follows. As a vector space, it equals*

$$\mathrm{gr}\, U(\mathfrak{g}) = \mathrm{gr}_0\, U(\mathfrak{g}) \oplus \mathrm{gr}_1\, U(\mathfrak{g}) \oplus \mathrm{gr}_2\, U(\mathfrak{g}) \oplus \dots$$

where $\mathrm{gr}_n\, U(\mathfrak{g}) := U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ *for each* $n \geq 0$ *and* $U_{-1}(\mathfrak{g}) := \{0\}$.
The multiplication in $U(\mathfrak{g})$ *defines bilinear maps*

$$\mathrm{gr}_n\, U(\mathfrak{g}) \times \mathrm{gr}_m\, U(\mathfrak{g}) \to \mathrm{gr}_{n+m}\, U(\mathfrak{g})$$

and then, by bi-additive extension, a multiplication on $\mathrm{gr}\, U(\mathfrak{g})$.
This makes $\mathrm{gr}\, U(\mathfrak{g})$ *into an associative unital graded algebra.*

Fix an ordered basis $\{x_i\}$ of $\mathfrak{g}$. Denote the image of this basis in $U(\mathfrak{g})$ by $\{y_i\}$. For every finite sequence $I = (i_1, \dots, i_m)$ of indices, let $y_I = y_{i_1} y_{i_2} \dots y_{i_m} \in U(\mathfrak{g})$.

Lemma 1.12.
(a) *The vector space* $U_m(\mathfrak{g})$ *is generated by* y_I *for all increasing sequences* I *of length at most* m.
(b) *The algebra* $\mathrm{gr}\, U(\mathfrak{g})$ *is commutative.*

Proof. (a) The claim is clear without the adjective "increasing". It follows by induction on m that indeed we may take only increasing sequences.

(b) From part (a), we see that $\mathrm{gr}\, U(\mathfrak{g})$ is generated as a graded k-algebra by $\mathrm{gr}\, U_1(\mathfrak{g})$. But if $X = x + U_0(\mathfrak{g})$ and $Y = y + U_0(\mathfrak{g})$ are two elements in $\mathrm{gr}\, U_1(\mathfrak{g})$ with $x, y \in \mathfrak{g}$ then

$$XY - YX = xy - yx + U_1(\mathfrak{g}) = [x, y] + U_1(\mathfrak{g}) = 0$$

because $[x, y] \in \mathfrak{g}$ maps to $U_1(\mathfrak{g})$ in $U(\mathfrak{g})$. $\square$

We wish to show that the canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective and to understand the structure of the commutative algebra $\mathrm{gr}\, U(\mathfrak{g})$. The main technical result that we need is next.

Let $P = k[z_i]$ be the algebra of polynomials in the indeterminates z_i. Let P_m denote the subspace of polynomials of total degree less than or equal to m. If $I = (i_1, \dots, i_m)$ is a sequence of integers, denote z_I as before. It will be convenient to use the notation $i \leq I$ to mean $i \leq i_k$ for all $k = 1, \dots, m$.

Lemma 1.13 (Dixmier)**.** *For every* $m \geq 0$*, there exists a unique linear map* $f_m : \mathfrak{g} \otimes P_m \to P$ *such that:*
(A_m) $f_m(x_i \otimes z_I) = z_i z_I$ *for* $i \leq I$, $z_I \in P_m$*;*
(B_m) $f_m(x_i \otimes z_I) - z_i z_I \in P_k$ *for* $z_I \in P_k$, $k \leq m$*;*
(C_m) $f_m(x_i \otimes f_m(x_j \otimes z_J)) = f_m(x_j \otimes f_m(x_i \otimes z_J)) + f_m([x_i, x_j] \otimes z_J)$*, for* $z_J \in P_{m-1}$.
Moreover, the restriction of f_m *to* $\mathfrak{g} \otimes P_{m-1}$ *is* f_{m-1}.

(Dixmier, page 68). [3] To simplify notation, we will write

$$xz := f_m(x \otimes z) \quad \text{for} \quad x \in \mathfrak{g}, z \in P_m$$

when these have been defined; in this notation, the three conditions we need to establish become

[3] This proof is non-examinable.

(A_m) $x_i z_I = z_i z_I$ whenever $i \leq I$ and $|I| \leq m$,
(B_m) $x_i z_I - z_i z_I \in P_{|I|}$ for any i and any I such that $|I| \leq m$,
(C_m) $x_i(x_j z_J) = x_j(x_i z_J) + [x_i, x_j]z_J$, for any i, j and any J such that $|J| \leq m-1$.

In effect, the proof first constructs an action of the tensor algebra $T(\mathfrak{g})$ on P, and then verifies that this action descends to an action of $U(\mathfrak{g})$ on P by verifying condition (C_m).

Step 1. The proof is by induction on m. For $m = 0$, we define for any index i,

$$x_i 1 := z_i$$

then (A_0), (B_0) and (C_0) are all satisfied.

Step 2. Assume from now on that $m \geq 1$. Fix the index i and let I be an increasing sequence of indices with $|I| = m$; we will define $x_i z_I$. Write $I = (j, J)$ with $j \leq J$ and $|J| = m - 1$.

(a) If $i \leq j$, then we define

$$x_i z_I := z_i z_I. \tag{1.5.2}$$

(b) If $i > j$ then $x_i z_J - z_i z_J \in P_{m-1}$ by (B_{m-1}) and we define inductively

$$x_i z_I := z_i z_I + x_j(x_i z_J - z_i z_J) + [x_i, x_j]z_J. \tag{1.5.3}$$

Step 3. With these definitions, we see that (A_m) is satisfied, and it follows from (1.5.2) that (B_m) is satisfied if $i \leq I$. Suppose that $I = (j, J)$ where $j \leq J$ but $i > j$. Now (B_{m-1}) implies that $\mathfrak{g}P_{m-1} \subseteq P_m$. Since $x_i z_J - z_i z_J \in P_{|J|} = P_{m-1}$ by (B_{m-1}), it follows that $x_j(x_i z_J - z_i z_J) \in P_m$. Similarly, because $z_J \in P_{m-1}$ as $|J| = m - 1$, we have $[x_i, x_j]z_J \in P_m$. Applying (1.5.3) we see that in this case,

$$x_i z_I - z_i z_I = x_j(x_i z_J - z_i z_J) + [x_i, x_j]z_J \in P_m.$$

as required for (B_m).

Step 4. It remains to check that (C_m) is also satisfied. So, let the indices i, j be given and let J be an increasing sequence of indices with $|J| = m - 1$. We split the problem into 4 cases.

(a). Suppose that $j \leq J$ and $i > j$. Then by (1.5.2), $x_j z_J = z_j z_J = z_I$ where $I := (j, J)$. Then by (1.5.3),

$$x_i(x_j z_J) = x_i z_I = z_i z_I + x_j(x_i z_J - z_i z_J) + [x_i, x_j]z_J.$$

But $x_j(z_i z_J) = z_j z_i z_J$ by (1.5.2) because we're assuming $j < i$ and $j \leq J$, and this equals $z_i z_j z_J = z_i z_I$ because P is commutative. So the first term cancels with the third term and we obtain (C_m), namely

$$x_i(x_j z_J) = x_j(x_i z_J) + [x_i, x_j]z_J. \tag{1.5.4}$$

(b). Suppose now that $i \leq J$ and $j > i$; then by swapping i and j in (1.5.4) we obtain

$$x_j(x_i z_J) = x_i(x_j z_J) + [x_j, x_i]z_J.$$

Rearranging this equation and using $[x_j, x_i] = -[x_i, x_j]$ shows that (C_m) is also satisfied in this case.

(c). Because (C_m) is trivially satisfied when $i = j$, it follows from (a, b) that (C_m) holds if $i \leq J$ or $j \leq J$.

(d). Suppose finally that $J = (k, K)$ with $k \leq K$ and $k < i$ and $k < j$. Then $|K| = m - 2$ so by (C_{m-1}),

$$x_j z_J = x_j(x_k z_K) = x_k(x_j z_K) + [x_j, x_k]z_K. \tag{1.5.5}$$

Now by (B_{m-1}), $x_j z_K = z_j z_K + w$ for some $w \in P_{m-2}$. Note that by cases (a,b) above, we can apply (C_m) to $x_i(x_k(z_j z_K))$ because $k \leq K$ and $k < j$; we can also apply (C_{m-1}) to $x_i(x_k w)$. Therefore (C_m) applies to $x_i(x_k(x_j z_K))$ and gives

$$x_i(x_k(x_j z_K)) = x_k(x_i(x_j z_K)) + [x_i, x_k](x_j z_K). \tag{1.5.6}$$

On the other hand, it follows from (C_{m-1}) that

$$x_i([x_j, x_k]z_K) = [x_j, x_k](x_i z_K) + [x_i, [x_j, x_k]]z_K. \tag{1.5.7}$$

Applying x_i to equation (1.5.5) and using (1.5.6) and (1.5.7), we obtain

$$x_i(x_j z_J) \quad = \quad x_k(x_i(x_j z_K)) + [x_i, x_k](x_j z_K) \quad + \quad [x_j, x_k](x_i z_K) + [x_i, [x_j, x_k]]z_K.$$

Swap i and j in this equation and subtract: the two middle terms cancel and we obtain

$$x_i(x_j z_J) - x_j(x_i z_J) \quad = \quad x_k(x_i(x_j z_K) - x_j(x_i z_K)) \quad + \quad ([x_i, [x_j, x_k]] - [x_j, [x_i, x_k]])\, z_K.$$

Apply (C_{m-1}) to the first term and the Jacobi identity to the second term to obtain

$$x_i(x_j z_J) - x_j(x_i z_J) = x_k([x_i, x_j]z_K) + [[x_i, x_j], x_k]z_K.$$

Applying (C_{m-1}) again finally shows that the right hand side equals $[x_i, x_j](x_k z_K)$; since $k \leq K$ this equals $[x_i, x_j](z_k z_K) = [x_i, x_j]z_J$ by (1.5.2). This establishes (C_m) in the final case, and completes the proof. ☐

Proposition 1.14. *The set $\{y_I \mid I$ increasing sequence$\}$ is a basis of $U(\mathfrak{g})$.*

Proof. By Lemma 1.13, there is a representation $\rho : \mathfrak{g} \to \mathrm{End}(P)$ such that $\rho(x_i)z_I = z_i z_I$ for all $i \leq I$. By Lemma 1.10, this extends to a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to \mathrm{End}(P)$ such that

$$\phi(y_i)z_I = z_i z_I, \text{ for all } i \leq I.$$

From this, we deduce recursively that, if I is an increasing sequence, then $\phi(y_I)1 = z_I$. Since $\{z_I\}$ are linearly independent in P, it follows that $\{y_I\}$ is a linearly independent set in $U(\mathfrak{g})$. But we already know that it is also a generating set, hence a basis. ☐

Corollary 1.15. *The canonical map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Proof. Clear from Proposition 1.14. ☐

In light of this result, from now on identify $\mathfrak{g}$ with its image in $U(\mathfrak{g})$ and drop ι (and the y's) from notation.

Corollary 1.16. *Let $(x_1, \ldots, x_n)$ be an ordered basis of $\mathfrak{g}$. Then $\{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} : k_i \in \mathbb{N}\}$ is a basis of $U(\mathfrak{g})$.*

Proof. This is just a rephrasing of Proposition 1.14 in the case when $\dim \mathfrak{g} < \infty$. ☐

Since $\iota : \mathfrak{g} \to \mathrm{gr}\, U(\mathfrak{g})$ is an injection, we can uniquely extend it to a canonical homomorphism $\iota : S(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{g})$. (Both algebras are commutative!) Clearly, ι maps $S^m(\mathfrak{g})$ to $\mathrm{gr}_m U(\mathfrak{g})$ for each $m \geq 0$.

Theorem 1.17 (Poincaré-Birkhoff-Witt Theorem)**.** *The canonical homomorphism $\iota : S(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{g})$ is an isomorphism of graded algebras.*

Proof. It follows from Proposition 1.14 that $\{y_I : I$ is increasing and $|I| = m\}$ maps to a basis of $\mathrm{gr}_m U(\mathfrak{g}) = U_m(\mathfrak{g})/U_{m-1}(\mathfrak{g})$. Now if $I = (i_1, i_2, \ldots, i_m)$ then

$$\iota(y_I + U_{m-1}(\mathfrak{g})) = \prod_{j=1}^{m} \iota(y_{i_j} + U_0(\mathfrak{g})) = \prod_{j=1}^{m} z_{i_j} = z_I$$

so ι maps this basis of $\mathrm{gr}_m U(\mathfrak{g})$ to the basis $\{z_I : |I| = m\}$ of $S^m(\mathfrak{g})$. Hence ι is an isomorphism. ☐

The PBW theorem allows us to identify canonically $\mathrm{gr}\, U(\mathfrak{g})$ with $S(\mathfrak{g})$.

Definition 1.18. *Suppose that k has characteristic zero. Let $\widetilde{\phi} : S(\mathfrak{g}) \to T(\mathfrak{g})$ be the* symmetrizing map*:*

$$x_1 x_2 \ldots x_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}. \tag{1.5.8}$$

Denote $\phi : S(\mathfrak{g}) \to U(\mathfrak{g})$ the composition of the map $\widetilde{\phi}$ with the projection onto $U(\mathfrak{g})$.

Notice that $x_{\sigma(1)} \cdots x_{\sigma(m)} \equiv x_1 \cdots x_m \mod U_{m-1}(\mathfrak{g})$ for any $\sigma \in S_m$, whence

$$\phi(x_1 \cdots x_m) \equiv x_1 \cdots x_m \mod U_{m-1}(\mathfrak{g})$$

for any $x_1, \ldots, x_m \in \mathfrak{g}$. This means that ϕ preserves the filtrations on $S(\mathfrak{g})$ and $U(\mathfrak{g})$ and induces the identity map on the associated graded vector spaces. Therefore ϕ is an isomorphism of linear spaces. We emphasize that it is not an isomorphism of algebras! This is obvious, since $S(\mathfrak{g})$ is commutative, but $U(\mathfrak{g})$ is not.

Remark 1.19. One can show that ϕ is in fact an isomorphism of $\mathfrak{g}$-modules (exercise). By taking the $\mathfrak{g}$-invariants (the copies of the trivial representation), we obtain a linear isomorphism $\phi : S(\mathfrak{g})^{\mathfrak{g}} \to Z(\mathfrak{g})$, where $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ is the centre of $U(\mathfrak{g})$. But again, ϕ is just an isomorphism of linear spaces and not of algebras in general, even though now both algebras are commutative.

1.6. **The principal anti-automorphism of $U(\mathfrak{g})$.** Consider the *opposite ring* $U(\mathfrak{g})^{\mathrm{op}}$ which is $U(\mathfrak{g})$ as a k-vector space, with multiplication $a \cdot b := ba$. Consider the linear map ${}^T : \mathfrak{g} \to U(\mathfrak{g})^{\mathrm{op}}$ which sends $x \in \mathfrak{g}$ to $-x \in U(\mathfrak{g})^{\mathrm{op}}$. We observe that for any $x, y \in \mathfrak{g}$ we have

$$x^T \cdot y^T - y^T \cdot x^T = (-y)(-x) - (-x)(-y) = yx - xy = [y,x] = -[x,y] = [x,y]^T.$$

Therefore by Lemma 1.10, ^{T} extends to a k-algebra homomorphism ${}^T : U(\mathfrak{g}) \to U(\mathfrak{g})^{\mathrm{op}}$ satisfying

$$(x_1 x_2 \cdots x_n)^T = (-1)^n x_n x_{n-1} \cdots x_1 \quad \text{for all} \quad x_1, \ldots, x_n \in \mathfrak{g}.$$

We can also view ^{T} as an *anti-automorphism*, by which mean a k-linear map $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ such that

$$\tau(xy) = \tau(y)\tau(x) \quad \text{for all} \quad x, y \in U(\mathfrak{g}).$$

Definition 1.20. *${}^T : U(\mathfrak{g}) \to U(\mathfrak{g})$ is called the* principal anti-automorphism *of $U(\mathfrak{g})$.*

If $\rho : \mathfrak{g} \to \mathrm{End}_k V$ is a Lie algebra representation, we have the *contragredient* representation $\rho^* : \mathfrak{g} \to \mathrm{End}_k(V^*)$ corresponding to the dual $\mathfrak{g}$-module V^*. Then by Lemma 1.10, ρ^* extends uniquely to a k-algebra homomorphism $\rho^* : U(\mathfrak{g}) \to \mathrm{End}_k(V^*)$. We can understand this extension explicitly using the principal anti-automorphism as follows:

$$\rho^*(u)(f)(v) = f(\rho(u^T)(v)) \quad \text{for all} \quad u \in U(\mathfrak{g}), v \in V, f \in V^*.$$

2. Representations of $\mathfrak{sl}(2)$

From now on, the ground field k is assumed to have characteristic zero. In this section, we study finite dimensional representations of $\mathfrak{g} = \mathfrak{sl}(2)$.

