C2.3 Representations of semisimple Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 2

All Lie algebras are defined over an algebraically closed field k of characteristic zero.

1. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra with a nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form $B : \mathfrak{g} \times \mathfrak{g} \to \mathsf{k}$. Let $\{x_i \mid 1 \leq i \leq n\}$ and $\{y_i \mid 1 \leq i \leq n\}$ be dual bases of $\mathfrak{g}$ with respect to B. Define the *Casimir element* of $\mathfrak{g}$ (with respect to B) to be $C := \sum_{i=1}^{n} x_i y_i \in U(\mathfrak{g})$.

(i) Show that the definition of C does not depend on the choice of dual bases. Prove that C is in $Z(\mathfrak{g})$, the centre of $U(\mathfrak{g})$.

(ii) Suppose that $\mathfrak{g}$ is a simple Lie algebra. Show that if C and C' are Casimir elements with respect to the forms B and B', respectively, than C' is a scalar multiple of C. [Hint: there is only one nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form up to scalar. Why?]

(iii) Suppose L is a finite dimensional simple $\mathfrak{g}$-module. Deduce that C acts on L by a scalar.

(iv) Let $\mathfrak{g} = \mathfrak{sl}(n)$ and take $B(x, y) = \mathrm{tr}(xy)$, for $x, y \in \mathfrak{g}$. Verify that B is a nondegenerate symmetric bilinear $\mathfrak{g}$-invariant form and define C with respect to convenient dual bases. How does C differ from C', the Casimir element defined with respect to the Killing form?

(v) Let V be a finite dimensional simple $\mathfrak{g} = \mathfrak{sl}(n)$-module with highest weight λ. Compute the scalar by which the Casimir element C from (iv) acts on V.

Let $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, Φ the system of roots $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix a base of simple roots Π and the corresponding positive roots Φ^+.

2. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in \mathfrak{h}^*$. Show that:

(a) For every $\alpha \in \Pi$, $s_\alpha(\rho) = \rho - \alpha$.

(b) For every $\alpha \in \Pi$, $\rho(h_\alpha) = 1$, where h_α is the coroot of α.

(c) Write $\Pi = \{\alpha_1, \dots, \alpha_k\}$ and define $\omega_i \in \mathfrak{h}^*$ by $\omega_i(h_{\alpha_j}) = \delta_{ij}$ for each i, j. Show that $\{\omega_1, \dots, \omega_k\}$ is a $\mathbb{Z}$-basis of the weight lattice P of $\mathfrak{g}$.

(d) Show that $\rho = \sum_{i=1}^{k} \omega_i$.

3. Let P be the weight lattice and let Q be the root lattice of $\mathfrak{g} = \mathfrak{sl}(n)$. Show that $P/Q \cong \mathbb{Z}/n\mathbb{Z}$.

4. Let $\mathfrak{g} = \mathfrak{sl}(3)$ and let $\alpha := \epsilon_1 - \epsilon_2$ and $\beta := \epsilon_2 - \epsilon_3$.

(a) Show that $\{\alpha, \beta\}$ forms a base for $\mathfrak{g}$ and write down Φ^+ and Φ^- for this base.

(b) What is the angle between α and β?

(c) Show that W is isomorphic to the symmetric group S_3.

(d) Calculate ω_1, ω_2 and ρ in terms of α and β.

(e) What is the highest weight of the adjoint representation of $\mathfrak{g}$?

5. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\mathfrak{a}, \mathfrak{b}$ be subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ as vector spaces. Let $\lambda : \mathfrak{b} \to \mathsf{k}$ be a Lie algebra homomorphism and let k_λ be the 1-dimensional $\mathfrak{b}$-module defined by this character. Prove that the induced module $U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathsf{k}_\lambda$ is free of rank 1 as a $U(\mathfrak{a})$-module.