C2.3 Representations of semisimple Lie algebras

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Problem Sheet 4

Throughout, $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field k of characteristic zero.

1. Let V and W be two finite dimensional $\mathfrak{g}$-modules. Prove that $\mathsf{ch}_{V\otimes W} = \mathsf{ch}_V \cdot \mathsf{ch}_W$.

2. Let $\omega_1, \ldots, \omega_n$ be the fundamental weights of $\mathfrak{g}$. Show that every finite dimensional simple $\mathfrak{g}$-module occurs as a direct summand in a suitable tensor product (repetitions allowed) of the simple modules $L(\omega_1), \ldots, L(\omega_n)$. We call these simple modules *the fundamental representations of* $\mathfrak{g}$.

3. Use Weyl's dimension formula to show that for every natural number k, there exists a simple $\mathfrak{g}$-module of dimension k^r, where r is the number of positive roots of $\mathfrak{g}$.

4. The *length* of an element $w \in W$ is the smallest $n \in \mathbb{N}$ such that w can be written as a product of n simple reflections. Prove that $\ell(w) = |\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}|$.

[*Hint: Sheet 2 Question 2 may be useful.*]

5. (i) Let L be a finite dimensional $\mathfrak{g}$-module. Show that L is simple if and only if L^* is simple.

(ii) Let $L(\lambda)$ be a simple $\mathfrak{g}$-module with highest weight $\lambda \in P^+$. Show that the dual $L(\lambda)^*$ is isomorphic to $L(-w_0(\lambda))$, where w_0 is the Weyl group element sending the positive roots Φ^+ to $-\Phi^+$.

6. Let $\mathfrak{g} = \mathfrak{sl}(n)$.

(i) Use Weyl's dimension formula to calculate $\dim L(\omega_i)$ for each $1 \leq i \leq n-1$.

(ii) Why is the adjoint representation $\mathfrak{g}$ irreducible?

(iii) Find non-negative integers $k_1, \ldots, k_{n-1}$ such that $\mathfrak{g} \cong L(k_1\omega_1 + \cdots + k_{n-1}\omega_{n-1})$ as $\mathfrak{g}$-modules.

7. Define the Casimir element C in $U(\mathfrak{g})$ with respect to the Killing form of a semisimple Lie algebra $\mathfrak{g}$. Compute $\chi_\lambda(C)$ for the infinitesimal character defined by $\lambda \in \mathfrak{h}^*$, and the image of C under the Harish-Chandra isomorphism.

[*Hint: Sheet 2 Question 1 may be useful.*]

8. Let $\mathfrak{g} = \mathfrak{sl}(3)$ and $L(\omega_1)$, $L(\omega_2)$ the two fundamental representations. Verify:

(i) $L(\omega_1)^* \cong L(\omega_2)$.

(ii) Kostant's multiplicity formula, and

(iii) Weyl's character formula for these two representations.

9. Let $M(\lambda)$ be a Verma module for the semisimple Lie algebra $\mathfrak{g}$, and let L be a finite dimensional module. Consider the tensor product $T = M(\lambda) \otimes L$; it is in category $\mathcal{O}$ by Proposition 4.6(3). Prove that there exists a chain of submodules:

$$0 = T_{n+1} \subset T_n \subset T_{n-1} \subset \cdots \subset T_2 \subset T_1 = T,$$

where $n = \dim L$, and $T_i/T_{i+1} \cong M(\lambda + \mu_i)$, where $\mu_1, \ldots, \mu_n$ are the weights of L (counted with multiplicity) in an appropriate order.

[*Hint: try to generalise Sheet 3 Question 3(c) and try to use formal characters.*]

P.T.O.

10 (*Optional. But if you wish to have more practice more with finite dimensional representations and learn more examples...*)**.** Let $\mathfrak{g} = \mathfrak{sp}(2n)$ realized as the space of matrices $X \in \mathfrak{gl}(2n)$ such that $X^t J + JX = 0$, where X^t is the transpose matrix, and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$; here I_n is the $n \times n$ identity matrix.

(i) Show that every $X \in \mathfrak{g}$ is of the form $X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$, where B and C are symmetric $n \times n$ matrices and A is an arbitrary $n \times n$ matrix.

(ii) Let $\mathfrak{h}$ be the subalgebra consisting of diagonal matrices. Determine the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ and the Cartan decomposition.

(iii) Choose the system of positive roots such that the corresponding root vectors lie in matrices of the form $\begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix}$, where A is an upper triangular matrix and B is a symmetric matrix as before.

(iv) Determine the fundamental weights.

(v) Let $V = \mathsf{k}^{2n}$ be the standard representation of $\mathfrak{g}$ — it is the restriction of the natural representation of $\mathfrak{gl}(2n)$ to $\mathfrak{g}$. Show that V is an irreducible $\mathfrak{g}$-representation and it is in fact a fundamental representation.

(vi) Show that $\bigwedge^2 V$ decomposes as $W \bigoplus \mathsf{k}$, where k is the trivial representation and W is an irreducible (fundamental) representation.

(vii) For $\mathfrak{sp}(4)$, describe all the weights of the fundamental representations V and W and verify that the Weyl dimension formula holds for V and W.

(viii) In $\mathfrak{sp}(2n)$, show that the k-th fundamental representation is contained in $\bigwedge^k V$ and in fact it is precisely the kernel of the *contraction* map $\phi_k : \bigwedge^k V \to \bigwedge^{k-2} V$ defined by

$$\phi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k,$$

where Q is the skew-symmetric form defining $\mathfrak{g}$, i.e., $Q(v, u) = v^t J u$.

[*For this exercise, you may consult Section 16 in Fulton-Harris "Representation Theory", especially for the structural results on roots and the Cartan decomposition.*]