2.1. **Weights and weight vectors.** The Lie algebra $\mathfrak{sl}(2)$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a + d = 0$ (trace zero). The *standard basis* of $\mathfrak{sl}(2)$ is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.1.1}$$

The relations between the basis elements are

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Lemma 2.1. *The following identities hold in $U(\mathfrak{sl}(2))$:*

$$[h, e^k] = 2ke^k, \ [h, f^k] = -2kf^k, \ [e, f^k] = kf^{k-1}(h - (k-1)),$$

for all integers $k \geq 1$.

Proof. By induction on k, using the identity $[a, bc] = [a,b]c + b[a,c]$ valid in any associative ring. □

An important role in the representation theory of $\mathfrak{sl}(2)$ is played by the Casimir element.

Definition 2.2. *The* Casimir element *of $U(\mathfrak{sl}(2))$ is $C := h^2 + 2h + 4fe \in U(\mathfrak{sl}(2))$.*

Note that C it is unique only up to a scalar multiple, as we will see when we discuss the general theory for a semisimple Lie algebra.

Lemma 2.3. *The Casimir element C belongs to the centre of $U(\mathfrak{sl}(2))$.*

Proof. It follows from Definition 1.9 that $U(\mathfrak{sl}(2))$ is generated by e, h, f. So it is sufficient to check that C commutes with e and f, because then it also commutes with $h = ef - fe$. This is a direct calculation:

$$[C, e] = [h^2, e] + 2[h, e] + 4[fe, e],$$

and $[h^2, e] = h[h,e] + [h, e[h = h(2e) + (2e)h = 4he - 4e$. Moreover, $[h,e] = 2e$ and $[fe, e] = [f, e]e = -he$. This shows that the sum above is zero indeed, and $[C, f] = 0$ is similar. □

Definition 2.4. *Let V be an $\mathfrak{sl}(2)$-module.*

(1) *A vector $v \in V$ is called a* weight vector *if it is an eigenvector for the action of h:*

$$h \cdot v = \lambda v \quad \textit{for some} \quad \lambda \in k.$$

(2) *If $v \neq 0$ is a weight vector, we call the corresponding eigenvalue λ a* weight.

(3) *For each* $\lambda \in k$, *define*

$$V_\lambda^{\rm ss} = \{v \in V \mid (h-\lambda)v = 0\} \quad \subseteq \quad V_\lambda = \{v \in V \mid \exists N > 0 \text{ such that } (h-\lambda)^N v = 0\},$$

and call them the λ-weight space *and the* generalized λ-weight space, *respectively.*

(4) *If* $V_\lambda^{\rm ss} = V_\lambda$ *for all* λ, *we say that* h acts semisimply on V.

Lemma 2.5. *Let* V *be an* $\mathfrak{sl}(2)$*-module. Then:*

(1) $e \cdot V_\lambda \subseteq V_{\lambda+2}$ *and* $e \cdot V_\lambda^{\rm ss} \subseteq V_{\lambda+2}^{\rm ss}$,
(2) $f \cdot V_\lambda \subseteq V_{\lambda-2}$ *and* $f \cdot V_\lambda^{\rm ss} \subseteq V_{\lambda+2}^{\rm ss}$.

Proof. Suppose $v \in V_\lambda$ is given. Then there exists $N > 0$ such that $(h-\lambda)^N v = 0$. Notice that in $U(\mathfrak{sl}(2))$, $(h-2)e = eh$ which means that $(h-2)^j e = eh^j$ for all j. Then $(h-\lambda-2)^N e \cdot v = [(h-2)-\lambda]^N e \cdot v = e(h-\lambda)^N v = 0$, which means that $e \cdot v \in V_{\lambda+2}$.

The case of $V_\lambda^{\rm ss}$ is when $N = 1$. The statement about f is completely similar. □

Definition 2.6. *A vector* $v \neq 0$ *in* V *is called a* highest weight vector *if* $v \in V_\lambda^{\rm ss}$ *for some* λ *and* $e \cdot v = 0$.

If V is a finite dimensional $\mathfrak{sl}(2)$-module, then Lemma 2.5 implies that highest weight vectors do exist.

Lemma 2.7. *Let* V *be an* $\mathfrak{sl}(2)$*-module which admits a highest weight vector* $v \in V$ *of weight* λ. *Consider the sequence of vectors* $v_0 = v$, $v_k = f^k \cdot v$, *for* $k \geq 0$. *Then:*

(a) $h \cdot v_k = (\lambda - 2k)v_k$, $e \cdot v_0 = 0$, $e \cdot v_{k+1} = (k+1)(\lambda - k)v_k$, $f \cdot v_k = v_{k+1}$, *for* $k \geq 0$.
(b) *The subspace* $L \subset V$ *spanned by the vectors* $\{v_k \mid k \geq 0\}$ *is an* $\mathfrak{sl}(2)$*-submodule and all nonzero vectors* v_k *are linearly independent.*
(c) *Suppose that* $v_k = 0$ *for some* k. *Then there exists* $\ell \in \mathbb{Z}_{\geq 0}$ *such that* $\lambda = \ell$, $v_k \neq 0$ *for* $0 \leq k \leq \ell$ *and* $v_k = 0$ *for all* $k > \ell$.

Proof. (a) Use Lemma 2.1. (b) This follows from (a) because the eigenvectors v_k have distinct eigenvalues. (c) Let ℓ be the first index such that $v_{\ell+1} = 0$. Then $0 = e \cdot v_{\ell+1} = (\ell+1)(\lambda-\ell)v_\ell$. Since our ground field k has characteristic zero, $\ell + 1 \neq 0$ so $(\lambda - \ell)v_\ell = 0$. Since $v_\ell \neq 0$ we conclude that $\lambda = \ell$. □

2.2. **Irreducible finite dimensional $\mathfrak{sl}(2)$-modules.** For every $\ell \geq 0$, we construct an irreducible representation $V(\ell)$ of dimension $\ell + 1$ generated by a highest weight vector of weight ℓ.

Algebraic construction. The relations in Lemma 2.7 tell us how to define the module $V(\ell)$. Let $V(\ell)$ be the span of $\{v_0, v_1, \ldots, v_\ell\}$ and define the $\mathfrak{sl}(2)$-action by:

$$h \cdot v_k = (\ell - 2k)v_k,\ e \cdot v_0 = 0,\ e \cdot v_{k+1} = (k+1)(\ell - k)v_k,\ f \cdot v_k = v_{k+1},\ k \geq 0. \tag{2.2.1}$$

(By convention, $v_{\ell+1} = 0$ in the above equations.) We can compute directly that these formulas define an action of $\mathfrak{sl}(2)$.

Lemma 2.8. *The module* $V(\ell)$ *just defined is irreducible.*

Proof. Suppose that $M \neq 0$ is a submodule of $\mathfrak{sl}(2)$. Let $0 \neq \sum_{i=0}^{\ell} a_i v_i$ be a vector in M. Apply f to it: $f \cdot \sum_{i=0}^{\ell} a_i v_i = \sum_{i=1}^{\ell} a_{i-1} v_i$, which has to be an element of M too. Applying f repeatedly, we get that $a_0 v_\ell$ belongs to M and so $v_\ell \in M$. Then also $\sum_{i=0}^{\ell-1} a_i v_i$ is in M and repeat the process to show that all v_i are in M. So $M = V(\ell)$. □

Geometric construction.[4] Consider the "natural" $\mathfrak{g} = \mathfrak{sl}(2)$-action on $V(1)$. Fix a basis $\{x, y\}$ for $V(1)$ with x as the highest weight vector and $y = f \cdot x$. Now extend this action to the polynomial algebra $k[x, y] = S(V(1))$; the definition of the $\mathfrak{g}$-action on $S(V(1))$ from Lemma 1.7 shows that this action is by *derivations*. On the other hand, every derivation D of $k[x, y]$ is determined by $D(x)$ and $D(y)$ since $D = D(x)\partial_x + D(y)\partial_y$. Thus we see that the action can be written as follows:

$$e \mapsto x\partial_y, \quad h \mapsto x\partial_x - y\partial_y, \quad f \mapsto y\partial_x. \tag{2.2.2}$$

One may also verify directly that these assignments respect the $\mathfrak{sl}(2)$ relations.

[4] The reason we refer to this construction as geometric is the following. Consider the group $G = SL(2, \mathbb{C})$ of 2×2 matrices of determinant one acting via matrix multiplication on the space $\mathbb{C}^2 = \{(x, y)\}$. There is an induced action on polynomials in x and y and the action of the Lie algebra $\mathfrak{g}$ defined ad-hoc in this paragraph is in fact the differential (in the Lie groups sense) of the natural action of G on polynomials.

It is clear that these three operators preserve the total degree of any monomial. Therefore, the subspace $V(\ell)$ of homogeneous polynomials of degree ℓ is invariant under this action. Notice that $V(\ell)$ is the span of $\{x^\ell, x^{\ell-1}y, x^{\ell-2}y^2, \dots, y^\ell\}$, so it is $\ell+1$ dimensional. We have

$$e \cdot x^\ell = 0, \quad h \cdot (x^{\ell-i}y^i) = (\ell - 2i)x^{\ell-i}y^i, \quad f \cdot (x^{\ell-i}y^i) = (\ell - i)x^{\ell-i-1}y^{i+1},$$

so that the correspondence

$$x^{\ell-i}y^i \longleftrightarrow (\ell - i)! v_i$$

defines an isomorphism between this realization of the module of $V(\ell)$ and the algebraic one defined before.

Theorem 2.9.

(1) *Every $\mathfrak{sl}(2)$-module V with $0 < \dim V < \infty$ contains a submodule isomorphic to one of the $V(\ell)$'s.*
(2) *The Casimir element C acts on $V(\ell)$ by $\ell(\ell+2)$.*
(3) *The modules $V(\ell)$ are irreducible, distinct, and exhaust all (isomorphism classes of) finite dimensional irreducible $\mathfrak{sl}(2)$-modules.*

Proof. (1) Consider all eigenvalues of V with respect to the action of h. Since V is finite dimensional, there exists an eigenvalue λ such that $\lambda + 2$ is not an eigenvalue. Let $v_0 \neq 0$ be an eigenvector for this λ. By Lemma 2.7, $\lambda = \ell$ for some ℓ and $L = V(\ell) \subset V$.

(2) We compute directly that $C \cdot v_0 = \ell(\ell+2)v_0$. If v is some other vector in $V(\ell)$, there exists $x \in U(\mathfrak{sl}(2))$ such that $v = x \cdot v_0$. Then $C \cdot v = Cx \cdot v_0 = xC \cdot v_0 = \ell(\ell+2)x \cdot v_0 = \ell(\ell+2)v$.

(3) We know that $V(\ell)$ is irreducible by Lemma 2.8. Since the scalar by which C acts on each $V(\ell)$ determines ℓ uniquely, it follows that these modules are non-isomorphic. And if V is an irreducible $\mathfrak{sl}(2)$-module then V contains a copy of some $V(\ell)$ by (1) and therefore must be equal to it by irreducibility. $\square$

Remark 2.10. From the construction of the modules $V(\ell)$, we see that on every $V(\ell)$, the element h acts semisimply and the weights are $\{\ell, \ell-2, \ell-4, \dots, -\ell+2, -\ell\}$ and each weight space is one dimensional.

2.3. **Complete reducibility.** In this subsection, we prove directly that every finite dimensional $\mathfrak{sl}(2)$-module is completely reducible. This completes the classification of finite dimensional $\mathfrak{sl}(2)$-modules.

Proposition 2.11. *Every finite dimensional $\mathfrak{sl}(2)$-module V is isomorphic to a direct sum of modules $V(\ell)$, $\ell \geq 0$. In particular, V is completely reducible.*

Proof. (Bernstein) We'll use a general criterion whose proof is an exercise: if every module of length 2 is completely reducible, then every module of finite length is completely reducible. This reduces the proof to the case when V has length 2 with simple submodule $S = V(\ell)$ and simple quotient $Q = V/S \cong V(k)$.

If $k \neq \ell$, then the Casimir element acts with different eigenvalues on S and Q. Therefore, V splits into a direct sum of two generalized eigenspaces for C, one with eigenvalue $\ell(\ell+1)$ and the other with eigenvalue $k(k+1)$. Since C is central in $U(\mathfrak{sl}(2))$, both of these eigenspaces are $\mathfrak{sl}(2)$-submodules and we are done.

Assume $k = \ell$. Decompose V into generalized h-eigenspaces $V = \oplus V_i$. By assumption, $i \in \{-\ell, \ell+2, \dots, \ell-2, \ell\}$ and $\dim V_i = 2$. We claim that $f^\ell : V_\ell \to V_{-\ell}$ is a linear isomorphism. Let $0 \neq v \in V_\ell$ be given. If $v \in S$, then v is a highest weight vector with weight ℓ and so $f^\ell v \neq 0$. Otherwise, $v + S \neq S$ in $Q = V/S$, but $v + S \in Q_\ell$, so $f^\ell(v+S) \neq S$, implying that $f^\ell v \notin S$.

Now consider the identity from Lemma 2.1

$$ef^{\ell+1} - f^{\ell+1}e = \ell f^\ell(h - \ell)$$

acting on V_ℓ. Since the left hand side is 0, the right hand side must be 0 too. But f^ℓ is invertible on V_ℓ as we argued before, which means that $h = \ell \cdot \operatorname{Id}$ on V_ℓ. In other words, $V_\ell = V_\ell^{\mathrm{ss}}$. But this gives two linearly independent highest weight vectors with weight ℓ in V, and V decomposes as the sum of the $\mathfrak{sl}(2)$-submodules that these two vectors generate. $\square$

Corollary 2.12. *Let V be a finite dimensional $\mathfrak{sl}(2)$-module.*

(1) *h acts semisimply on V with integer weights.*
(2) *For every weight $i \geq 0$, $f^i : V_i \to V_i$ and $e^i : V_{-i} \to V_i$ are linear isomorphisms.*

Proof. Both claims follow from the complete reducibility of V and the corresponding statements (which we know are true) for the modules $V(\ell)$. $\square$

3. Some basic facts about semisimple Lie algebras

In this section, we recall a few basic definitions and results about the structure of semisimple Lie algebras. These results will be used in the sequel. We do not give proofs of these facts, but many missing proofs can be found in the lecture notes for C2.1 Lie Algebras, which are available here:

`https://courses.maths.ox.ac.uk/node/view_material/42444`

$\mathfrak{g}$ will be a finite dimensional Lie algebra over an algebraically closed field k of characteristic zero.

3.1. **Nilpotent and solvable Lie algebras.** The *lower central series* of $\mathfrak{g}$ is the decreasing chain of ideals $C^0\mathfrak{g} \supseteq C^1\mathfrak{g} \supseteq C^2\mathfrak{g} \supseteq \cdots \supseteq C^i\mathfrak{g} \supseteq \ldots$ defined inductively by $C^0\mathfrak{g} = \mathfrak{g}$ and $C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}]$ for $i \geq 1$. We say that $\mathfrak{g}$ is nilpotent if there exists $N > 0$ such that $C^N\mathfrak{g} = 0$.

The *derived series* of $\mathfrak{g}$ is the decreasing chain of ideals $D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq D^2\mathfrak{g} \supseteq \cdots \supseteq D^i\mathfrak{g} \supseteq \ldots$ defined inductively by $D^0\mathfrak{g} = \mathfrak{g}$ and $D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}]$ for $i \geq 1$. We say that $\mathfrak{g}$ is *solvable* if there exists $N > 0$ such that $D^N\mathfrak{g} = 0$. The ideal $D\mathfrak{g} := D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is called the *derived subalgebra* of $\mathfrak{g}$.

Since for every i, $D^i\mathfrak{g} \subseteq C^i\mathfrak{g}$, it is clear that every nilpotent Lie algebra is solvable. The converse is false.

Definition 3.1. *Let V be a finite dimensional vector space.*

1. *A* flag $\mathcal{F}$ *in V is a collection of vector subspaces $\mathcal{F} = (V_0 \subset V_1 \subset \cdots \subset V_m)$.*
2. *$\mathcal{F}$ is a* complete flag *if $\dim V_i = i$ for all i.*
3. *The* stabiliser *of $\mathcal{F}$ is $\mathfrak{b}_\mathcal{F} := \{x \in \mathfrak{gl}(V) \mid x(V_i) \subset V_i, \forall i\}$.*
4. *We also define $\mathfrak{n}_\mathcal{F} := \{x \in \mathfrak{gl}(V) \mid x(V_i) \subset V_{i-1}, \forall i \geq 1\}$.*

Suppose $\mathcal{F}$ is complete. If we choose a basis $\{e_1, \ldots, e_n\}$ of V and set $V_i = \operatorname{span}\{e_1, \ldots, e_i\}$, then $\mathfrak{b}_\mathcal{F}$ is identified with the algebra of upper triangular matrices with respect to this basis, while $\mathfrak{n}_\mathcal{F}$ is the algebra of strictly upper triangular matrices.

Example 3.2. Suppose that $\mathcal{F}$ is a flag.

1. $\mathfrak{n}_\mathcal{F} \subset \mathfrak{b}_\mathcal{F}$ are Lie subalgebras of $\mathfrak{gl}(V)$.
2. $\mathfrak{n}_\mathcal{F}$ is a nilpotent Lie algebra.
3. If $\mathcal{F}$ is a complete flag, then $\mathfrak{b}_\mathcal{F}$ is a solvable Lie algebra (but not nilpotent unless $\dim V = 1$).
4. If $\mathcal{F}$ is a complete flag, then $\mathfrak{n}_\mathcal{F}$ is the derived subalgebra of $\mathfrak{b}_\mathcal{F}$.

It is easy to see that if I and J are two solvable ideals of $\mathfrak{g}$, then $I + J$ is also a solvable ideal. This implies that there exists a unique maximal solvable ideal in $\mathfrak{g}$, called the *radical* of $\mathfrak{g}$, $\operatorname{rad}(\mathfrak{g})$.

Definition 3.3.

1. *$\mathfrak{g}$ is called* semisimple *if $\operatorname{rad}(\mathfrak{g}) = 0$.*
2. *$\mathfrak{g}$ is called* simple *if $\mathfrak{g}$ is non-abelian and $\mathfrak{g}$ doesn't have any proper ideals.*

3.2. **The Killing form.**

Definition 3.4. *The* Killing form *of $\mathfrak{g}$ is the pairing*

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to k, \quad \kappa(x, y) = \operatorname{tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)), \; x, y \in \mathfrak{g}. \tag{3.2.1}$$

It is a symmetric, bilinear form on $\mathfrak{g}$ and it is $\mathfrak{g}$-invariant, meaning that

$$\kappa([x, y], z) + \kappa(y, [x, z]) = 0, \quad x, y, z \in \mathfrak{g}. \tag{3.2.2}$$

The following result is important:

Theorem 3.5 (Cartan's criteria)**.**

1. *$\mathfrak{g}$ is semisimple if and only κ is nondegenerate.*
2. *$\mathfrak{g}$ is solvable if and only if $\kappa_{|D\mathfrak{g} \times D\mathfrak{g}} = 0$.*

Proof. (1) This is Theorem 13.2.

(2) This is Theorem 11.3. □

Corollary 3.6. *$\mathfrak{g}$ is semisimple if and only if f it is a direct sum of simple ideals. The decomposition into a sum of simple ideals is unique.*

Proof. This is Proposition 14.3. □

Example 3.7. The classical Lie algebras $\mathfrak{sl}(V)$, $\mathfrak{so}(V)$ (defined with respect to a nondegenerate symmetric bilinear form), and $\mathfrak{sp}(V)$ (with respect to a nondegenerate skew-symmetric bilinear form) are all simple, hence semisimple, Lie algebras.

3.3. **Maximal toral subalgebras.** Because we are working over a ground field k which we assume to be algebraically closed, a linear endomorphism $T : V \to V$ of a finite dimensional k-vector space is semisimple if and only if it is diagonalisable.

Definition 3.8. *We say that a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is* toral *if $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is semisimple for all $x \in \mathfrak{h}$.*

Lemma 3.9. *Every toral Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is abelian.*

Proof. Suppose for a contradiction that $\mathfrak{h}$ is not abelian. Then there exists $y \in \mathfrak{h}$ such that the endomorphism $S := \mathrm{ad}(y)_{|\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$ is non-zero. Choose a non-zero eigenvalue λ of S and a corresponding eigenvector $x \in \mathfrak{h}$ so that $S(y) = [y, x] = \lambda x$, and consider $T := \mathrm{ad}(x)_{|\mathfrak{h}}$. Then $T^2(y) = [x, [x, y]] = [x, -\lambda x] = 0$. But T is also semisimple, and this forces $T(y) = [y, x] = 0$, a contradiction. $\square$

Definition 3.10. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. We say that $\mathfrak{h}$ is a* Cartan subalgebra *if*

(1) *$\mathfrak{h}$ is nilpotent;*
(2) *$\mathfrak{h}$ is self-normalizing, i.e., $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$ where $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid \mathrm{ad}(x)(\mathfrak{h}) \subset \mathfrak{h},$ for all $x \in \mathfrak{g}\}$.*

Definition 3.11. *Let $\mathfrak{g}$ be a Lie algebra and let $x \in \mathfrak{g}$ be* ad*-nilpotent. The map*

$$e^{\mathrm{ad}(x)} : \mathfrak{g} \to \mathfrak{g}$$

is called an elementary automorphism *of $\mathfrak{g}$.*

Since $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent and we are working over a field k of characteristic zero, the exponential series $e^{\mathrm{ad}(x)}$ makes sense. Since $\mathrm{ad}(x)$ is a derivation of the Lie algebra $\mathfrak{g}$, $e^{\mathrm{ad}(x)}$ is in fact a Lie algebra automorphism of $\mathfrak{g}$.

Theorem 3.12. *Let $\mathfrak{g}$ be a semisimple Lie algebra.*

(1) *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal toral subalgebra if and only if it is a Cartan subalgebra.*
(2) *Cartan subalgebras exist.*
(3) *For any two Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{g}$, there exist elementary automorphisms $\theta_1, \cdots, \theta_m$ of $\mathfrak{g}$ such that $\mathfrak{h}_2 = (\theta_1 \circ \theta_2 \circ \cdots \circ \theta_m)(\mathfrak{h}_1)$.*

Proof. (1) This is [Hu1, Corollary 15.3]; also see Lemma 17.2.
(2) This is Lemma 8.3(2).
(3) This is [Hu1, Corollary 16.4]. $\square$

To construct a Cartan subalgebra, consider for each $x \in \mathfrak{g}$ the generalized 0-eigenspace of $\mathrm{ad}(x)$

$$\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} \mid \mathrm{ad}(x)^N y = 0, \text{ for some } N > 0\}.$$

One can show that $\mathfrak{g}_{0,x}$ is always a Lie subalgebra. An element x is called *regular* if $\dim \mathfrak{g}_{0,x}$ is minimal among all such subalgebras. One can show that every subalgebra $\mathfrak{g}_{0,x}$, where x is regular, is a Cartan subalgebra — see Lemma 8.3.

Example 3.13. If $\mathfrak{g} = \mathfrak{s}l(n)$, then the usual choice of a maximal toral subalgebra is $\mathfrak{h}$ consisting of diagonal, trace 0, matrices.

3.4. **Cartan decomposition.** From now on, $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h}$ is a fixed maximal toral subalgebra. The main tool for the structure of $\mathfrak{g}$ is the *Cartan decomposition.* Decompose $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian and the restriction of the adjoint representation to $\mathfrak{h}$ is semisimple, basic linear algebra tells us that $\mathfrak{g}$ decomposes into a direct sum of $\mathfrak{h}$-eigenspaces:

$$\mathfrak{g} = \bigoplus_{\chi \in \mathfrak{h}^*} \mathfrak{g}_\chi, \quad \mathfrak{g}_\chi = \{x \in \mathfrak{g} \mid [h, x] = \chi(h)x, \ h \in \mathfrak{h}\}.$$

Proposition 3.14. $\mathfrak{g}_0 = \mathfrak{h}$.

Proof. See [Hu1, Proposition 8.2]. $\square$

Definition 3.15.

(1) $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$ *is the set of* roots *of* $\mathfrak{g}$ *(with respect to* $\mathfrak{h}$*).*
(2) *The* Cartan decomposition *or* root space decomposition *of* $\mathfrak{g}$ *is*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha. \tag{3.4.1}$$

(3) *The spaces* $\mathfrak{g}_\alpha$ *are called* root spaces *and every nonzero vector in* $\mathfrak{g}_\alpha$ *is called a* root vector.

Example 3.16. Let $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{h}$ be the diagonal matrices of trace 0. In coordinates, we may think of $\mathfrak{h}$ as $\mathfrak{h} = \{(a_1, \dots, a_n) \in k^n \mid \sum_i a_i = 0\}$. The dual space $\mathfrak{h}^*$ is naturally identified with

$$\mathfrak{h}^* = k\langle \epsilon_1, \dots, \epsilon_n\rangle / \langle \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n\rangle,$$

where $\epsilon_i : \mathfrak{h} \to k$ is defined by $\epsilon_i(a_1, a_2, \dots, a_n) = a_i$. The roots are

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\} \subset \mathfrak{h}^*.$$

The Cartan decomposition is $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i\neq j} \mathfrak{g}_{\epsilon_i - \epsilon_j}$, with $\mathfrak{g}_{\epsilon_i-\epsilon_j} = k \cdot E_{ij}$, where E_{ij} is the elementary matrix that has 1 on the (i, j) position and 0 everywhere else.

Here is a list of the main facts about this decomposition. The main tool for proving the nontrivial statements is the Cartan criterion for semisimplicity, Theorem 3.5(1).

Theorem 3.17.

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, $\alpha, \beta \in \Phi \cup \{0\}$.
(2) $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *unless* $\beta = -\alpha$.
(3) *The restriction of* κ *to* $\mathfrak{h} \times \mathfrak{h}$ *is nondegenerate.*
(4) *The restriction of* κ *to* $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ *is non-degenerate for all* $\alpha \in \Phi$.
(5) Φ *spans* $\mathfrak{h}^*$.
(6) $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.
(7) *The Killing form on* $\mathfrak{h}$ *can be computed by the formula*

$$\kappa(h, h') = \sum_{\alpha\in\Phi} \alpha(h)\alpha(h'), \quad h, h' \in \mathfrak{h}. \tag{3.4.2}$$

(8) *For every* $\alpha \in \Phi$, *there exist vectors* $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_\alpha$, $h_\alpha := [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$ *such that* $\{e_\alpha, h_\alpha, e_{-\alpha}\}$ *satisfy the* $\mathfrak{sl}(2)$*-relations. We have* $\alpha(h_\alpha) = 2$.

Proof. (1) If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ then $[x, y] \in \mathfrak{g}_{\alpha+\beta}$ because

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y] \quad \text{for all} \quad h \in \mathfrak{h}.$$

(2,3) This is Proposition 17.1.

(4) From (2), $\mathfrak{g}_\alpha$ is κ-orthogonal to all other $\mathfrak{g}_\beta$ except possibly $\mathfrak{g}_{-\alpha}$. So if $\kappa(x, \mathfrak{g}_{-\alpha}) = 0$ for some $x \in \mathfrak{g}_\alpha$ then $\kappa(x, \mathfrak{g}) = 0$ which forces $x = 0$ because κ is non-degenerate by Theorem 3.5(1).

(5,6) See Proposition 17.5(2) and Lemma 17.7.

(7) The matrix of $\mathrm{ad}(h) \circ \mathrm{ad}(h')$ is diagonal with entry $\alpha(h)\alpha(h')$ corresponding to any basis vector of $\mathfrak{g}_\alpha$. But $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$ by (5). Now take the trace.

(8) See Proposition 17.5(4). ☐

Definition 3.18. *We let* $\mathfrak{sl}_\alpha$ *denote the copy of* $\mathfrak{sl}(2)$ *spanned by the* $\mathfrak{sl}(2)$*-triple* $\{e_\alpha, h_\alpha, e_{-\alpha}\}$.

It follows from Theorem 3.17 that $\mathfrak{g}$ is spanned by all of these copies of $\mathfrak{sl}(2)$. In this way, we may regard $\mathfrak{g}$ as being "glued" out of finitely many copies of $\mathfrak{sl}(2)$. This idea can be made more precise by considering the *Serre presentation* of $\mathfrak{g}$ — see [Hu1, Theorem 18.3].

By Theorem 3.17(3), the Killing form is nondegenerate on $\mathfrak{h}$. Let $\lambda \mapsto t_\lambda$ be the isomorphism $\mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}$ induced by it, so that $\kappa(t_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$.

Lemma 3.19.

(1) *For all* $\alpha \in \Phi$, $\alpha(t_\alpha) \neq 0$ *and* $h_\alpha = \frac{2}{\alpha(t_\alpha)} t_\alpha$.
(2) *The coroots* $\{h_\alpha : \alpha \in \Phi\}$ *span* $\mathfrak{h}$.
(3) *If* $\alpha, \beta, \alpha + \beta \in \Phi$ *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
(4) $\beta - \beta(h_\alpha)\alpha \in \Phi$ *for all* $\alpha, \beta \in \Phi$.

Proof. (1) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$. Using the symmetry and $\mathfrak{g}$-invariance of κ, we have

$$\kappa([x,y],h) = \kappa(h,[x,y]) = \kappa([h,x],y) = \kappa(\alpha(h)x,y) = \alpha(h)\kappa(x,y) = \kappa(x,y)\kappa(t_\alpha,h).$$

Therefore $[x,y]-\kappa(x,y)t_\alpha$ is orthogonal to $\mathfrak{h}$ and is therefore 0 by Theorem 3.17(3): this shows that $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] \subseteq kt_\alpha$. On the other hand, this space is spanned by h_α by Theorem 3.17(6,8) so we can write $h_\alpha = c_\alpha t_\alpha$ for some non-zero $c_\alpha \in k$. But then $2 = \alpha(h_\alpha) = c_\alpha \alpha(t_\alpha)$.

(2) We know that $\{t_\alpha : \alpha \in \Phi\}$ spans $\mathfrak{h}$ by Theorem 3.17(3,5). Now apply part (1).

(3) Regard $\mathfrak{g}$ as an $\mathfrak{sl}_\alpha$-module via the adjoint representation and consider $V := U(\mathfrak{sl}_\alpha)\cdot e_\beta$. Then V is a cyclic finite dimensional $U(\mathfrak{sl}_\alpha)$-module and therefore has a highest weight. Since V is finite dimensional, it is isomorphic to a $V(\ell)$ by Lemma 2.7, and now $\{e_{\beta+m\alpha} : \beta+m\alpha \in \Phi\}$ form a basis for V. So (2.2.1) implies that $e_\alpha \cdot e_\beta \neq 0$ if $\mathfrak{g}_{\alpha+\beta} \neq \{0\}$.

(4) See Proposition 17.8. □

3.5. **Root lattices, weight lattices and weights.**

Definition 3.20.

(1) *The element* $h_\alpha \in \mathfrak{h}$ *from Theorem 3.17(8) is called the* coroot *corresponding to* $\alpha \in \Phi$.
(2) $Q^\vee := \mathbb{Z}\{h_\alpha : \alpha \in \Phi\}$ *is called the* coroot lattice.
(3) $Q := \mathbb{Z}\Phi := \left\{\sum_{\alpha\in\Phi} a_\alpha \alpha \mid a_\alpha \in \mathbb{Z}\right\} \subset \mathfrak{h}^*$ *is called the* root lattice.
(4) $P := \{\chi \in \mathfrak{h}^* \mid \chi(h_\alpha) \in \mathbb{Z}$ *for all* $\alpha \in \Phi\} \subset \mathfrak{h}^*$ *is called the* integral weight lattice.

Definition 3.21. *Let V be an $\mathfrak{g}$-module and let $\lambda \in \mathfrak{h}^*$.*

(1) *The* generalized λ-weight space *of V is*

$$V_\lambda = \{v \in V : (h-\lambda(h))^n \cdot v = 0 \quad \textit{for sufficiently large} \quad n \in \mathbb{N}\}.$$

(2) $V_\lambda^{\mathrm{ss}} := \{v \in V : h\cdot v = \lambda(h)v$ *for all* $h \in \mathfrak{h}\}$ *is the* λ-weight space *of V.*
(3) *We say that V is* $\mathfrak{h}$-semisimple *if* $V_\lambda = V_\lambda^{\mathrm{ss}}$ *for all* $\lambda \in \mathfrak{h}^*$ *and* $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$.
(4) *The set of* weights *of V is by definition* $\Psi(V) := \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\}$.

Lemma 3.22. *Let V be a finite dimensional $\mathfrak{g}$-module. Then every weight of V is integral: $\Phi(V) \subset P$.*

Proof. Recall from Definition 3.18 that for every $\alpha \in \Phi^+$, we have a copy of $\mathfrak{sl}(2)$ in $\mathfrak{g}$ called $\mathfrak{sl}_\alpha$. Regard V as a finite dimensional $\mathfrak{sl}_\alpha$-module. Then if $\lambda \in \Psi(V)$, h_α acts on V_λ by the scalar $\lambda(h_\alpha)$. But this scalar must be an integer by Corollary 2.12(1). Hence $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. □

Corollary 3.23. *$Q \subseteq P$, that is: $\beta(h_\alpha) \in \mathbb{Z}$ for all $\alpha,\beta \in \Phi$.*

Proof. Note that $\Phi \cup \{0\} = \Psi(\mathfrak{g})$ when $\mathfrak{g}$ is viewed as a $\mathfrak{g}$-module via the adjoint representation. Therefore $\Phi \subseteq \Psi(\mathfrak{g}) \subset P$ by Lemma 3.22. □

Example 3.24. Suppose that $\mathfrak{g} = \mathfrak{sl}(n)$. The lattices appearing in Definition 3.20 are:

$$\begin{aligned} Q^\vee &= \left\{(a_1,\dots,a_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i = 0\right\}; \\ Q &= \left\{\sum_{i=1}^n a_i\epsilon_i \mid a_i \in \mathbb{Z},\ \sum_{i=1}^n a_i = 0\right\}/\mathbb{Z}(\epsilon_1+\epsilon_2+\cdots+\epsilon_n); \\ P &= (\mathbb{Z}\epsilon_1 \oplus\cdots\oplus \mathbb{Z}\epsilon_n)/\mathbb{Z}(\epsilon_1+\epsilon_2+\cdots+\epsilon_n). \end{aligned} \tag{3.5.1}$$

In this case, $Q \subsetneq P$ and, in fact $P/Q \cong \mathbb{Z}/n\mathbb{Z}$ — see Sheet 2 Question 3.

3.6. **The Weyl group.** We will now define a very important group that will play a large role in the representation theory of $\mathfrak{g}$. First, define a nondegenerate bilinear form $(-,-)$ on $\mathfrak{h}^*$ by "transport of structure":

$$(\lambda,\mu) := \kappa(t_\lambda,t_\mu) = \lambda(t_\mu) = \mu(t_\lambda) \quad \text{for all} \quad \lambda,\mu \in \mathfrak{h}^*. \tag{3.6.1}$$

Lemma 3.25. *κ is positive definite on $Q^\vee$ and $(-,-)$ is positive definite on P.*

Proof. Since $\kappa(h,h) = \sum_{\alpha\in\Phi}\alpha(h)^2$ and since Φ spans $\mathfrak{g}^*$ by Theorem 3.17(7,5), κ is positive definite on $Q^\vee$. Therefore $(-,-)$ is positive definite on P because $(\lambda,\lambda) = \kappa(t_\lambda,t_\lambda) > 0$ whenever $\lambda \neq 0$. □

Definition 3.26.

(1) *For every* $\alpha \in \Phi$, *the* reflection in the hyperplane perpendicular to α *is :*

$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*, \quad s_\alpha(\chi) = \chi - \frac{2(\chi, \alpha)}{(\alpha, \alpha)}\alpha. \tag{3.6.2}$$

(2) *The* Weyl group *of* $\mathfrak{g}$ *is the subgroup of* $GL(\mathfrak{h}^*)$ *generated by* $\{s_\alpha : \alpha \in \Phi\}$.

s_α fixes the hyperplane $\alpha^\perp \subset \mathfrak{h}^*$ and sends α to $-\alpha$, so it really is the reflection in the hyperplane $\alpha^\perp$.

Lemma 3.27.

(1) *For all* $\chi \in \mathfrak{h}^*$ *and all* $\alpha \in \Phi$, *we have* $\frac{2(\chi,\alpha)}{(\alpha,\alpha)} = \chi(h_\alpha)$.
(2) W *is a finite subgroup of the orthogonal group* $O(\mathfrak{h}^*, (-,-))$.

Proof. (1) Using Lemma 3.19(1) together with (3.6.1), we calculate

$$\frac{2(\chi, \alpha)}{(\alpha, \alpha)} = \frac{2\kappa(t_\alpha, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \frac{2\chi(t_\alpha)}{\alpha(t_\alpha)} = \chi\left(\frac{2t_\alpha}{\alpha(t_\alpha)}\right) = \chi(h_\alpha).$$

(2) Let $\alpha, \beta \in \Phi$. Then using part (1) together with Lemma 3.19(4), we have

$$s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha \in \Phi.$$

So, every s_α preserves $\Phi \subset \mathfrak{h}^*$, and thus we get a permutation action $W \to \mathsf{Sym}(\Phi)$. The kernel of this action is trivial since Φ spans $\mathfrak{h}^*$ by Theorem 3.17(5). So W is isomorphic to a subgroup of the finite group $\mathsf{Sym}(\Phi)$ and is hence finite. In an inner product space of dimension n, each reflection in a hyperplane has matrix $\operatorname{diag}(-1, 1, \cdots, 1)$ with respect to an appropriate basis, and is hence orthogonal. So $W \subset O(\mathfrak{h}^*, (-,-))$. $\square$

If $\alpha, \beta \in \Phi$, then $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \beta(h_\alpha) := \langle \alpha, \beta \rangle$ is an integer by Corollary 3.23, called a *Cartan integer*.

3.7. **Positive roots.**

Definition 3.28. *We say that the subset* $\Pi \subset \Phi$ *is a* base *for* Φ *if*

(1) Π *is a basis for* $\mathfrak{h}^*$,
(2) $\Phi \subset \mathbb{Z}\Pi$.

Lemma 3.29. *The root system* Φ *has at least one base* Π.

Proof. Work with the real vector space $\mathfrak{a}^* := \mathbb{R} \otimes_{\mathbb{Z}} Q$. Then using Lemma 3.25, $(-,-)$ extends to an inner product on $\mathfrak{a}^*$, and we can consider for every non-zero $v \in \mathfrak{a}^*$ the half-space $\{w \in \mathfrak{a}^* : (v, w) > 0\}$. Then $\Phi = \Phi^+(v) \cup \Phi^-(v)$ where $\Phi^+(v) = \{\alpha \in \Phi : (v, \alpha) > 0\}$ and $\Phi^-(v) := \{\alpha \in \Phi : (v, \alpha) < 0\}$. Then it is shown in Proposition 19.9 that the set $\Pi(v) \subset \Phi^+(v)$ consisting of *indecomposable roots* in $\Phi^+(v)$ is a base for Φ, and in fact, *every* base for Φ arises in this way. $\square$

Definition 3.30. *Let* $\Pi \subset \Phi$ *be a base.*

(1) *Elements of* Π *are called* simple roots.
(2) $\Phi^+ := \mathbb{N}\Pi \cap \Phi$ *is the set of* positive roots.
(3) $\Phi^- := (-\mathbb{N})\Pi \cap \Phi$ *is the set of* negative roots.

Thus, every choice of base Π for Φ induces a decomposition $\Phi = \Phi^+ \cup \Phi^-$.

Definition 3.31. *Fix a choice of base* $\Pi \subset \Phi$.

(1) *Let* $Q^+ := \mathbb{N}\Pi \subset Q$.
(2) *Define a partial order* $\leq$ *on* $\mathfrak{h}^*$ *by* $\alpha \leq \beta \Leftrightarrow \beta - \alpha \in Q^+$.

We always have $\Pi \subset \Phi^+ \subset Q^+$.

4. The category $\mathcal{O}$

4.1. **Definitions.** Let $\mathfrak{g}$ be a semisimple Lie algebra over k, an algebraically closed field of characteristic zero. Fix a maximal toral subalgebra $\mathfrak{h}$ and let Φ be the roots of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the Cartan decomposition. Fix a choice of base $\Pi \subset \Phi$ and let Φ^+ be the corresponding positive roots. We retain all the other notation from the previous section. Denote

$$\mathfrak{n}^+ = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+. \tag{4.1.1}$$

It is easy to prove the following lemma by using the commutation relations between $\mathfrak{h}$, e_α, $\alpha \in \Phi^+$.

Lemma 4.1.
1. *The subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent.*
2. *The* Borel subalgebra $\mathfrak{b}$ *is solvable and its derived subalgebra is $\mathfrak{n}^+$.*

As a consequence of the PBW Theorem 1.17, we have the following *triangular decomposition* of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+). \tag{4.1.2}$$

We will use this decomposition repeatedly in this section.

Lemma 4.2. *Let $\lambda \in \mathfrak{h}^*$ and let $\alpha \in \Phi$. Then*
1. *$e_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$, and*
2. *$e_\alpha \cdot V_\lambda^{\mathrm{ss}} \subseteq V_{\lambda+\alpha}^{\mathrm{ss}}$.*

Definition 4.3. *The category $\mathcal{O}$ of $\mathfrak{g}$ is the full subcategory of (left) $U(\mathfrak{g})$-modules whose objects M satisfy the following conditions:*

($\mathcal{O}1$) *M is a finitely generated $U(\mathfrak{g})$-module;*
($\mathcal{O}2$) *M is $\mathfrak{h}$-semisimple;*
($\mathcal{O}3$) *M is locally $\mathfrak{n}^+$-finite, i.e., for every $v \in M$, the subspace $U(\mathfrak{n}^+) \cdot v$ is finite dimensional.*

Example 4.4. Recall the modules $V(\ell)$ constructed for $\mathfrak{sl}(2)$ with basis $\{v_0, v_1, \dots, v_\ell\}$. Given the explicit construction, we can see immediately that these modules are in the category $\mathcal{O}$.

Lemma 4.5. *Every finite dimensional $\mathfrak{g}$-module is in $\mathcal{O}$.*

Proof. If V is finite dimensional, then ($\mathcal{O}1$) and ($\mathcal{O}3$) are automatic. For ($\mathcal{O}2$), regard V as a finite dimensional $\mathfrak{sl}_\alpha$-module; V is h_α-semisimple by Corollary 2.12. But now a standard linear algebra fact implies that the action the span of the h_α's is semisimple, because the h_α's commute. Therefore all of $\mathfrak{h}$ acts semisimply on V because it is spanned by the coroots $\{h_\alpha \mid \alpha \in \Phi\}$ by Lemma 3.19(2). □

We record some of the immediate properties of $\mathcal{O}$ in the next proposition.

Proposition 4.6.
1. *$\mathcal{O}$ is a Noetherian category, i.e., every $M \in \mathcal{O}$ is a Noetherian $U(\mathfrak{g})$-module.*
2. *$\mathcal{O}$ is closed under taking submodules, quotients, and finite direct sums. Hence $\mathcal{O}$ is an abelian category.*
3. *If $M \in \mathcal{O}$ and L is finite dimensional, then $L \otimes M \in \mathcal{O}$.*
4. *If $M \in \mathcal{O}$, then M is finitely generated as a $U(\mathfrak{n}^-)$-module.*

Proof. (1) $U(\mathfrak{g})$ is Noetherian[5] and M is a finitely generated $U(\mathfrak{g})$-module. Therefore M is Noetherian.

(2) The only statement that needs explanation is the fact that ($\mathcal{O}1$) holds for submodules. But this is precisely because of the Noetherian property from (1): every submodule of a finitely generated module is finitely generated.

(3) The tensor product $L \otimes M$ satisfies ($\mathcal{O}2$) and ($\mathcal{O}3$). To prove finite generation, let $\{v_1, \dots, v_n\}$ be a basis of L and let $m_1, \dots, m_k$ generate M. Then $\{v_i \otimes m_j\}$ generates $L \otimes M$. To see this, let N be the submodule this set generates. Since every $v \in L$ can be written as $v = \sum a_i v_i$ with $a_i \in k$, we see that all simple tensors of the form $v \otimes m_j$, $v \in L$, are also in N. If $x \in \mathfrak{g}$, we calculate

$$x \cdot (v \otimes m_j) = x \cdot v \otimes m_j + v \otimes x \cdot m_j \in N.$$

[5] This is proved in the "Noncommutative rings" lectures — see Corollary 1.28(a) — so we won't repeat the proof here.

The first term is in N, so $v \otimes x \cdot m_j \in N$. Repeating this, we see that $v \otimes u \cdot m_j \in N$ for all $u \in U(\mathfrak{g})$. But then $L \otimes M \subset N$, which concludes the proof.

(4) Because of the axioms, we see that M is generated as a $U(\mathfrak{g})$-module by a finite dimensional $U(\mathfrak{b})$-module V. By the PBW Theorem 1.17, we have $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{b}^+)$. So, a basis of V generates M as a $U(\mathfrak{n}^-)$-module. □

4.2. **Highest weight modules.**

Definition 4.7. *Let M be a $U(\mathfrak{g})$-module. A nonzero vector $v \in M$ is called a* highest weight vector *(of weight λ) if $v \in M_\lambda$ for some $\lambda \in \Psi(M)$ and $\mathfrak{n}^+ \cdot v = 0$. The last condition is equivalent with $e_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$.*

Definition 4.8. *A $U(\mathfrak{g})$-module M is called a* highest weight module *of weight λ if there exists $v \in M_\lambda$ such that $\mathfrak{n}^+ \cdot v = 0$ and $M = U(\mathfrak{g}) \cdot v$. The last condition is equivalent with $M = U(\mathfrak{n}^-) \cdot v$ by the triangular decomposition.*

We list several immediate properties of highest weight modules.

Lemma 4.9. *Let M be a highest weight module with highest weight λ.*

(a) *Each nonzero quotient of M is also a highest weight module of weight λ.*
(b) *M is $\mathfrak{h}$-semisimple.*
(c) *$\Psi(M) \subset \lambda - Q^+$.*
(d) *$\dim M_\mu < \infty$ for all $\mu \in \Psi(M)$ and $\dim M_\lambda = 1$.*
(e) *$M \in \mathcal{O}$.*

Proof. (a) This is clear.

(b) Choose an ordering of the positive roots: $\alpha_1, \alpha_2, \ldots, \alpha_m$. Then M is spanned by the vectors $e_{-\alpha_1}^{i_1} \cdots e_{-\alpha_m}^{i_m} \cdot v$. Each such vector has weight $\lambda - \sum_{j=1}^m i_j \alpha_j$. Hence M is $\mathfrak{h}$-semisimple.

(c) Note that $e_{-\alpha_1}^{i_1} \ldots e_{-\alpha_m}^{i_m} \cdot v \in M_{\lambda - \sum_{j=1}^m i_j \alpha_j}$.

(d) $\beta \in Q^+$ has only finitely many ways of expressing it as an $\mathbb{N}$-linear combination of elements of Φ^+.

(e) Axiom ($\mathcal{O}3$) follows from Lemma 4.2: every $\alpha \in \Phi^+$ maps M_μ to $M_{\mu+\alpha}$. □

As a consequence, we can see that the highest weight modules are the building blocks of category $\mathcal{O}$:

Proposition 4.10. *Let $M \neq 0$ be a module in $\mathcal{O}$. There exists a finite filtration*

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

of modules in $\mathcal{O}$ such that M_i/M_{i+1} is a highest weight module.

Proof. M is generated by finitely many weight vectors $v_{\lambda_1}, \ldots, v_{\lambda_\ell}$. Set $V = U(\mathfrak{n}^+) \cdot \langle v_{\lambda_1}, \ldots, v_{\lambda_\ell} \rangle$. Because of ($\mathcal{O}3$), V is finite dimensional and, of course, $M = U(\mathfrak{g}) \cdot V = U(\mathfrak{n}^-) \cdot V$. Go by induction on $\dim V$.

Take $v \in V$ a weight vector for a maximal weight (among the weights that occur in V). Then v must be a highest weight vector in M. Set $M_1 = U(\mathfrak{g}) \cdot v$ which is a submodule of M and a highest weight module. Next $\overline{M} = M/M_1$ is generated by $\overline{V} = V/V \cap M_1$. Then $\dim \overline{V} < \dim V$ because $0 \neq v \in V \cap M_1$ and we are done by induction. □

Corollary 4.11. *Let M be a module in $\mathcal{O}$.*

(1) *For every $\lambda \in \Psi(M)$, the weight space M_λ is finite dimensional.*
(2) *There exist $\lambda_1, \ldots, \lambda_m \in \Psi(M)$ such that $\Psi(M) \subset \bigcup_{i=1}^m (\lambda_i - Q^+)$.*
(3) *M is $Z(\mathfrak{g})$-finite, i.e., for every $v \in M$, $Z(\mathfrak{g}) \cdot v$ is finite dimensional.*

Proof. (1,2) Note that if $0 \to N \to M \to M/N \to 0$ is an exact sequence in $\mathcal{O}$ then $\Psi(M) = \Psi(N) \cup \Psi(M/N)$. Now apply Proposition 4.10 and Lemma 4.9(c,d).

(3) If $v \in M$ we may write v as a sum of weight vectors. It is sufficient to prove the claim when $v \in M_\lambda$. Since $z \in Z(\mathfrak{g})$ commutes with $\mathfrak{h}$, we see that $z \cdot v \in M_\lambda$ as well. But M_λ is finite dimensional by (1), so $Z(\mathfrak{g}) \cdot v$ is finite dimensional. □

Example 4.12. We can verify that when $\mathfrak{g} = \mathfrak{sl}(2)$, the tensor product $M(\lambda) \otimes M(\mu)$ is *not* in $\mathcal{O}$.

Proposition 4.13. *Let $M \in \mathcal{O}$ be a highest weight module.*

(1) *M has a unique maximal submodule and hence a unique simple quotient. In particular, M is indecomposable.*
(2) *Let $\lambda \in \mathfrak{h}^*$ be given. All simple highest weight modules of weight λ are isomorphic. If M is a simple highest weight module of weight λ, then $\dim \mathrm{End}_{U(\mathfrak{g})}(M) = 1$.*

Proof. (1) If N is a proper submodule of M then $N \in \mathcal{O}$ (as $M \in \mathcal{O}$), hence N is $\mathfrak{h}$-semisimple. Write $N = \bigoplus_{\mu \in \Psi(N) \subset \Psi(M)} N_\nu$. Since M_λ is one-dimensional and every vector in M_λ generated M, it follows that $\lambda \notin \Psi(N)$. This implies that the sum of all proper submodules of M is still proper (λ is not a weight for any of them), and therefore there is a unique maximal submodule.

(2) Suppose M_1 and M_2 are two simple highest weight modules of the same weight λ. Let v_1, v_2 be highest weight vectors for M_1 and M_2, respectively. Then $v = v_1 + v_2$ is also a highest weight vector in $M = M_1 \oplus M_2$. Denote $N = U(\mathfrak{g}) \cdot v \subset M$. Then N is a highest weight module of weight λ. The two canonical projections give projections $N \to M_1$ and $N \to M_2$. Hence M_1 and M_2 are both simple quotients of N. By (1), $M_1 \cong M_2$.

For the second part, let M be a simple highest weight module of weight λ and let $\phi : M \to M$ be a nonzero $\mathfrak{g}$-homomorphism. Since M is simple, ϕ must be an isomorphism. Then it maps M_λ to M_λ. Fix a highest weight vector $v \in M_\lambda$. Since M_λ is one-dimensional, $\phi(v) = cv$ for some constant $c \in k$. But since v generates M, it follows that $\phi = c \cdot \mathrm{Id}$. □

4.3. **Verma modules.** Recall the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Since $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}$, we have the natural projection $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n}^+ \cong \mathfrak{h}$, which is a Lie algebra homomorphism. For every $\lambda \in \mathfrak{h}^*$, denote by k_λ the one-dimensional $\mathfrak{b}$-representation pulled back via this projection. Then $\mathfrak{n}^+$ acts by 0 on k_λ. We can also regard k_λ as a $U(\mathfrak{b})$-module.

Definition 4.14. *Let $\lambda \in \mathfrak{h}^*$ be given. Define the* Verma module of highest weight λ *to be*

$$M(\lambda) = U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} k_\lambda.$$

This is a left $U(\mathfrak{g})$-module under the natural action (left multiplication) of $U(\mathfrak{g})$.

Define the vector $v = 1 \otimes 1 \in M(\lambda)$. This is a highest weight vector of weight λ and $M(\lambda) = U(\mathfrak{g}) \cdot v$. This means that $M(\lambda)$ is indeed a highest weight module of weight λ. An alternative definition goes as follows. Let $I(\lambda) := U(\mathfrak{g})\mathfrak{n}^+ + \sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h))$. Then

$$M(\lambda) \cong U(\mathfrak{g})/I(\lambda).$$

Lemma 4.15 (Universal property)**.** *Suppose M is a highest weight module of weight λ. Then there exists a surjective $\mathfrak{g}$-linear map $p : M(\lambda) \twoheadrightarrow M$.*

Proof. Let $v' \in M$ be a highest weight vector. The assignment $v \mapsto v'$ extends to a $U(\mathfrak{g})$-homomorphism $M(\lambda) \to M$ which is surjective since v' generates M. Alternatively, start with the projection $\widetilde{p} : U(\mathfrak{g}) \to M$, $1 \mapsto v$. Since the left ideal $I(\lambda)$ kills M, $\widetilde{p}$ factors through $U(\mathfrak{g})/I(\lambda) \to M$. □

In other words, $M(\lambda)$ is the universal highest weight module with highest weight λ. We can also apply Proposition 4.13 to $M(\lambda)$.

Definition 4.16. *Let $\lambda \in \mathfrak{h}^*$.*

(1) *Let $N(\lambda)$ denote the unique maximal submodule of $M(\lambda)$.*
(2) *Let $L(\lambda) := M(\lambda)/N(\lambda)$ denote the unique simple quotient module of $M(\lambda)$.*

Theorem 4.17.

(1) *Every simple module in $\mathcal{O}$ is isomorphic to a module $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*
(2) *$L(\lambda) \cong L(\mu)$ if and only if $\lambda = \mu$.*

Proof. (1) Let M be a simple module in $\mathcal{O}$. It follows from Proposition 4.10 that M is a highest weight module of weight λ, where λ is a maximal weight in $\Psi(L)$. By Lemma 4.15, L is a quotient of the Verma module of $M(\lambda)$, hence $L \cong L(\lambda)$.

(2) Suppose that $L(\lambda) \cong L(\mu)$. Then $\lambda \in \Psi(L(\mu)) \subseteq \mu - Q^+$ by Lemma 4.9(c). Similarly, $\mu \in \lambda - Q^+$, so $\lambda - \mu \in Q^+ \cap -Q^+ = \{0\}$. So, $\lambda = \mu$. □

So, Theorem 4.17 and Proposition 4.13(2) imply that $\dim \mathrm{Hom}_{U(\mathfrak{g})}(L(\lambda), L(\mu)) = \Pi_{\lambda\mu}$ for any $\lambda, \mu \in \mathfrak{h}^*$.

4.4. **Finite dimensional modules.** We have now established a one-to-one correspondence

$$\mathfrak{h}^* \longrightarrow \{\text{simple modules in } \mathcal{O}\}, \quad \lambda \mapsto L(\lambda).$$

We would like to determine which modules $L(\lambda)$ are finite dimensional.

Definition 4.18.

(1) *The weight* $\lambda \in \mathfrak{h}^*$ *is said to be* integral dominant *if* $\lambda(h_\alpha) \in \mathbb{N}$ *for all* $\alpha \in \Phi^+$.
(2) P^+ *will denote the set of integral dominant weights.*

We will frequently drop the adjective "integral" in this definition.

Theorem 4.19. *Let* $\lambda \in \mathfrak{h}^*$. *Then the simple module* $L(\lambda) \in \mathcal{O}$ *is finite dimensional if and only if* $\lambda \in P^+$.

For the proof, we need to analyze first the structure of $M(\lambda)$. First, we prove the following basic

Lemma 4.20. *Let* $\lambda \in \mathfrak{h}^*$. *Then*

(1) $M(\lambda)$ *is a free* $U(\mathfrak{n}^-)$*-module of rank* 1, *generated by any highest weight vector.*
(2) $\Psi(M(\lambda)) = \lambda - Q^+$.

Proof. (1) Let $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$ and let $f_i := e_{-\alpha_i}$ so that $\{f_1, \ldots, f_m\}$ is a basis for $\mathfrak{n}^-$. The PBW Theorem 1.17 tells us that the monomials $\{f^n : n \in \mathbb{N}^m\}$ form a free basis for $U(\mathfrak{g})$ as a *right* $U(\mathfrak{b})$-module. Now applying $- \otimes_{U(\mathfrak{b})} k_\lambda$ shows that the monomials $\{f^n \otimes 1 : n \in \mathbb{N}^m\}$ form a basis for $M(\lambda)$ as a k-vector space. In other words, $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank 1.

(2) We know that $\Psi(M(\lambda)) \subseteq \lambda - Q^+$ by Lemma 4.9(c). Since $f^n \otimes 1 \neq 0$ for each $n \in \mathbb{N}^m$, we see that $M(\lambda)_{\lambda-\beta} \neq 0$ for any $\beta \in \mathbb{N}\Phi^+ = Q^+$. So, $\lambda - Q^+ \subseteq \Psi(M(\lambda))$. □

Next, we look for highest weight vectors in $M(\lambda)$ with weight $\mu < \lambda$: we have seen this idea already in the case of $\mathfrak{sl}(2)$. The key calculation is in the following proposition.

Proposition 4.21. *Let* $M(\lambda)$ *be a Verma module,* $\lambda \in \mathfrak{h}^*$, *and let* $v \in M(\lambda)$ *be a highest weight vector of weight* λ. *Let* $\alpha \in \Phi^+$ *be a positive root. If* $n := \lambda(h_\alpha) \in \mathbb{N}$, *then* $e_{-\alpha}^{n+1} \cdot v$ *is a highest weight vector of weight* $\mu := \lambda - (n+1)\alpha < \lambda$.

Proof. Write $f := e_{-\alpha}$ and $v' = f^{n+1} \cdot v$. Then v' is a μ-weight vector by Lemma 4.2(2), and we need to check that $\mathfrak{n}^+ \cdot v' = 0$. Let $\beta \in \Phi^+$, and suppose first that $\beta \in \Pi$.

If $\beta \neq \alpha$, then $f = e_{-\alpha}$ and e_β commute, because $\beta - \alpha$ is not a root [6], so

$$e_\beta \cdot v' = e_\beta \cdot f^{n+1} \cdot v = f^{n+1} e_\beta \cdot v = 0.$$

If $\beta = \alpha$, then using $\lambda(h_\alpha) = n$ together with Lemma 2.1 we see that

$$e_\alpha \cdot v' = e_\alpha e_{-\alpha}^{n+1} \cdot v = [e_\alpha, e_{-\alpha}^{n+1}] \cdot v + e_{-\alpha}^{n+1} e_\alpha \cdot v = (n+1) e_{-\alpha}^n (h_\alpha - n) \cdot v + 0 = 0$$

Finally if $\beta \in \Phi^+$ is not necessarily simple, then it can be written as a sum of finitely many simple roots, which means that e_β can be written as a repeated commutator of simple root vectors by Lemma 3.19(3). Hence it also must kill v'. □

Corollary 4.22. *Let* $\alpha \in \Phi^+$, *suppose that* $n = \lambda(h_\alpha) \in \mathbb{N}$ *and let* $\mu = \lambda - (n+1)\alpha$. *Then there exists an injective homomorphism* $M(\mu) \hookrightarrow M(\lambda)$, *whose image lies in the unique maximal submodule* $N(\lambda)$ *of* $M(\lambda)$.

Proof. If v' is a highest weight vector of $M(\mu)$ of weight μ, then the homomorphism is defined by sending $v' \mapsto e_{-\alpha}^{n+1} \cdot v$, where v is a highest weight vector of weight λ in $M(\lambda)$ and extending as a $U(\mathfrak{g})$-homomorphism. The fact that this is injective follows from the fact that the two Verma modules are free of rank one over $U(\mathfrak{n}^-)$ by Lemma 4.20(1). Since the image of the homomorphism is a proper submodule of $M(\lambda)$, it must lie in the unique maximal submodule. □

To complete the proof of Theorem 4.19 we need the following technical

Lemma 4.23. *Let* $\{f_1, \ldots, f_m\}$ *be a basis for* $\mathfrak{n}^-$ *and let* $n_1, \ldots, n_m \in \mathbb{N}$ *be given. Let* $U := U(\mathfrak{n}^-)$. *Then the left ideal* $I := U f_1^{n_1} + U f_2^{n_2} + \cdots + U f_m^{n_m}$ *has finite codimension in* U.

[6] otherwise β would not be an indecomposable root

Proof. Let W be the k-linear span of all monomials $f^\alpha := f_1^{\alpha_1} \cdots f_m^{\alpha_m}$ where $\alpha \in \mathbb{N}^m$ satisfies $0 \leq \alpha_i < n_i$ for all i. It is enough to show that $I + W = U$. Let $\{U_n : n \geq 0\}$ be the PBW filtration on U; we will show by induction on n that $U_n \subseteq I + W$ for all $n \geq 0$. The base case $n = 0$ is trivial since $1 \in W$, so suppose that $n \geq 1$ and that $U_{n-1} \subseteq I + W$. Let $\alpha \in \mathbb{N}^m$ be given with $|\alpha| := \alpha_1 + \cdots + \alpha_m = n$. If $\alpha_i < n_i$ for all $i = 1, \ldots, m$, then $f^\alpha \in W$. Otherwise, if $\alpha_i \geq n_i$ for some i, then using the fact that $\operatorname{gr} U$ is commutative — see Lemma 1.12(b) — we have

$$f^\alpha \equiv f_1^{\alpha_1} \cdots f_{i-1}^{\alpha_{i-1}} f_{i+1}^{\alpha_{i+1}} \cdots f_m^{\alpha_m} f_i^{\alpha_i - n_i} \cdot f_i^{n_i} \mod U_{n-1}.$$

Hence $f^\alpha \in I + U_{n-1} \subseteq I + W$ by induction. Thus we see that $U_n = k\{f^\alpha : |\alpha| = n\} + U_{n-1} \subseteq I + W$. □

Proof of Theorem 4.19. Let $L(\lambda) = U(\mathfrak{g}) \cdot v$ for some highest weight vector v.

Suppose that $L(\lambda)$ is finite dimensional. Then $\Psi(L(\lambda)) \subseteq P$ by Lemma 3.22, so $\lambda \in P$. For any $\alpha \in \Phi^+$, since $\mathfrak{g}_\alpha \cdot v \subseteq \mathfrak{n}^+ \cdot v_\lambda = 0$, we see that v is a highest weight vector for $\mathfrak{sl}_\alpha$. But then $\lambda(h_\alpha) \in \mathbb{N}$ by Lemma 2.7(c) and thus $\lambda \in P^+$.

Conversely, suppose that $\lambda(h_\alpha) \in \mathbb{N}$ for all $\alpha \in \Phi^+$. Then $e_{-\alpha}^{\lambda(h_\alpha)+1} \cdot v = 0$ by Corollary 4.22. Let $I := \{u \in U(\mathfrak{n}^-) : u \cdot v = 0\}$; this is a left ideal of $U(\mathfrak{n}^-)$ which contains $e_{-\alpha}^{\lambda(h_\alpha)+1}$ for all $\alpha \in \Phi^+$. Therefore I has finite codimension in $U(\mathfrak{n}^-)$ by Lemma 4.23. But $U(\mathfrak{n}^-)/I$ is isomorphic to $U(\mathfrak{n}^-) \cdot v = L(\lambda)$, so $L(\lambda)$ must be finite dimensional. □

We can now state the classification of irreducible finite dimensional $\mathfrak{g}$-modules.

Corollary 4.24 (Cartan-Weyl)**.**

1. *Every irreducible finite dimensional $\mathfrak{g}$-module V is isomorphic to $L(\lambda)$ for some $\lambda \in P^+$.*
2. *Every finite dimensional $\mathfrak{g}$-module is a direct sum of simple modules $L(\lambda)$, $\lambda \in P^+$.*

Proof. (1) By Lemma 4.5, V lies in category $\mathcal{O}$. Hence $V \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$ by Theorem 4.17. But then λ must be integral dominant by Theorem 4.19.

(2) Apply (1) together with Weyl's theorem on complete reducibility.[7] □

We now have a classification via integral dominant highest weights of the simple finite dimensional $\mathfrak{g}$-modules. Other typical information that one would still like to have in representation theory is:

- the dimension of $L(\lambda)$,
- the formal character of $L(\lambda)$,
- models (or explicit realizations) of $L(\lambda)$.

We will obtain satisfactory answers for the first two topics, but the third topic, except for some particular examples, is beyond the scope of this course.

5. The centre of $U(\mathfrak{g})$

5.1. **Central characters.** Recall the notation $Z(\mathfrak{g})$ for the centre of $U(\mathfrak{g})$. We wish to understand the structure of $Z(\mathfrak{g})$. We will first look at the action of $Z(\mathfrak{g})$ on modules in $\mathcal{O}$, but we begin in greater generality. If A is an associative k-algebra, every A-module M has an associated representation $\rho_M : A \to \operatorname{End}_k(M)$, given by $\rho(a)(m) = a \cdot m$. Moreover, if Z is the centre of A then $\rho_M(Z)$ is contained in $\operatorname{End}_A(M)$.

Definition 5.1. *The* central character *of M is the restriction of ρ_M to Z:*

$$\chi_M := \rho_{M|Z} : Z \to \operatorname{End}_A(M)$$

In this notation, the centre Z of A acts on M via the central character χ_M:

$$z \cdot m = \chi_M(z)(m) \quad \text{for all} \quad z \in Z, m \in M.$$

In situations of interest to us, our A-modules M have small A-linear endomorphism rings.

Lemma 5.2. *Let $\lambda \in \mathfrak{h}^*$. Then every $z \in Z(\mathfrak{g})$ acts by a scalar $\chi_{M(\lambda)}(z) \in k$ on the Verma module $M(\lambda)$.*

Proof. Proposition 4.13(2) tells us that $\operatorname{End}_{U(\mathfrak{g})}(M(\lambda)) = k$. So,

$$\chi_\lambda := \chi_{M(\lambda)} : Z(\mathfrak{g}) \to \operatorname{End}_{U(\mathfrak{g})}(M(\lambda))$$

takes values in k, and $z \cdot m = \chi_\lambda(z)m$ for all $z \in Z(\mathfrak{g})$ and $m \in M(\lambda)$. □

[7] See Theorem 15.10 for a proof of Weyl's complete reducibility theorem is using the action of the Casimir operator.

Definition 5.3. *Pick a basis $\{f_1,\ldots,f_m\}$ for $\mathfrak{n}^-$, $\{h_1,\ldots,h_r\}$ for $\mathfrak{h}$ and $\{e_1,\ldots,e_m\}$ for $\mathfrak{n}^+$. Then by the PBW Theorem 1.17, the monomials $\{f^a h^b e^c, a\in\mathbb{N}^m, b\in\mathbb{N}^r, c\in\mathbb{N}^m\}$ forms a basis for $U(\mathfrak{g})$.*

(1) *Define* pr $: U(\mathfrak{g})\to U(\mathfrak{h})$ *to be the unique k-linear map which sends all such monomials to zero,* unless $a=c=0$ *in which case* pr *sends the monomial h^b to itself.*
(2) *The* Harish Chandra homomorphism *is the restriction $\xi_{|Z(\mathfrak{g})}$ of* pr *to $Z(\mathfrak{g})$.*

Note that pr does not depend on the choice of bases made: *any* choice of bases induces a decomposition

$$U(\mathfrak{g})=\mathfrak{n}^-U(\mathfrak{g})\oplus U(\mathfrak{h})\oplus U(\mathfrak{g})\mathfrak{n}^+$$

of vector spaces, and pr is simply the projection onto the middle factor along the other two summands. Clearly pr is a linear surjective map.

Definition 5.4. *Let V be a vector space. A* polynomial function *on V is an element of the symmetric algebra $S(V^*)$. We will sometimes write*

$$\mathcal{O}(V):=S(V^*)$$

Example 5.5. Let $\{v_1,v_2\}$ be a basis for V. Then every linear function of the form

$$\lambda_1v_1+\lambda_2v_2\mapsto\lambda_1c_1+\lambda_2c_2$$

for some $c_1,c_2\in k$ is also a polynomial function on V: in fact this function is simply $c_1f_1+c_2f_2$ where $\{f_1,\ldots,f_2\}$ is the dual basis for V^*. And then a general polynomial function on V has the form

$$\lambda_1v_1+\lambda_2v_2\mapsto\sum_{i,j=0}^{N}c_{ij}\lambda_1^i\lambda_2^j$$

for some $c_{ij}\in k$; this is in fact precisely $\sum_{i,j=0}^N c_{ij}f_1^if_2^j$ in $S(V^*)$.

The following fact holds because our ground field k is infinite.

Lemma 5.6. *Suppose that $f\in\mathcal{O}(V)$ is such that $f(v)=0$ for all $v\in V$. Then $f=0$.*

We will identify $U(\mathfrak{h})=S(\mathfrak{h})$ with $\mathcal{O}(\mathfrak{h}^*)=S(\mathfrak{h}^{**})$. Let $\mathrm{ev}_\lambda:\mathcal{O}(\mathfrak{h}^*)\to k$ be the evaluation at λ homomorphism: this is a very convenient geometric way of thinking about the unique extension of the linear functional $\lambda:\mathfrak{h}\to k$ to a k-algebra homomorphism $\mathrm{ev}_\lambda:S(\mathfrak{h})\to k$.

Proposition 5.7. *We have $\chi_\lambda=\mathrm{ev}_\lambda\circ\xi$ for all $\lambda\in\mathfrak{h}^*$.*

Proof. Write $\Phi^+=\{\alpha_1,\ldots,\alpha_m\}$ and choose $0\neq e_i\in\mathfrak{g}_{\alpha_i}$ for each i. Let $v\in M(\lambda)_\lambda$ be a highest weight vector and let $z\in Z(\mathfrak{g})$. Consider a monomial $m:=f^ah^be^c$ appearing in z. If $c\neq 0$, then $m\cdot v=0$. If $c=0$, then $m\cdot v\in M(\lambda)_{\lambda-\sum_{i=1}^m a_i\alpha_i}$. On the other hand,

$$z\cdot v=\chi_\lambda(z)v\in M(\lambda)_\lambda$$

forces $m\cdot v=0$ whenever $c=0$ and $a\neq 0$. So, $m\cdot v=0$ unless $a=c=0$, so that

$$z\cdot v=\mathrm{pr}(z)\cdot v=\mathrm{pr}(z)(\lambda)v=\mathrm{ev}_\lambda(\mathrm{pr}(z))v\quad\text{for all}\quad z\in Z(\mathfrak{g}). \tag{5.1.1}$$

Thinking of $S(\mathfrak{h})$ as $\mathcal{O}(\mathfrak{h}^*)$, this shows that $\mathrm{ev}_\lambda\circ\xi=\chi_\lambda$. □

Corollary 5.8. *The map $\xi:Z(\mathfrak{g})\to S(\mathfrak{h})$ is an algebra homomorphism.*

Proof. By Proposition 5.7, $\xi(z)(\lambda)=\chi_\lambda(z)$ for all $\lambda\in\mathfrak{h}^*$ and $z\in Z(\mathfrak{g})$. Given $z,w\in Z(\mathfrak{g})$ the element $\xi(zw)-\xi(z)\xi(w)$ of $\mathcal{O}(\mathfrak{h}^*)$ vanishes at all $\lambda\in\mathfrak{h}^*$ because χ_λ is an algebra homomorphism for all $\lambda\in\mathfrak{h}^*$. So Lemma 5.6 implies that $\xi(zw)=\xi(z)\xi(w)$. □

5.2. **The Image of Harish-Chandra.** To understand infinitesimal characters further, we need to know when $\chi_\lambda=\chi_\mu$ for $\lambda,\mu\in\mathfrak{h}^*$. Recall from Sheet 2 Question 2 that $\rho:=\frac{1}{2}\sum\limits_{\alpha\in\Phi^+}\alpha\in P$.

Definition 5.9. *Define the* dot-action *of W on $\mathfrak{h}^*$ by*

$$w\bullet\lambda=w(\lambda+\rho)-\rho,\quad w\in W,\lambda\in\mathfrak{h}^*. \tag{5.2.1}$$

We say that λ and μ are linked *if $\mu=w\bullet\lambda$ for some $w\in W$.*

Notice that μ and λ are linked if and only if $\mu+\rho$ and $\lambda+\rho$ are in the same W-orbit in $\mathfrak{h}^*$ under the natural action of W. In particular, "linkage" is an equivalence relation on $\mathfrak{h}^*$.

Proposition 5.10. *If $\lambda, \mu \in P$ are linked then $\chi_\lambda = \chi_\mu$.*

Proof. Suppose first that $\mu = s_\alpha \bullet \lambda$ for some $\alpha \in \Pi$. Then $\lambda(h_\alpha) \in \mathbb{Z}$ as $\lambda \in P$. Now $\rho(h_\alpha) = 1$ by Sheet 2 Question 2(b) because $\alpha \in \Pi$ is a simple root, so

$$\mu = s_\alpha(\lambda+\rho) - \rho = \lambda + \rho - \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)}\alpha - \rho = \lambda - (\lambda+\rho)(h_\alpha)\alpha = \lambda - (\lambda(h_\alpha)+1)\alpha$$

by Definition 3.26(1) and Lemma 3.27(1). So, if $\lambda(h_\alpha) \geq 0$ then $M(\mu) \hookrightarrow M(\lambda)$ by Corollary 4.22. But then $Z(\mathfrak{g})$ acts on $M(\mu)$ through the character χ_λ as well as χ_μ, so that $\chi_\lambda = \chi_\mu$. If $\lambda(h_\alpha) = -1$ then $\mu = \lambda$ and there is nothing to prove. If $\lambda(h_\alpha) \leq -2$ then $\lambda = s_\alpha \bullet \mu$ and $\mu(h_\alpha) = -\lambda(h_\alpha) - 2 \geq 0$ because $\alpha(h_\alpha) = 2$ by Theorem 3.17(8), so we are back in the first case with the roles of λ and μ reversed.

Finally, suppose that $\mu = w \bullet \lambda$ for some $w \in W$. Since w is by Definition 3.26(2) a product of simple reflections s_α, we see that $\chi_\mu = \chi_\lambda$ in this case as well. □

Proposition 5.11. *The image of the Harish-Chandra homomorphism $\xi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ lies in the subalgebra $S(\mathfrak{h})^{W\bullet} = \mathcal{O}(\mathfrak{h}^*)^{W\bullet}$ of invariant polynomials for the shifted W-action.*

Proof. The shifted W-action on $\mathcal{O}(\mathfrak{h}^*)$ is defined in the usual way by

$$(w \bullet f)(\lambda) = f(w^{-1} \bullet \lambda) \quad \text{for all} \quad w \in W, f \in \mathcal{O}(\mathfrak{h}^*), \lambda \in \mathfrak{h}^*.$$

Now, let $z \in Z(\mathfrak{g})$ and $w \in W$. Then for any $\lambda \in P$ we have

$$\xi(z)(\lambda) = \chi_\lambda(z) = \chi_{w\bullet\lambda}(z) = \xi(z)(w \bullet \lambda)$$

by Proposition 5.7 and Proposition 5.10, so $w^{-1} \bullet \xi(z) - \xi(z)$ vanishes on P for any $w \in W$. Since P is Zariski dense in $\mathfrak{h}^*$, $\xi(z) \in \mathcal{O}(\mathfrak{h}^*)$ is invariant for the shifted W-action for any $z \in Z(\mathfrak{g})$. □

Corollary 5.12. *If $\lambda, \mu \in \mathfrak{h}^*$ are linked, then $\chi_\lambda = \chi_\mu$.*

Proof. Say $\mu = w \bullet \lambda$. Let $z \in Z(\mathfrak{g})$; then $w^{-1} \bullet z = z$ by Proposition 5.11, so

$$\chi_\lambda(z) = \xi(z)(\lambda) = (w^{-1} \bullet \xi(z))(\lambda) = \xi(z)(\mu) = \chi_\mu(z)$$

by Proposition 5.7. □

Example 5.13. Let $\mathfrak{g} = \mathfrak{sl}(2)$ and consider its Casimir element $C := h^2 + 2h + 4fe \in Z(\mathfrak{g})$.

(1) Because $fe \in \mathfrak{n}^- U(\mathfrak{g})$ we have $\mathrm{pr}(C) = h^2 + 2h$.
(2) Let $h^* \in \mathfrak{h}^*$ be defined by $h^*(h) = 1$ and let $\Phi^+ = \{\alpha\}$. Then since $h_\alpha = h$ we have $\alpha(h) = \alpha(h_\alpha) = 2$.
(3) Thus $\alpha = 2h^*$ and $\rho = \frac{1}{2}\alpha = h^*$.
(4) The Weyl group $W = \langle s \rangle$ where $s = s_\alpha$ sends α to $-\alpha$ so that s is the negation map on $\mathfrak{h}^*$.
(5) The shifted W-action on $S(\mathfrak{h}) = k[h]$ is given by

$$(s \bullet h)(\lambda) = h(s \bullet \lambda) = h(s(\lambda+\rho) - \rho) = h(-\lambda - 2\rho) = -\lambda(h) - 2 = (-h-2)(\lambda) \quad \text{for all} \quad \lambda \in \mathfrak{h}^*.$$

So $s \bullet h = -h - 2$.
(6) Therefore $s \bullet \xi(C) = s \bullet (h(h+2)) = (-h-2)(-h) = \xi(C)$ and $\xi(C) \in S(\mathfrak{h})^{W\bullet}$.

Theorem 5.14 (Harish-Chandra)**.** *The algebra homomorphism $\xi : Z(\mathfrak{g}) \to S(\mathfrak{h})^{W\bullet}$ is an isomorphism.*

Proof. We view $Z(\mathfrak{g})$ as the subspace of $\mathfrak{g}$-invariants for the adjoint representation of $\mathfrak{g}$ in $U(\mathfrak{g})$: $Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$.

Let $\zeta : S(\mathfrak{g}) \to S(\mathfrak{h})$ be the unique k-algebra homomorphism which kills $\mathfrak{n}^- \oplus \mathfrak{n}^+$ and which restricts to the identity map on $S(\mathfrak{h})$ — this is a commutative analogue of the Harish-Chandra projection from Definition 5.3, $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h})$. Note that pr respects filtrations on $U(\mathfrak{g})$ and $U(\mathfrak{h})$, and that the diagram

$$\begin{tikzcd}
\mathrm{gr}(U(\mathfrak{g})^{\mathfrak{g}}) \arrow[r, "\mathrm{gr}\,\xi"] \arrow[d, "\cong"'] & \mathrm{gr}(U(\mathfrak{h})^{W\bullet}) \arrow[d] \\
S(\mathfrak{g})^{\mathfrak{g}} \arrow[r, "\zeta_{|S(\mathfrak{g})^{\mathfrak{g}}}"'] & S(\mathfrak{h})^{W}
\end{tikzcd}$$

is commutative; here we identify $\operatorname{gr} U(\mathfrak{g})$ with $S(\mathfrak{g})$ using the isomorphism from Theorem 1.17. Recall the symmetrisation map $\phi : S(\mathfrak{g}) \to U(\mathfrak{g})$ from Definition 1.18. The arrow on the left is an isomorphism because for every homogeneous $u \in S^n(\mathfrak{g})^{\mathfrak{g}}$, the symmetrised element $\phi(u)$ lies in $U(\mathfrak{g})^{\mathfrak{g}}$ and satisfies $\phi(u) \equiv u \mod F_{n-1}U(\mathfrak{g})$. The arrow on the right is an isomorphism because the shifted W-action on $U(\mathfrak{h})$ induces the usual W-action on $\operatorname{gr} U(\mathfrak{h}) \cong S(\mathfrak{h})$.

The Killing isomorphism $\kappa : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^*$ extends to a k-algebra isomorphism $\kappa : S(\mathfrak{g}) \xrightarrow{\cong} \mathcal{O}(\mathfrak{g})$. By Theorem 3.17(3), we also get a k-algebra isomorphism $\kappa : S(\mathfrak{h}) \xrightarrow{\cong} \mathcal{O}(\mathfrak{h})$. Let $\theta : \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{h})$ be the restriction map which sends a polynomial map on $\mathfrak{g}$ to its restriction to $\mathfrak{h}$; then the diagram

$$\begin{tikzcd} S(\mathfrak{g}) \arrow[r, "\zeta"] \arrow[d, "\kappa"'] & S(\mathfrak{h}) \arrow[d, "\kappa"] \\ \mathcal{O}(\mathfrak{g}) \arrow[r, "\theta"'] & \mathcal{O}(\mathfrak{h}) \end{tikzcd}$$

is commutative. To see this, note that the k-algebra homomorphisms $\theta \circ \kappa$ and $\kappa \circ \zeta$ both kill $\mathfrak{n}^+ \oplus \mathfrak{n}^-$, and restrict to $\kappa : \mathfrak{h} \to \mathfrak{h}^*$ on $\mathfrak{h} \subset \mathfrak{g}$. Now we can apply the *Chevalley Restriction Theorem* 5.15 below to deduce that $\operatorname{gr} \xi$ is an isomorphism. Hence ξ is also an isomorphism. □

Theorem 5.15 (Chevalley). *The restriction map $\theta : \mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{h})$, $f \mapsto f_{|\mathfrak{h}}$, induces an isomorphism*

$$\theta : \mathcal{O}(\mathfrak{g})^{\mathfrak{g}} \quad \xrightarrow{\cong} \quad \mathcal{O}(\mathfrak{h})^W.$$

Proof. See [Be, Theorem 9.1] or [Hu1, §23.1]. □

Proposition 5.16. *Let $\lambda, \mu \in \mathfrak{h}^*$. Then $\chi_\lambda = \chi_\mu$ if and only if $W \bullet \lambda = W \bullet \mu$.*

Proof. In view of Corollary 5.12, we must show that if $W \bullet \lambda \neq W \bullet \mu$, then $\chi_\lambda \neq \chi_\mu$. Now $W \bullet \lambda$ and $W \bullet \mu$ are finite disjoint subsets of $\mathfrak{h}^*$, so by the Chinese Remainder Theorem we can find some $f \in \mathcal{O}(\mathfrak{h}^*)$ which is 1 on $W \bullet \lambda$ and 0 on $W \bullet \mu$. Now let $g := \sum_{w \in W} w \bullet f$: then $g \in U(\mathfrak{h})^{W\bullet}$ is still zero on $W \bullet \mu$ and non-zero on $W \bullet \lambda$. Using Theorem 5.14, choose $z \in Z(\mathfrak{g})$ such that $\xi(z) = g$. Then we deduce from Proposition 5.7 that $z \in \ker \chi_\mu$ but $z \notin \ker \chi_\lambda$, so $\chi_\lambda \neq \chi_\mu$ as claimed. □

Lemma 5.17. *Let R be a commutative ring and let G be a finite group acting on R by automorphisms. Then R is integral over the invariant ring R^G.*

Proof. Given $r \in R$, let $f(t) := \prod_{g \in G}(t - g \cdot r)$. Then $f(t) \in R^G[t]$ is monic and $f(r) = 0$. □

Theorem 5.18. *The Harish Chandra isomorphism induces a natural bijection*

$$\xi^* : \mathfrak{h}^*/W^\bullet \quad \xrightarrow{\cong} \quad \operatorname{MaxSpec}(Z(\mathfrak{g})), \qquad W \bullet \lambda \mapsto \ker \chi_\lambda.$$

Proof. Corollary 5.12 tells us $\ker \chi_\lambda$ only depends on $W \bullet \lambda$, so ξ^* is well-defined. Proposition 5.16 implies that ξ^* is injective. Now $S(\mathfrak{h})$ is integral over $S(\mathfrak{h})^{W\bullet}$ by Lemma 5.17. Using Theorem 5.14 together with the Going Up Theorem — Theorem 7.8(a) — for every maximal ideal M of $Z(\mathfrak{g})$ we can find a maximal ideal P of $S(\mathfrak{h})$ such that $M = \xi^{-1}(P)$. Since k is algebraically closed, $P = \ker \operatorname{ev}_\lambda$ for some $\lambda \in \mathfrak{h}^*$ by the Nullstellensatz, see for example Theorem 4.3. Therefore $M = \xi^{-1}(\ker \operatorname{ev}_\lambda) = \ker(\operatorname{ev}_\lambda \circ \xi) = \ker \chi_\lambda$ by Proposition 5.7, so ξ^* is surjective. □

The following is known about the relation between $S(\mathfrak{h})$ and its invariant subring $S(\mathfrak{h})^W$.

Theorem 5.19 (Chevalley-Shephard-Todd).

(1) *$S(\mathfrak{h})^W$ is a polynomial algebra in $\dim \mathfrak{h}$ variables.*
(2) *$S(\mathfrak{h})$ is a finitely generated and free $S(\mathfrak{h})^W$-module.*

5.3. **Composition series.** Recall that we know that every $M \in \mathcal{O}$ is a Noetherian module.

Proposition 5.20. *Each $M \in \mathcal{O}$ is Artinian.*

Proof. By Proposition 4.10 we may assume that $M = M(\lambda)$ is a Verma module. Suppose for a contradiction that we have a strictly descending chain of $U(\mathfrak{g})$-submodules $M(\lambda) = N^0 > N^1 > N^2 > \cdots$ of $M(\lambda)$. By refining the chain, we may assume that each subquotient $S^i := N^i/N^{i+1}$ is non-zero and *cyclic* as a $U(\mathfrak{g})$-module, i.e. it is a highest weight module. Let μ_i be the highest weight of S_i. Then $\chi_{\mu_i} = \chi_\lambda$, so $\mu_i \in W \bullet \lambda$ by Proposition 5.16. The short exact sequence $0 \to N^{i+1}_{\mu_i} \to N^i_{\mu_i} \to S^i_{\mu_i} \to 0$ shows that $N^{i+1}_{\mu_i} < N^i_{\mu_i}$, so

$$\bigoplus_{\mu \in W\bullet\lambda} N^i_\mu \quad > \quad \bigoplus_{\mu\in W\bullet\lambda} N^{i+1}_\mu \quad \text{for all} \quad i \geq 0.$$

Let $V := \sum_{w\in W} M(\lambda)_{w\bullet\lambda}$; then $V \cap N_0 > V \cap N_1 > \cdots$ is a strictly descending chain of linear subspaces of V. This is impossible because $\dim V < \infty$ by Lemma 4.9(d). $\square$

Corollary 5.21. *Let $M \in \mathcal{O}$.*

(1) *M admits a finite composition series.*
(2) *The set of Jordan-Hölder factors of M is $\{L(\lambda_1), \ldots, L(\lambda_k)\}$ for some $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$.*

Proof. (1) M is Noetherian by Proposition 4.6(1) and Artinian by Proposition 5.20. Hence the Jordan-Hölder Theorem applies and M has a finite composition series. (2) This follows from Theorem 4.17(1). $\square$

Definition 5.22. *For each $\mu \in \mathfrak{h}^*$, let $[M : L(\mu)]$ denote the multiplicity with which $L(\mu)$ appears in a composition series of $M \in \mathcal{O}$. The numbers $[M(\lambda) : L(\mu)]$ are called the* Kazhdan-Lusztig *multiplicities.*

By the Jordan-Hölder theorem, this does not depend on the choice of composition series of M.

6. Character formulas

The goal is to obtain the **Weyl character formula** and the **dimension formula** for the finite dimensional simple modules $L(\lambda)$, $\lambda \in P^+$. We follow the expositions by Bernstein [Be] and Fulton-Harris [FH].

6.1. **Formal characters.** Recall from Corollary 4.11(1) that each weight space of every object of category $\mathcal{O}$ is finite dimensional. Therefore the following definition makes sense.

Definition 6.1. *The* formal character *of a module $M \in \mathcal{O}$ is the function*

$$\mathsf{ch}_M : \mathfrak{h}^* \to \mathbb{Z}, \quad \mathsf{ch}_M(\lambda) = \dim M_\lambda < \infty.$$

We have the following basic properties of formal characters.

Lemma 6.2. *Let $M \in \mathcal{O}$.*

(1) *ch_M is well-defined, i.e. $\mathsf{ch}_M(\lambda) \in \mathbb{N}$ for all $\lambda \in \mathbb{N}$.*
(2) *$\mathsf{ch}_M = 0$ if and only if $M = 0$.*
(3) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in $\mathcal{O}$, then $\mathsf{ch}_{M_2} = \mathsf{ch}_{M_1} + \mathsf{ch}_{M_3}$.*

What does ch_V look like when V is a finite dimensional $\mathfrak{g}$-module?

Definition 6.3. *Let $\mathfrak{a}$ be an algebra. An $\mathfrak{a}$-module M is called* $\mathfrak{a}$-finite *if it is a sum of finite dimensional $\mathfrak{a}$-modules.*

We have already encountered this notion in axiom ($\mathcal{O}3$) for category $\mathcal{O}$. The idea behind $\mathfrak{a}$-finite module is that we can extend to this setting the local properties of finite dimensional modules.

Lemma 6.4. *Let $\alpha \in \Pi$ be given and suppose that $M \in \mathcal{O}$ is an $\mathfrak{sl}_\alpha$-finite module. Then $\dim M_\mu = \dim M_{s_\alpha(\mu)}$ for every weight μ of M.*

Proof. Decompose $M = \bigoplus_{k\in\mathbb{Z}} M_k$ with respect to the action of h_α. Here M_k is the k-eigenspace of h_α. By Corollary 2.12(2), we know that e^k_α induces a linear isomorphism between the k-eigenspace and the $(-k)$-eigenspace of any finite dimensional $\mathfrak{sl}_\alpha$-module. But then this can also be applied to our $\mathfrak{sl}_\alpha$-finite module M. Denote $j_k : M_k \to M_{-k}$ the resulting linear isomorphism induced by the action of $e^k_{-\alpha}$. We can decompose

$$M_k = \bigoplus_{\mu\in\mathfrak{h}^*, \mu(h_\alpha)=k} M_\mu, \quad M_{-k} = \bigoplus_{\mu'\in\mathfrak{h}^*, \mu'(h_\alpha)=-k} M_{\mu'} = \bigoplus_{\mu\in\mathfrak{h}^*, \mu(h_\alpha)=k} M_{s_\alpha(\mu)}.$$

Now, if $v_\mu \in M_\mu$, we have $h_\alpha \cdot e^k_{-\alpha} \cdot v_\mu = [h_\alpha, e^k_{-\alpha}] \cdot v_\mu + e^k_{-\alpha} h_\alpha \cdot v_\mu = (\mu - k\alpha)(h_\alpha) e^k_{-\alpha} \cdot v_\mu = s_\alpha(\mu) e^k_{-\alpha} \cdot v_\mu$, since $k = \mu(h_\alpha)$. This shows that j_k maps M_μ to $M_{s_\alpha(\mu)}$. But then it has to induce a linear isomorphism between M_μ to $M_{s_\alpha(\mu)}$. □

Theorem 6.5. *Let $\lambda \in P^+$. Then $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ for all $\mu \in P$ and all $w \in W$.*

Proof. Suppose $\lambda \in P^+$. Then $L(\lambda)$ is finite dimensional by Theorem 4.19, so it is $\mathfrak{sl}_\alpha$-finite for any $\alpha \in \Pi$. Then $\dim L(\lambda)_\mu = \dim L(\lambda)_{s_\alpha(\mu)}$ for all $\mu \in \Psi(L(\lambda))$. Since α was arbitrary and the s_αs generate W, it follows that $\dim L(\lambda)_\mu = \dim L(\lambda)_{w(\mu)}$ for all $\mu \in \Psi(L(\lambda))$. □

The Weyl group W acts on $\mathfrak{h}^*$ via the natural action $w(\chi)$, $w \in W$, $\chi \in \mathfrak{h}^*$. This induces an action (the left regular action) on functions in the standard way:

$$(wf)(\chi) = f(w^{-1}(\chi)),\ w \in W,\ f : \mathfrak{h}^* \to k.$$

Theorem 6.5 can be restated more succinctly as follows:

Corollary 6.6. *The formal character of any finite dimensional $\mathfrak{g}$-module V is W-invariant:*

$$w\, \mathsf{ch}_V = \mathsf{ch}_V \quad \textit{for all} \quad w \in W.$$

Proof. By Corollary 4.24 and Lemma 6.2(3), we may assume that $V \cong L(\lambda)$ for some $\lambda \in P^+$. The result follows from Theorem 6.5. □

6.2. **Characters of $M(\lambda)$ and $L(\lambda)$.** The formal character of Verma modules $M(\lambda)$ is easily computable.

Definition 6.7.

(1) *The* Kostant partition function $K : \mathfrak{h}^* \to \mathbb{Z}$ *is given by $K(\mu) =$ the number of ways in which μ can be written as $\mu = \sum_{\alpha \in \Phi^+} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}_{\geq 0}$.*
(2) *The* negative Kostant partition function *is $p : \mathfrak{h}^* \to \mathbb{Z}$, $p(\mu) = K(-\mu)$.*

Lemma 6.8. *For every $\lambda \in \mathfrak{h}^*$ we have $\dim M(\lambda)_\mu = K(\lambda - \mu)$ for all $\mu \in \mathfrak{h}^*$, and in particular, $p = \mathsf{ch}_{M(0)}$.*

Proof. See Problem Sheet 3. □

Here is the first indication that formal characters are useful: $L(\lambda)$ is very mysterious, but its formal character is easily understood provided the Kazhdan-Lusztig multiplicities are known.

Proposition 6.9. *Let $\lambda \in \mathfrak{h}^*$.*

(1) *We have $\mathsf{ch}_{M(\lambda)} = \sum\limits_{\mu \in W \bullet \lambda} [M(\lambda) : L(\mu)] \mathsf{ch}_{L(\mu)}$.*
(2) *There exists a set of integers $\{b_{\lambda,\mu} : \mu \in W \bullet \lambda\}$ such that $\mathsf{ch}_{L(\lambda)} = \sum\limits_{\mu \in W \bullet \lambda} b_{\lambda,\mu} \mathsf{ch}_{M(\mu)}$.*

To prove this, we need the following Lemma.

Lemma 6.10. *Suppose that $M = M(\lambda)$ is a Verma module and let $\mu \in \mathfrak{h}^*$.*

(1) *If $[M(\lambda) : L(\mu)] > 0$ then $\mu \leq \lambda$ and $\mu \in W \bullet \lambda$.*
(2) *$[M(\lambda) : L(\lambda)] = 1$.*

Proof. (1) If $L(\mu)$ appears as a subquotient of $M(\lambda)$, then $\chi_\mu = \chi_\lambda$, so $\mu \in W \bullet \lambda$ by Proposition 5.16. Also $1 = \dim L(\mu)_\mu \leq \dim M(\lambda)_\mu$ forces $\mu \in \Psi(M(\lambda)) = \lambda - Q^+$ by Lemma 4.20.

(2) Note that $\lambda \notin \Psi(N(\lambda))$. Hence $[N(\lambda) : L(\lambda)] = 0$ and $[M(\lambda) : L(\lambda)] = 1$. □

Proof of Proposition 6.9. (1) Apply Lemma 6.2(3) to a composition series of $M(\lambda)$.

(2) Write $u_\mu = \mathsf{ch}_{M(\mu)}$ and $v_\mu = \mathsf{ch}_{L(\mu)}$ for each $\mu \in W \bullet \lambda$. Then by (1)

$$u_\mu = \sum_{\nu \in W \bullet \lambda} a_{\mu,\nu} v_\nu \quad \text{for all } \mu \in W \bullet \lambda$$

where $a_{\mu,\nu} = [M(\mu) : L(\nu)]$. This number is zero *unless* $\nu \leq \mu$, by Lemma 6.10(1). Using Lemma 6.10(2) we see that the matrix $(a_{\mu,\nu})$ is uni-lower-triangular with entries in $\mathbb{Z}$. So its inverse $(b_{\mu,\nu})$ exists and has the same properties. Therefore

$$v_\mu = \sum_{\nu \in W \bullet \lambda} b_{\mu,\nu} u_\nu \quad \text{for all } \mu \in W \bullet \lambda$$

and the result follows by taking $\mu = \lambda$. □

6.3. **Weyl character formula.**

Definition 6.11.

(1) *For every function* $f : \mathfrak{h}^* \to \mathbb{Z}$*, define the* support *of* f *to be*
$$\mathsf{supp} f = \{\chi \in \mathfrak{h}^* \mid f(\chi) \neq 0\}.$$
(2) *Define* $\mathcal{E}$ *to be the set of functions* $f : \mathfrak{h}^* \to \mathbb{Z}$ *such that there exist* $\lambda_1, \dots, \lambda_m \in \mathfrak{h}^*$ *such that*
$$\mathsf{supp} f \quad \subseteq \quad \bigcup_{i=1}^{m} \lambda_i - Q^+.$$
(3) *For every* $\mu \in \mathfrak{h}^*$*, define the* delta function at μ *to be* $\delta_\mu \in \mathcal{E}$ *defined by* $\delta_\mu(\chi) := \delta_{\mu,\chi}$.
(4) *The set* $\mathcal{E}$ *can be endowed with the* convolution product*, for* $f, g \in \mathcal{E}$*:*
$$(f \star g)(\mu) = \sum_{\chi \in \mathfrak{h}^*} f(\chi) g(\mu - \chi) = \sum_{\chi \in \mathfrak{h}^*} f(\mu - \chi) g(\chi). \tag{6.3.1}$$

Example 6.12. $\mathsf{supp}\ \mathsf{ch}_{M(\lambda)} = \lambda - Q^+$. *In particular,* $p \in \mathcal{E}$ *and* $\mathsf{supp}\ p = -Q^+$.

Because of the support condition on the elements of $\mathcal{E}$, there are only finitely many nonzero elements in the sum, hence the convolution is well defined.

Lemma 6.13.

(1) $(\mathcal{E}, +, \star)$ *is an associative and commutative ring with identity* δ_0.
(2) $\delta_\mu \star \delta_\lambda = \delta_{\mu+\lambda}$*, for all* $\mu, \lambda \in \mathfrak{h}^*$.

Lemma 6.14. $\mathsf{ch}_{M(\lambda)} = p \star \delta_\lambda \in \mathcal{E}$.

Proof. By Lemma 6.8, $\dim M(\lambda)_\mu = K(\lambda - \mu) = p(\mu - \lambda)$. Hence
$$(p \star \delta_\lambda)(\mu) = \sum_{\chi \in \mathfrak{h}^*} p(\mu - \chi)\delta_\lambda(\chi) = p(\mu - \lambda) = \dim M(\lambda)_\mu = \mathsf{ch}_{M(\lambda)}(\mu) \quad \text{for all} \quad \mu \in \mathfrak{h}^*. \qquad \square$$

Corollary 6.15. $\mathsf{ch}_M \in \mathcal{E}$ *for any* $M \in \mathcal{O}$.

Proof. If M is a submodule or a quotient module of a Verma module $M(\lambda)$, then $\mathsf{supp}\ \mathsf{ch}_M \subseteq \mathsf{supp}\ \mathsf{ch}_{M(\lambda)}$ which has the right shape by Lemma 6.14. Now apply Proposition 4.10 together with Lemma 6.2(3). $\square$

In the light of Lemma 6.14, it is desirable to find the inverse (if it exists) of p in $\mathcal{E}$.

Definition 6.16 (Weyl denominator)**.** *Define* $\Delta := \prod_{\alpha \in \Phi^+} (\delta_{\alpha/2} - \delta_{-\alpha/2}) \in \mathcal{E}$.

The reason for the name "denominator" comes from our next result, which informally states $p = \frac{\delta_\rho}{\Delta}$.

Lemma 6.17. *We have* $p \star \Delta \star \delta_{-\rho} = \delta_0$*, or equivalently* $p \star \Delta = \delta_\rho$.

Proof. Set $a_\alpha = \delta_0 + \delta_{-\alpha} + \delta_{-2\alpha} + \cdots + \delta_{-n\alpha} + \cdots \in \mathcal{E}$ for every $\alpha \in \Phi^+$. Then $p = \prod_{\alpha \in \Phi^+} a_\alpha$. Next notice that $a_\alpha \star (\delta_0 - \delta_{-\alpha}) = a_\alpha - a_\alpha \star \delta_{-\alpha} = \delta_0$. But then, as $\delta_{-\rho} = \prod_{\alpha \in \Phi^+} \delta_{-\alpha/2}$, we have
$$p \star \Delta \star \delta_{-\rho} = p \star \prod_{\alpha \in \Phi^+} (\delta_0 - \delta_{-\alpha}) = \prod_{\alpha \in \Phi^+} a_\alpha \star (\delta_0 - \delta_{-\alpha}) = \delta_0. \qquad \square$$

Let $\det(w)$ denote the determinant of $w \in W$ viewed as a linear transformation of $\mathfrak{h}^*$.

Lemma 6.18. *The function* Δ *is* W*-skew-invariant, i.e.,* $w\Delta = \det(w) \cdot \Delta$*, for all* $w \in W$.

Proof. It is sufficient to check that for every simple root α, $s_\alpha \Delta = -\Delta$. (Recall that $\det(s_\alpha) = -1$.) We know that s_α permutes the roots $\Phi^+ \setminus \{\alpha\}$ and $s_\alpha(\alpha) = -\alpha$. Hence
$$s_\alpha \Delta = \prod_{\beta \in \Phi^+} s_\alpha(\delta_{\beta/2} - \delta_{-\beta/2}) = (\delta_{-\alpha/2} - \delta_{\alpha/2}) \star \prod_{\beta \in \Phi^+ \setminus \{\alpha\}} (\delta_{\beta/2} - \delta_{-\beta/2}) = -\Delta. \qquad \square$$

Lemma 6.19. *If* $\lambda \in P^+$ *and* $w \in W$ *are such that* $w \bullet \lambda = \lambda$*, then* $w = 1$.

Proof. Suppose $1 \neq w \in W$. Then by Sheet 4 Question 4, there exists $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$. Write $\alpha = \sum_{\beta \in \Pi} n_\beta \beta$ with $n_\beta \in \mathbb{N}$ and not all $n_\beta = 0$. Then using Lemma 3.27(1) we have

$$(\lambda + \rho, \alpha) = \sum_{\beta \in \Pi} n_\beta \frac{(\beta, \beta)}{2} (\lambda + \rho)(h_\beta).$$

This expression is *strictly positive* because $\lambda \in P^+$ implies $\lambda(h_\beta) \geq 0$ for all $\beta \in \Pi$ and because some $n_\beta > 0$. If $w \bullet \lambda = \lambda$, then also $w^{-1}(\lambda + \rho) = \lambda + \rho$ and hence, using Lemma 3.27(2),

$$(\lambda + \rho, \alpha) = (w^{-1}(\lambda + \rho), \alpha) = (\lambda + \rho, w\alpha) < 0$$

by the previous equation applied to $w\alpha \in -\Phi^+$. This contradiction shows that in fact $w = 1$. $\square$

Theorem 6.20. *Suppose that $\lambda \in P^+$. Then $\Delta \star \mathsf{ch}_{L(\lambda)} = \sum\limits_{w \in W} \det(w)\, \delta_{w(\lambda+\rho)}$.*

Proof. By Lemma 6.19, the stabiliser of λ in W under the shifted action is trivial. By Proposition 6.9(2) together with Lemma 6.14 we have

$$\mathsf{ch}_{L(\lambda)} = \sum_{w \in W} b_{\lambda, w\bullet\lambda} \mathsf{ch}_{M(w\bullet\lambda)} = \sum_{w \in W} n_w \; p \star \delta_{w \bullet \lambda}$$

for some integers $n_w := b_{\lambda, w\bullet\lambda}$. Multiply both sides by Δ. Applying Lemma 6.17, we see that $\Delta \star p \star \delta_{w\bullet\lambda} = \delta_\rho \star \delta_{w\bullet\lambda} = \delta_{w(\lambda+\rho)}$. Therefore

$$\Delta \star \mathsf{ch}_{L(\lambda)} = \sum_{w \in W} n_w \delta_{w(\lambda+\rho)}$$

and it remains to show that $n_w = \det(w)$ for each $w \in W$. We claim that the left hand side of the identity is W-skew-invariant. Indeed, by Lemma 6.18 and Corollary 6.6,

$$w(\Delta \star \mathsf{ch}_{L(\lambda)}) = w\Delta \star w\mathsf{ch}_{L(\lambda)} = \det(w)\Delta \star w\mathsf{ch}_{L(\lambda)} = \det(w)\Delta \star \mathsf{ch}_{L(\lambda)}.$$

But then $\sum_{w \in W} n_w \delta_{w(\lambda+\rho)}$ is also W-skew-invariant, and since $n_1 = b_{\lambda,\lambda} = 1$, we get $n_w = \det(w)$. $\square$

Corollary 6.21 (Weyl's denominator formula)**.** $\prod\limits_{\alpha \in \Phi^+} (\delta_{\alpha/2} - \delta_{-\alpha/2}) = \Delta = \sum\limits_{w \in W} \det(w) \cdot \delta_{w(\rho)}$.

Proof. This is the case $\lambda = 0$ in Theorem 6.20. $\square$

Corollary 6.22. *Let $\lambda \in P^+$, and define $A_\lambda := \sum\limits_{w \in W} \det(w)\, \delta_{w(\lambda+\rho)}$.*

1. *(Weyl Character Formula) We have $A_\rho * \mathsf{ch}_{L(\lambda)} = A_{\lambda+\rho}$.*
2. *(BGG formula) We have $\mathsf{ch}_{L(\lambda)} = \sum\limits_{w \in W} \det(w)\; \mathsf{ch}_{M(w\bullet\lambda)}$.*
3. *(Kostant's multiplicity formula) $\mathsf{ch}_{L(\lambda)}(\mu) = \sum\limits_{w \in W} \det(w)\; K(w(\lambda+\rho) - (\mu+\rho))$.*

Proof. (1) Apply Theorem 6.20 together with Corollary 6.21.

(2) Multiply the formula from Theorem 6.20 by $p \star \delta_{-\rho}$ on the left. Then Lemma 6.17 tells us that

$$\mathsf{ch}_{L(\lambda)} = \sum_{w \in W} \det(w)\; p \star \delta_{w\bullet\lambda}$$

since $\delta_{w(\lambda+\rho)}\delta_{-\rho} = \delta_{w\bullet\lambda}$. Now use Lemma 6.14.

(3) is immediate from (2) together with Lemma 6.8. $\square$

Example 6.23. Let us consider the case of $\mathfrak{g} = \mathfrak{sl}(2)$. We identify, as we may, P^+ with $\mathbb{N}$ and then $\alpha = 2\rho$ is identified with $2 \in \mathbb{N}$. The Weyl group is $\{\pm 1\}$. Let $L(n)$, $n \in \mathbb{Z}_{\geq 0}$ be the simple module of dimension $n+1$. The weights of $L(n)$ are $n, n-2, n-4, \ldots, -n$. This means that in $\mathcal{E}$, the character of $L(n)$ equals

$$\mathsf{ch}_{L(n)} = \delta_n + \delta_{n-2} + \cdots + \delta_{-n+2} + \delta_{-n}.$$

On the other hand $\Delta = \delta_1 - \delta_{-1}$. Therefore, Corollary 6.22(1) becomes the easy identity

$$(\delta_n + \delta_{n-2} + \cdots + \delta_{-n+2} + \delta_{-n}) \star (\delta_1 - \delta_{-1}) = \delta_{n+1} - \delta_{-(n+1)}.$$

Corollary 6.22(2) in this case becomes $\mathsf{ch}_{L(n)} = \mathsf{ch}_{M(n)} - \mathsf{ch}_{M(-n-2)}$. Notice that $w \bullet n = -n-2$ when $w = -1 \in W$. This formula is a consequence of the short exact sequence

$$0 \longrightarrow M(-n-2) \longrightarrow M(n) \longrightarrow L(n) \longrightarrow 0.$$

6.4. **Weyl's dimension formula.** We wish to use the formula in Theorem 6.20 to deduce the dimension formula for the finite dimensional $\mathfrak{g}$-modules $L(\lambda)$. For this, we will find a much smaller ring than $\mathcal{E}$ that still allows an effective way of stating the character formula, namely the *group ring of integral weights*

$$\mathbb{Z}[P] = \bigoplus_{\lambda \in P} \mathbb{Z}[\lambda].$$

Here $[\lambda] \in \mathbb{Z}[P]$ is a formal symbol for each $\lambda \in P$, and $[\lambda] \cdot [\mu] = [\lambda + \mu]$.

Lemma 6.24. *The subring of $\mathcal{E}$ generated by $\{\delta_\lambda : \lambda \in P\}$ is isomorphic to $\mathbb{Z}[P]$.*

Proof. The map $\delta : P \to \mathcal{E}^\times$ given by $\lambda \mapsto \delta_\lambda$ is a group homomorphism by Lemma 6.13(1). Its image is linearly independent. Hence the extended ring homomorphism $\delta : \mathbb{Z}[P] \to \mathcal{E}$ is injective. $\square$

Definition 6.25.

(1) *The* augmentation homomorphism $\epsilon : \mathbb{Z}[P] \to \mathbb{Z}$ *is defined by* $\epsilon\left(\sum_{\lambda \in P} n_\lambda[\lambda]\right) = \sum_{\lambda \in P} n_\lambda$.

(2) *For each $\lambda \in P$, define* $F_\lambda := \sum_{w \in W} \det(w)[w(\lambda)] \in \mathbb{Z}[P]$.

In this language, the Weyl character formula says that $\mathsf{ch}_{L(\lambda)} = \delta\left(\frac{F_{\lambda+\rho}}{F_\rho}\right)$.

Recall that because k is a field of characteristic zero, the formal exponential series $e^{at} := \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}$ makes sense in the ring of formal power series $k[[t]]$, for any $a \in k$.

Proposition 6.26.

(1) *For each $\mu \in P$ there is a ring homomorphism*

$$\Psi_\mu : \mathbb{Z}[P] \to k[[t]] \qquad \text{defined by} \qquad \Psi_\mu([\lambda]) := e^{(\mu,\lambda)t} \qquad \text{for all} \quad \lambda \in P.$$

(2) *We have* $\Psi_\mu(F_\lambda) = \Psi_\lambda(F_\mu)$ *for all* $\lambda, \mu \in P$.

(3) *We have* $\Psi_\lambda(F_\rho) = \prod_{\alpha \in \Phi^+} (e^{(\alpha,\lambda)\frac{t}{2}} - e^{-(\alpha,\lambda)\frac{t}{2}})$ *for any* $\lambda \in P$.

(4) $\Psi_\lambda(F_\rho) = \left(\prod_{\alpha \in \Phi^+} (\alpha, \lambda)\right) t^{|\Phi^+|} \mod t^{|\Phi^+|+1}k[[t]]$ *for any* $\lambda \in P$.

(5) $\Psi_\lambda(S)(0) = \epsilon(S)$ *for all* $S \in \mathbb{Z}[P]$.

Proof. (1) The map $P \to k[[t]]^\times$ given by $\lambda \mapsto e^{(\mu,\lambda)t}$ is a group homomorphism. Now apply the universal property of the group ring $\mathbb{Z}[P]$.

(2) Because the bilinear form $(-,-)$ on $\mathfrak{h}^*$ is symmetric and W-invariant by Lemma 3.27(2), we have $(\mu, w\lambda) = (\lambda, w^{-1}\mu)$. Since $\det(w) = \det(w)^{-1}$ for any $w \in W$, we have

$$\Psi_\mu(F_\lambda) = \sum_{w \in W} \det(w) e^{(\mu, w\lambda)t} = \sum_{w \in W} \det(w^{-1}) e^{(\lambda, w^{-1}\mu)t} = \Psi_\lambda(F_\mu).$$

(3) The maps Ψ_λ and δ extend to $\mathbb{Z}[\frac{1}{2}P]$: there is a commutative diagram of commutative rings

$$\begin{tikzcd}
\mathbb{Z}[P] \arrow[r, "\Psi_\lambda"] \arrow[d, "\delta"'] \arrow[dr] & k[[t]] \\
\mathcal{E} & \mathbb{Z}\left[\frac{1}{2}P\right] . \arrow[l, dotted, "\delta"] \arrow[u, dotted, "\Psi_\lambda"']
\end{tikzcd}$$

Now $\delta(F_\rho) = \Delta = \delta\left(\prod_{\alpha \in \Phi^+} [\frac{\alpha}{2}] - [-\frac{\alpha}{2}]\right)$ by Corollary 6.21 and Definition 6.16. Since δ is injective, we obtain the formula $F_\rho = \prod_{\alpha \in \Phi^+} ([\frac{\alpha}{2}] - [-\frac{\alpha}{2}])$ valid in $\mathbb{Z}[\frac{1}{2}P]$. Now apply Ψ to this formula.

(4) For any $a \in k$, $e^{\frac{at}{2}} - e^{-\frac{at}{2}} \equiv at \mod t^2 k[[t]]$. Now apply part (3).

(5) Evaluation at zero is a ring homomorphism $k[[t]] \to k$, so it's enough to check the formula for $S = [\mu]$ with $\mu \in P$. But $\Psi_\lambda([\mu]) = e^{(\mu,\lambda)t}(0) = 1 = \epsilon([\mu])$ for any $\mu \in P$. $\square$

We can now deduce the dimension formula.

Theorem 6.27 (Weyl's dimension formula)**.** *Let* $\lambda \in P^+$. *Then* $\dim L(\lambda) = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$.

Proof. Since $L(\lambda)$ is finite dimensional by Theorem 4.19, its formal character already lies in $\mathbb{Z}[P]$: if

$$\ell(\lambda) := \sum_{\mu\in P} \dim L(\lambda)_\mu[\mu] \in \mathbb{Z}[P]$$

then because $\delta : \mathbb{Z}[P] \to \mathcal{E}$ is injective by Lemma 6.24, the Weyl Character Formula, Corollary 6.22(1) says

$$F_\rho \cdot \ell(\lambda) = F_{\lambda+\rho}.$$

Apply Ψ_ρ from Proposition 6.26(1) to both sides of this formula and use Proposition 6.26(2) to obtain

$$\Psi_\rho(F_\rho)\Psi_\rho(\ell(\lambda)) = \Psi_\rho(F_{\lambda+\rho}) = \Psi_{\lambda+\rho}(F_\rho).$$

Both sides are $k[[t]]$-multiples of $t^{|\Phi^+|}$ by Proposition 6.26(4). Compare the leading coefficients of both power series, i.e. divide both sides by $t^{|\Phi^+|}$ and then evaluate the result at zero. Proposition 6.26(4,5) then gives

$$\left(\prod_{\alpha\in\Phi^+}(\alpha,\lambda)\right)\cdot\epsilon(\ell(\lambda)) = \left(\prod_{\alpha\in\Phi^+}(\alpha,\lambda+\rho)\right).$$

Finally $\epsilon(\ell(\lambda)) = \sum_{\mu\in P} \dim L(\lambda)_\mu = \dim L(\lambda)$ by Definition 6.25 and we're done. □

6.5. **The BGG resolution.** Corollary 6.22(2) has a beautiful homological refinement. The first step is to determine the maximal submodule $N(\lambda)$ of $M(\lambda)$ when $\lambda \in P^+$.

Theorem 6.28. *If* $\lambda \in P^+$, *there exists an exact sequence*

$$\bigoplus_{\alpha\in\Pi} M(s_\alpha \bullet \lambda) \to M(\lambda) \to L(\lambda) \to 0.$$

We will not give a proof of this Theorem; note that our proof of Theorem 4.19 shows that the sum of the images of all the Verma modules $M(s_\alpha\bullet\lambda)$ where α ranges over *all* positive roots of $\mathfrak{g}$ has finite codimension in $M(\lambda)$ and is therefore contained in the unique maximal submodule $N(\lambda)$ of $M(\lambda)$. Theorem 6.28 strengthens this by showing that in fact $N(\lambda)$ equals the sum of these modules where we only take the sum over all *simple* roots α. The BGG resolution extends the sequence above to a resolution of $L(\lambda)$.

Theorem 6.29 (BGG resolution)**.** *Suppose* $\lambda \in P^+$. *Then there exists an exact sequence:*

$$0 \to M(w_0\bullet\lambda) \to \bigoplus_{\ell(w)=|\Phi^+|-1} M(w\bullet\lambda) \to \cdots \to \bigoplus_{\ell(w)=k} M(w\bullet\lambda) \to \cdots \to \bigoplus_{\alpha\in\Pi} M(s_\alpha\bullet\lambda) \to M(\lambda) \to L(\lambda) \to 0. \tag{6.5.1}$$

Proof. Here ℓ is the *length function* on W — see Sheet 4, Question 4 — and w_0 is the *longest element* of W. See [Hu2, Theorem 6.2] for the proof. □

As a consequence, the Euler-Poincaré principle implies that in $K(\mathcal{O})$:

$$\mathsf{ch}_{L(\lambda)} = \sum_{k=0}^{|\Phi^+|}(-1)^k \sum_{w\in W,\ell(w)=k} \mathsf{ch}_{M(w\bullet\lambda)} = \sum_{w\in W}(-1)^{\ell(w)}\mathsf{ch}_{M(w\bullet\lambda)}, \tag{6.5.2}$$

which is exactly the BGG formula, Corollary 6.22(2).

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